Numerical Comparisons for the Approximated First Order Polarization Tensor for Ellipsoids

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Abstract: *This paper discusses the first order polarization tensor for ellipsoids computed by our two previous methods. We compare the first order polarization tensor for several ellipsoids computed by both methods to the analytical solutions. The results show that the first order polarization tensor for several ellipsoids computed by one of the method are closer to the analaytical solution and thus, suggest that the method must be seriously considered to study PT in the future.*

1 Introduction

The study of the Polarization Tensor (PT) becomes very significant lately in the electric and electromagnetic applications. In Electrical Impedance Tomography (EIT) system, the PT is studied to improve the image reconstruction algorithm of a conducting inclusion. Besides, it can also be used to measure the effective properties of composites in the asymptotic models of dilute composite. These are possible to achieve as the PT contains significant information about geometric and physical properties of the subject. Detail discussions focusing on mathematical aspects of the PT for this purpose can be found in Ammari and Kang [1].

On the other hand, Marsh et al. [2] describe the location, orientation and material property of a target in a metal detector through the PT. Similarly, we have adapted the first

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order of the Generalized Polarization Tensor (GPT) from [1] in [3] to investigate whether it plays some role in characterization of objects by electrosensing fish. In these two applications, an object is described by fitting the PT for the object only without reconstrucing the image of the object as it offers lower computational efforts. This suggests the importance of computational aspects of the PT especially since it is difficult to determine the PT for any object by analytical approach. Therefore, the aim of this study is to discuss the convergence of the first order PT specifically for ellipsoids approximated by our two previous methods.

Generally, the first order PT for ellipsoids which are numerically computed according to the two previous methods in [4] and [5] respectively are compared to achieve this purpose. The first method in [4] are developed in our initial study on the first order PT and are applied in [3]. In addition, our second method in [5] are recently introduced to easily and fastly compute the first order PT in a software called BEM++ [6]. As our efforts in [4] is only to introduce briefly the implementation of BEM++ in computing the first order PT, our results there are not yet detailly analyzed. Thus, comparing the results from these two methods is the main agenda of this paper with the purpose to decide which method gives better approximations to the first order PT for future applications.

This paper now proceeds as follows. The next section reviews mathematical background and formulation about the first order PT. After that, section three briefly discusses about our two methods to approximate the first order PT. Numerical results are presented and discussed in section four. This study is then concluded in the last section of this paper.

2   Mathematical Formulation of the First Order Polarization Tensor

Consider a Lipschitz bounded domain B in $\mathbb{R}^3$ such that the origin $O \in B$ and let the conductivity of B be equal to k where $0 < k \neq 1 < +\infty$. Suppose that H is a harmonic function in $\mathbb{R}^3$ and u is the solution to the following problem

$$\begin{cases} \nabla \cdot (1 + (k-1)\chi(B)\nabla(u)) = 0 \text{ in } \mathbb{R}^3 \\ u(x) - H(x) = O(1/|x|^2) \text{ as } |x| \to \infty \end{cases} \tag{1}$$

where χ denotes the characteristic function of B. Formulation (1) actually appears in many industrial applications such as medical imaging, landmine detector and material sciences [1, 7, 8]. The PT is then defined through the following far-field expansion of u by [1]

$$(u - H)(x) = \sum_{|i|,|j|=1}^{+\infty} \frac{(-1)^{|i|}}{i!j!} \partial_x^i \Gamma(x) M_{ij}(k, B) \partial^j H(0) \text{ as } |x| \to +\infty \tag{2}$$

for $i = (i_1, i_2, i_3)$, $j = (j_1, j_2, j_3)$ multi indices, Γ is the fundamental solution of the Laplacian and $M_{ij}(k, B)$ is the generalized polarization tensor (GPT) for B. For some applications, (2) represents the perturbation of the voltage u caused by an object inclusion B while the first order of the GPT acts as dipole and shows conductivity distribution of B.

Alternatively, the definition of GPT in (2) is extended by Ammari and Kang [1] through an integral operator over the boundary of B by

$$M_{ij} = \int_{\partial B} y^j \phi_i(y) d\sigma(y) \tag{3}$$

where $\phi_i(y)$ is given by

$$\phi_i(y) = (\lambda I - \mathcal{K}_B^*)^{-1} (\nu_x \cdot \nabla x^i)(y) \tag{4}$$

for $x, y \in \partial B$ with ν_x is the outer unit normal vector to the boundary ∂B at x and λ is defined by $\lambda = (k+1)/2(k-1)$. $\mathcal{K}_B^*$ is a singular integral operator defined with Cauchy

principal value *P.V.* by

$$K_B^* \phi(x) = \frac{1}{4\pi} P.V. \int_{\partial B} \frac{\langle x - y, \nu_x \rangle}{|x - y|^3} \phi(y) d\sigma(y). \tag{5}$$

Consequently, the first order PT can be evaluated by using (3), (4) and (5) for $|i| = |j| = 1$ and by combining all possible values of i and j, the first order PT of an object B is a real 3×3 matrix in the form

$$M = \begin{bmatrix} M_{(1,0,0)(1,0,0)} & M_{(1,0,0)(0,1,0)} & M_{(1,0,0)(0,0,1)} \\ M_{(0,1,0)(1,0,0)} & M_{(0,1,0)(0,1,0)} & M_{(0,1,0)(0,0,1)} \\ M_{(0,0,1)(1,0,0)} & M_{(0,0,1)(0,1,0)} & M_{(0,0,1)(0,0,1)} \end{bmatrix}. \tag{6}$$

Besides, if B is an ellipsoid represented by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ in the Cartesian coordinate system where a, b and c each is the length of semi principal axes of B, the first order PT of B when the conductivity is k is given by [1] as

$$M(k, B) = (k - 1)|B| \begin{bmatrix} \frac{1}{(1-P)+kP} & 0 & 0 \\ 0 & \frac{1}{(1-Q)+kQ} & 0 \\ 0 & 0 & \frac{1}{(1-R)+kR} \end{bmatrix} \tag{7}$$

where $|B|$ is the volume of B while P, Q and R are constants defined by

$$P = \frac{bc}{a^2} \int_1^{+\infty} \frac{1}{t^2 \sqrt{t^2 - 1 + (\frac{b}{a})^2} \sqrt{t^2 - 1 + (\frac{c}{a})^2}} dt,$$

$$Q = \frac{bc}{a^2} \int_1^{+\infty} \frac{1}{(t^2 - 1 + (\frac{b}{a})^2)^{\frac{3}{2}} \sqrt{t^2 - 1 + (\frac{c}{a})^2}} dt, \tag{8}$$

$$R = \frac{bc}{a^2} \int_1^{+\infty} \frac{1}{\sqrt{t^2 - 1 + (\frac{b}{a})^2} (t^2 - 1 + (\frac{c}{a})^2)^{\frac{3}{2}}} dt.$$

In addition, the first order PT for the sphere B at conductivity k is also given in [1] and this can be easily obtained by setting $a = b = c$ in (7) which is in the form

$$M(k, B) = (k - 1)|B| \begin{bmatrix} \frac{3}{2+k} & 0 & 0 \\ 0 & \frac{3}{2+k} & 0 \\ 0 & 0 & \frac{3}{2+k} \end{bmatrix}. \tag{9}$$

3   Methodology

In order to approximate the first order PT in this study, we will use our two previous methods as mentioned before. The first method in [4] is developed by a simple quadrature rule of numerical integration. A code is then written in *Matlab* to perform the calculations by using $(3) - (5)$.

In contrast, our second method in [5] is developed based on Boundary Element Method (BEM) [9]. This is possible as every formula considered $(3) - (5)$ to compute the first order PT is in the form of boundary integral equations. Here, the software devoted to BEM called as *BEM++* is used to run the computations where the code is built from the combination of functions in *Python* and *C++*.

Before the first order PT for an object at specified conductivity can be approximated in both codes, a triangular mesh of the boundary of the considered object consisting sets
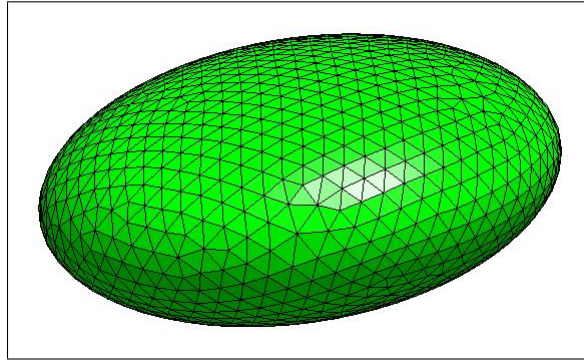
Figure 1: A triangularization of ellipsoid $\frac{x^2}{4} + \frac{y^2}{4} + z^2 = 1$ with 2608 elements by *Netgen*

of nodes and triangles (or elements) must be loaded to each code. Since we are considering objects of three dimensional domain, the boundary of the object is simply the surface of the object. In this study, the mesh is created by the software *Netgen* mesh generator [10] (see Figure 1 for example). Once the first order PT is successfully computed by both codes, we can then compare the results with the analytical solutions to decide which method gives better approximation. At the moment, we only consider the first order PT for ellipsoids since we have the analytical formula of the first order PT for them as given in (7).

4   Results and Discussions

4.1   Several approximations of the first order PT

We start by computing the first order PT for four types of ellipsoid at conductivity 1.5 by using analytical formula (7), our previous program in Matlab and our recent program in BEM++ and show the results in Table 1. Each ellipsoid is firstly triangularized with with a 'fine' mesh option in *Netgen* before numerically computed by Matlab and BEM++ where the number of triangles, N is also given in the table. For each ellipsoid in the table, the same triangularized ellipsoids are used in both Matlab and BEM++.

In order to easily compare the first order PT approximated by Matlab and BEM++, we then plot all elements of the first order PT from Table 1 in the same graph for each ellipsoid. These are shown in Figure 2, Figure 3, Figure 4 and Figure 5 respectively. In these figures, elements of the first order PT in the first row are denoted starting from the first column by 1, 2 and 3 followed by 4, 5 and 6 for the second row and 7, 8 and 9 for the third row.

Based on Figure 2, it can be seen that all elements of the approximated first order PT for $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$ computed either by Matlab or BEM++ are close to the elements of the analytical solutions except for element 5 which is the second diagonal of the first order PT. In this case, the element computed by Matlab is still far from the element of the analytic solution. For $\frac{x^2}{4} + \frac{y^4}{4} + z = 1$, both element 1 and element 5 of the first order PT approximated in Matlab are only slightly not equal to element 1 and element 5 of the first order PT for the analytical solution as shown in Figure 3.

Besides, element 1, element 5 and element 9 of the approximated first order PT for $x^2 + y^2 + \frac{z^2}{4} = 1$ computed by Matlab in Figure 4 has only small difference with the same elements of the analytical and BEM++'s solution. In contrasts, Figure 5 indicates that element 1, element 5 and element 9 of the first order PT for $x^2 + y^2 + y^2 = 1$ approximated either in Matlab or BEM++ has a big difference with the same elements of the analytical solution. However, these approximations look closer to the analytical solutions when computed by

Ellipsoid	Analytic	Matlab	BEM++
Scalene $\frac{x^2}{9}+\frac{y^2}{4}+z^2=1$ $N=7806$	$\begin{bmatrix} 11.6555 & 0 & 0 \\ 0 & 11.0856 & 0 \\ 0 & 0 & 9.7544 \end{bmatrix}$	$\begin{bmatrix} 11.5814 & 0.0007 & -0.0002 \\ 0.0015 & 10.8046 & 0.0126 \\ -0.0002 & 0.0496 & 9.6919 \end{bmatrix}$	$\begin{bmatrix} 11.6308 & 0.0001 & 0.0000 \\ 0.0001 & 11.0611 & 0.0000 \\ 0.0000 & 0.0000 & 9.7346 \end{bmatrix}$
Oblate $\frac{x^2}{4}+\frac{y^2}{4}+z^2=1$ $N=2608$	$\begin{bmatrix} 7.4920 & 0 & 0 \\ 0 & 7.4920 & 0 \\ 0 & 0 & 6.6299 \end{bmatrix}$	$\begin{bmatrix} 7.4090 & 0.0003 & -0.0001 \\ 0.0003 & 7.4091 & 0.0001 \\ -0.0002 & 0.0001 & 6.5779 \end{bmatrix}$	$\begin{bmatrix} 7.4515 & 0.0001 & 0.0000 \\ 0.0001 & 7.4515 & 0.0000 \\ 0.0000 & 0.0000 & 6.5957 \end{bmatrix}$
Prolate $x^2+y^2+\frac{z^2}{4}=1$ $N=1722$	$\begin{bmatrix} 3.4715 & 0 & 0 \\ 0 & 3.4715 & 0 \\ 0 & 0 & 3.8543 \end{bmatrix}$	$\begin{bmatrix} 3.4274 & 0.0001 & 0.0001 \\ 0.0001 & 3.4276 & 0.0007 \\ 0.0000 & 0.0002 & 3.7946 \end{bmatrix}$	$\begin{bmatrix} 3.4435 & 0.0000 & 0.0000 \\ 0.0000 & 3.4435 & 0.0001 \\ 0.0000 & 0.0001 & 3.8232 \end{bmatrix}$
Sphere $x^2+y^2+z^2=1$ $N=242$	$\begin{bmatrix} 1.7952 & 0 & 0 \\ 0 & 1.7952 & 0 \\ 0 & 0 & 1.7952 \end{bmatrix}$	$\begin{bmatrix} 1.6910 & 0.0000 & 0.0004 \\ 0.0000 & 1.6896 & -0.0001 \\ 0.0004 & -0.0001 & 1.6891 \end{bmatrix}$	$\begin{bmatrix} 1.7103 & 0.0000 & 0.0003 \\ 0.0000 & 1.7093 & -0.0001 \\ 0.0003 & -0.0001 & 1.7090 \end{bmatrix}$

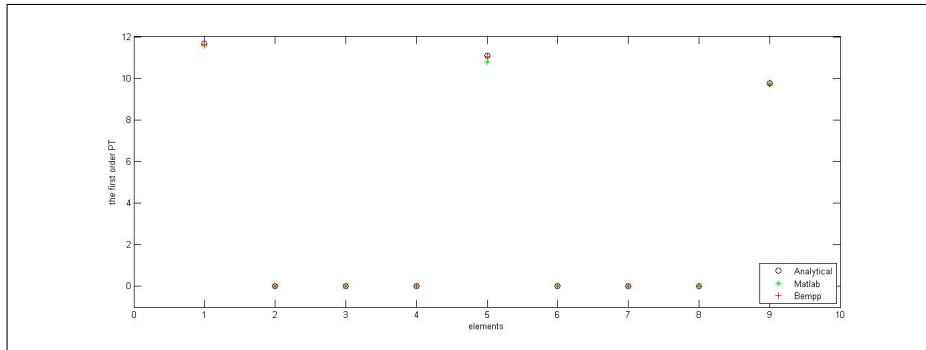Table 1: The First Order PT for Several Ellipsoids at conductivity 1.5

Figure 2: A comparison of the first order PT for ellipsoid $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$
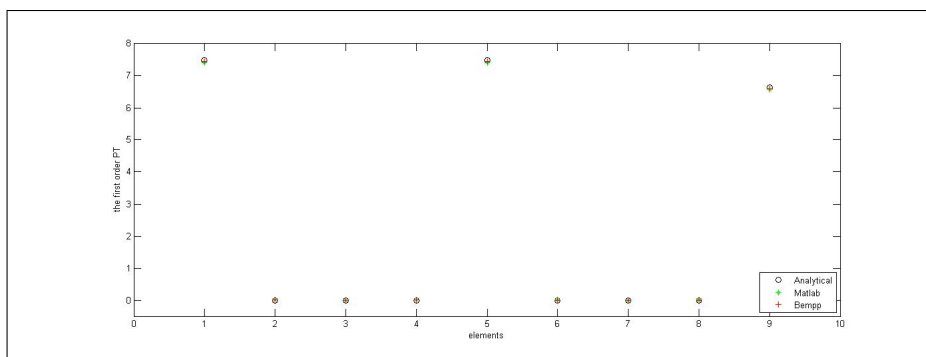

Figure 3: A comparison of the first order PT for ellipsoid $\frac{x^2}{4} + \frac{y^4}{4} + z = 1$
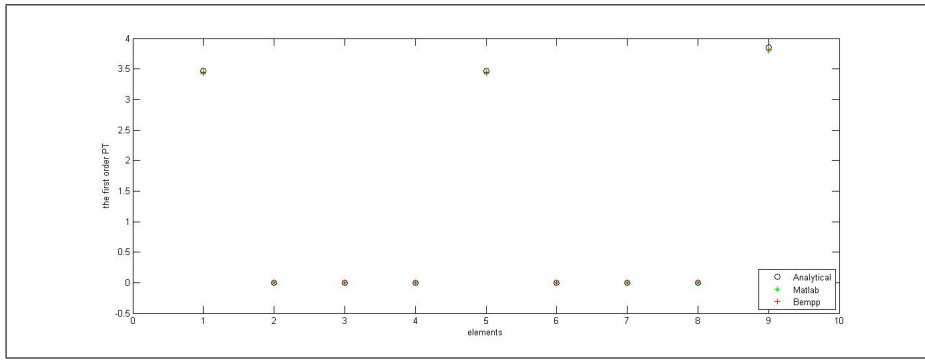
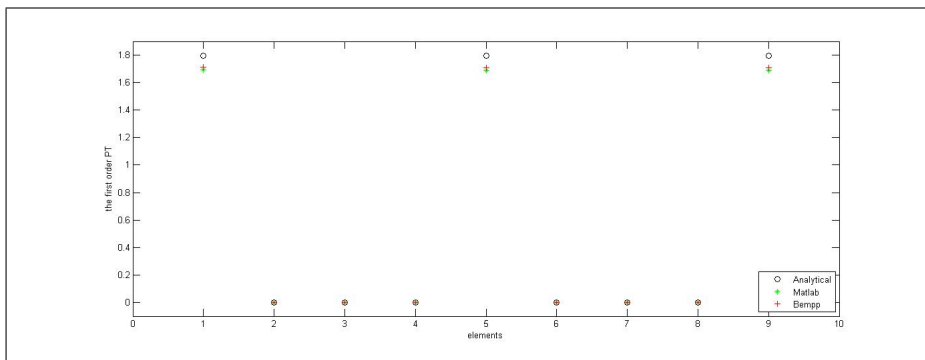Figure 4: A comparison of the first order PT for ellipsoid $x^2 + y^2 + \frac{z^2}{4} = 1$

Figure 5: A comparison of the first order PT for sphere $x^2 + y^2 + y^2 = 1$

BEM++.

We also want to highlight that in each figure, all non-diagonal elements (element 2, element 3, element 4, element 6, element 7 and element 8) of the approximated first order PT for every ellipsoid computed either by Matlab or BEM++ are almost equal to zero as required by the analytical solutions.

4.2   Increasing the number of triangles N

One possible way to theoretically improve the approximation of the first order PT for a triangularized object is to increase the number of triangles used during the triangularization. In this section, we will compare the convergence of the first order PT for the sphere $x^2 + y^2 + y^2 = 1$ at conductivity 1.5 approximated in Matlab and BEM++ to the analytical solution (9) for different number of triangles used to triangularized the sphere. For this purpose, five triangularized spheres $x^2 + y^2 + y^2 = 1$ with 242, 620, 2480, 4480 and 9920 triangles are considered.

Graphs in Figure 6 show all diagonal elements of the analytical and the computed first order PT by both Matlab and BEM++ for the sphere $x^2 + y^2 + y^2 = 1$ at conductivity 1.5 against the number of triangles used to triangularize the sphere. In order to compare the values, the first diagonal of the first order PT computed by the analytical formula, Matlab and BEM++ are firstly plotted in the same graph in Figure 6(a) followed by graph in Figure 6(b) and Figure 6(c) for the second and the third diagonal. Based on these graphs, it can be clearly seen that increasing the number of triangles will improve the convergence of every diagonal computed either by Matlab or BEM++ to the analytical solution but the

diagonals when approximated in BEM++ obviously are closer to the analytical solution then approximated in Matlab for any number of triangles used.

On the other hand, graphs in Figure 7 show every non-diagonal elements of the first order PT for the same sphere where each symmetrical pair of the non-diagonals are plotted in the same graph. According to Figure 7(a), the symmetrical elements are zero when computed by BEM++ and hence converge to the analytical solutions while they jump from -0.0001 to 0.0001 before converge to zero as the number of triangles increase when computed by Matlab. The next symmetrical elements in Figure 7(b) reduce directly from 0.0003 to zero when approximated in BEM++ but when approximated in Matlab, they once drop to zero from 0.0004 before increasing to about 0.0001. Similar convergence are achieved by the last symmetrical elements when computed by BEM++ and Matlab except that they rise from -0.0001 as the number of triangles increase and they reach to the analytical solution for two times when computed by Matlab. Technically, it is expected that these approximated non diagonal elements will stay at zero after sufficient triangles are used to triangularize the sphere.

4.3   Changing conductivity k

During our previous study in [4], we have found that the conductivity of the object also influence its first order PT. Thus, we will compare the first order PT for the sphere $x^2 + y^2 + y^2 = 1$ triangularized with 9920 triangles computed by Matlab and BEM++ to further investigate the convergence of the approximated to the analytical solution at different values of conductivity. The value 9920 are chosen for the triangles to ensure better approximation both in Matlab and BEM++. Here, we evaluate the first order PT at conductivities 1×10^{-6}, 5×10^{-5}, 0.01, 0.99995, 1.00004, 1.5, 100, 500, 1000 and 10000. Figure 8 and Figure 9 shows the results of our computations.
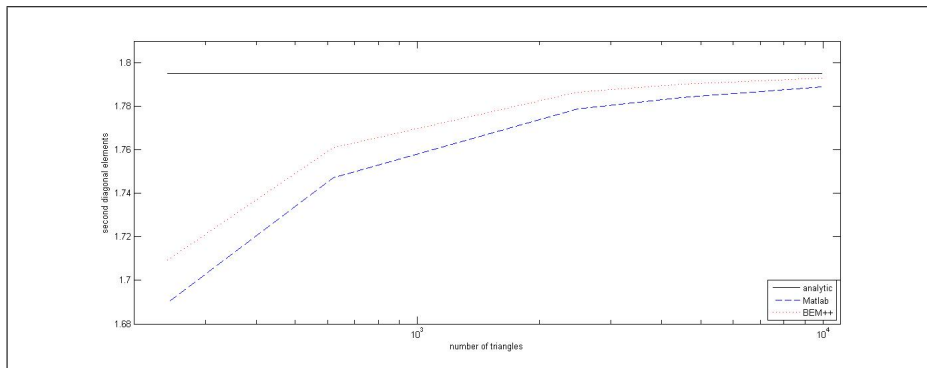
Figure 8 shows the diagonal elements of the approximated first order PT in Matlab and BEM++ together with the analytical values of the first order PT against the conductivities. Similar to Figure 6, the analytical values of the first order PT in each graph of the figure are the same but only change according to the conductivity as mentioned in formula (9). We can then see that every diagonal of the approximated first order PT computed either by Matlab or BEM++ converge to the analytical solution except for some high values of conductivity when computed by Matlab.

Besides, the non diagonal elements of the first order PT for the same sphere approximated in both Matlab and BEM++ only converge to zero as required by the analytical solution at conductivities near to 1 and this is shown in Figure 9. When computed by Matlab, the non-diagonal elements have a small difference with the analytical solution at conductivities less than one and have a greater difference when conductivities is greater than 1. In contrast, while the non-diagonal elements approximated in BEM++ have a slight difference with the analytical solutions at conductivities greater than 1, they converge to the analytical solution for conductivities less than 1 except for elements (1)(3) and (3)(1). Obviously, the non-diagonal elements computed by BEM++ are closer to the analytical solutions than the one computed by Matlab. For all cases, we conclude that BEM++ provides better approximation of the first order PT for this sphere at the chosen conductivities although some non-diagonal elements is slightly greater than zero of the analytical solution.

Based on the previous discussion, it is possible to increase the number of triangles used to triangularize the sphere to improve the results. However, our previous experience tells us that larger number of triangles will cause slower computation in both softwares and the machine needs more memory to compute the first order PT in Matlab. Faster computation can be achieved in BEM++ as it is equipped with *iterative solver* while Matlab uses *matrix operation* which will consume more memory. For completeness, we include in Table 2 the
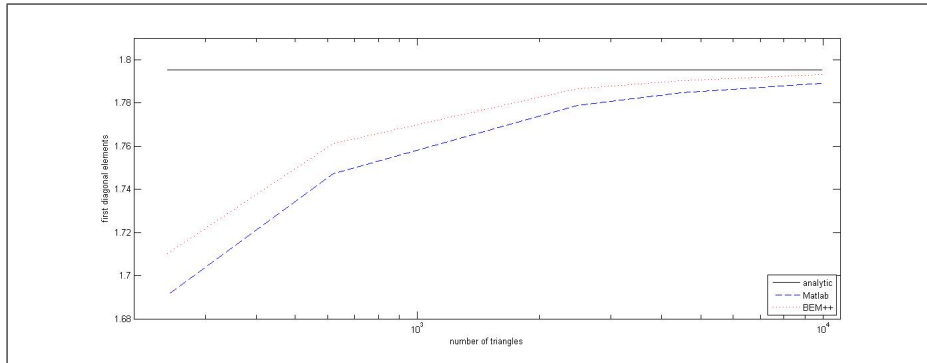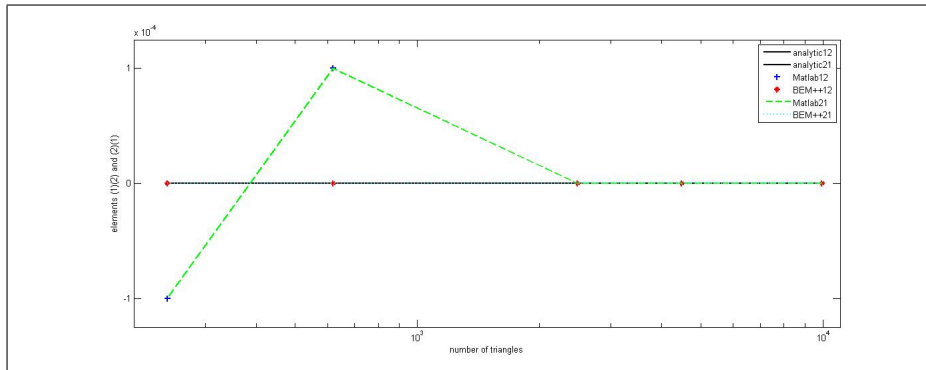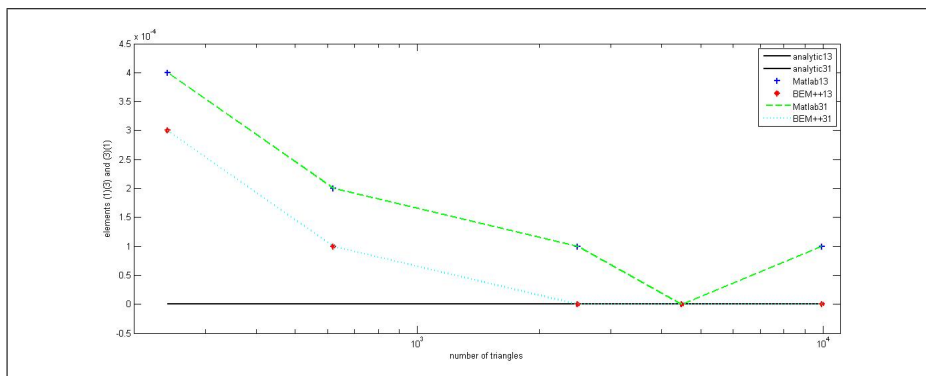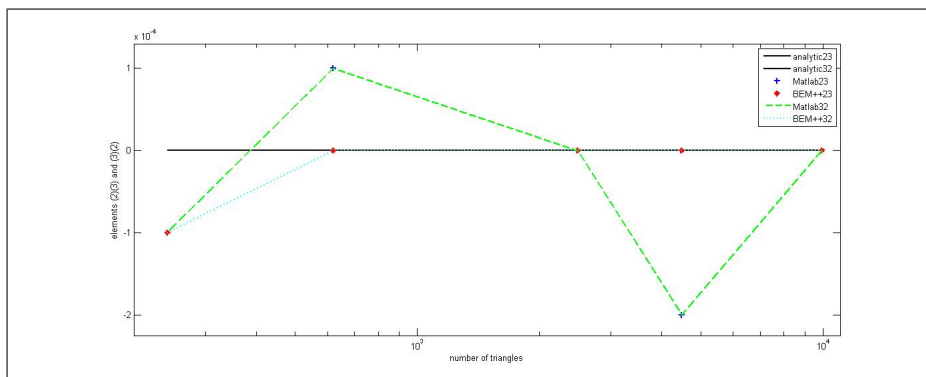
(a)

(b)
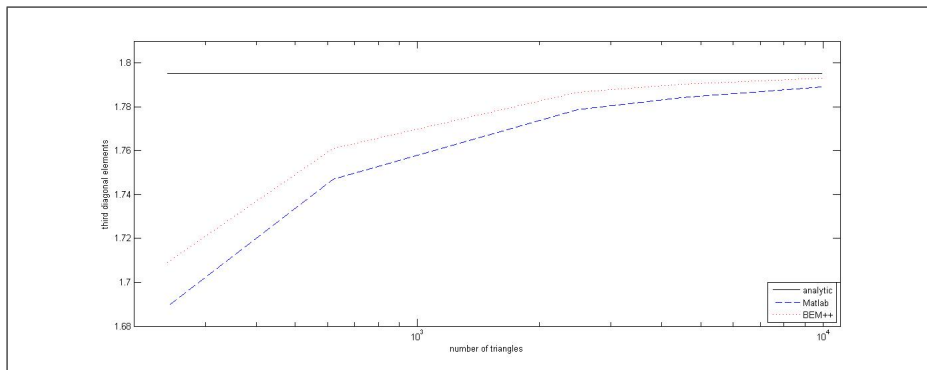
(c)

Figure 6: Diagonal elements of the approximated first order PT for a sphere through Matlab and BEM++ with diagonal elements of the analytical formula against number of triangles N
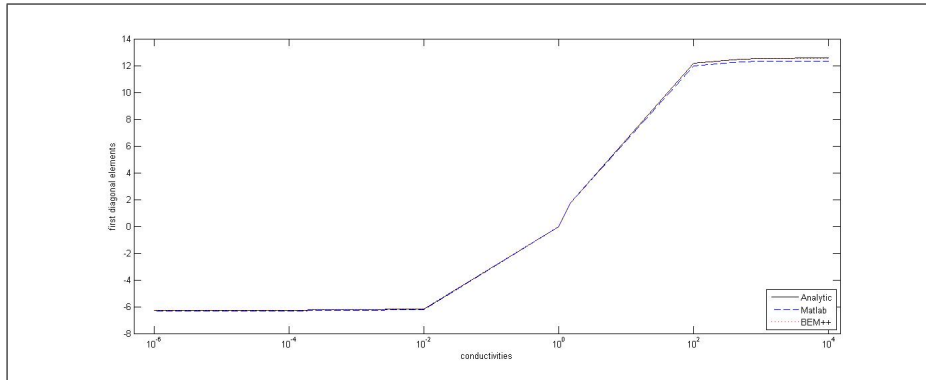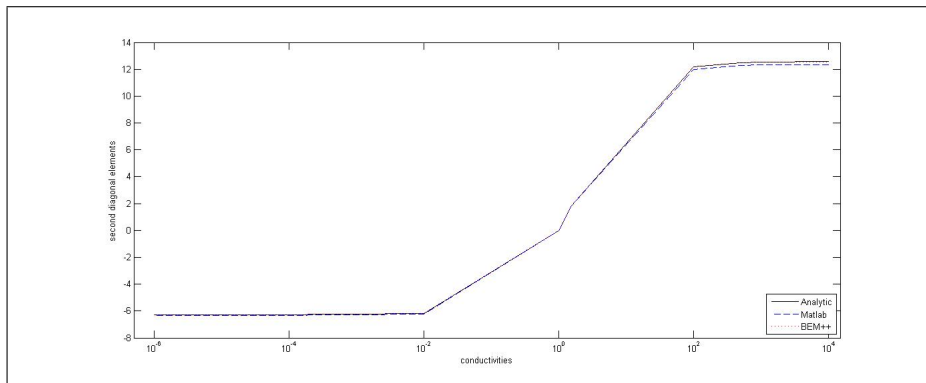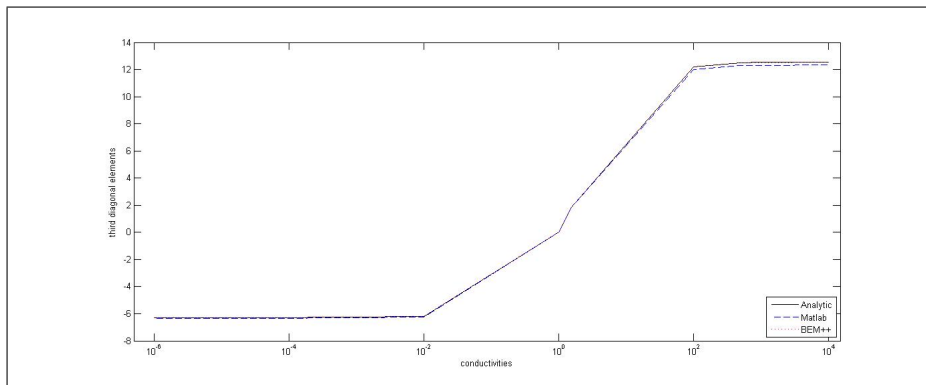
Figure 7: Non-diagonal elements of the approximated first order PT for a sphere through Matlab and BEM++ with non-diagonal elements of the analytical formula against number of triangles N
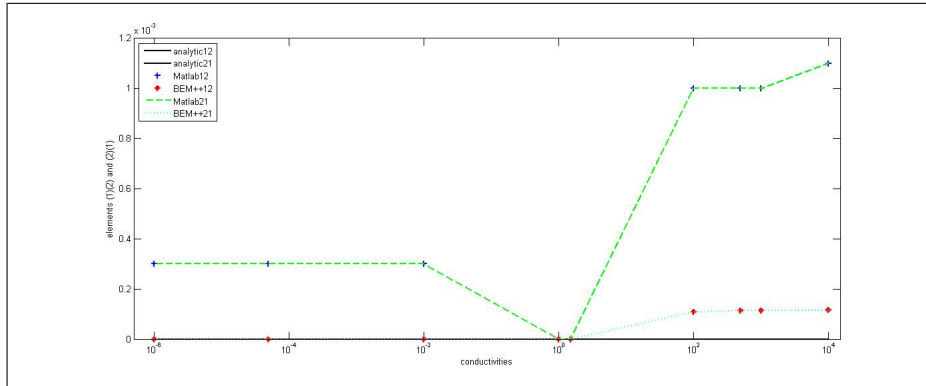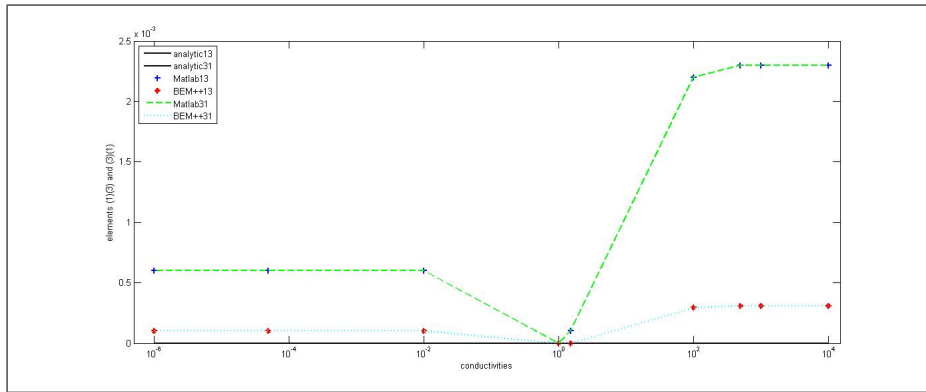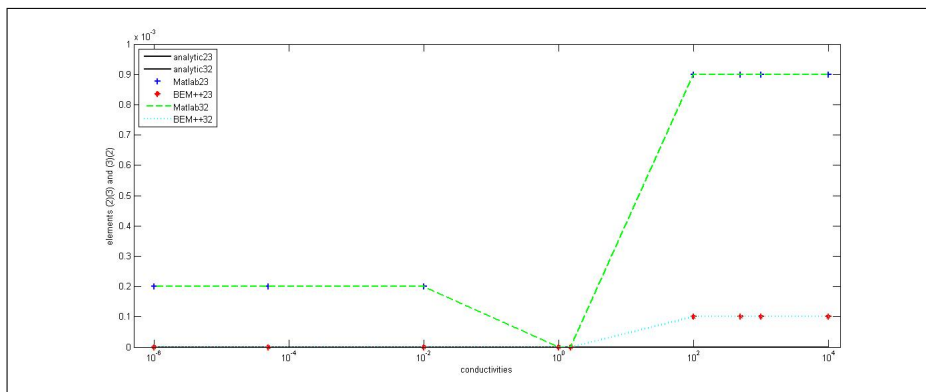
(a)

(b)

(c)

Figure 8: Diagonal elements of the approximated first order PT for the sphere $x^2+y^2+y^2 = 1$ triangularized with 9920 triangles through Matlab and BEM++ with diagonal elements of the analytical formula against conductivities k

(a)

(b)

(c)

Figure 9: Non-diagonal elements of the approximated first order PT for the sphere $x^2 + y^2 + y^2 = 1$ triangularized with 9920 triangles through Matlab and BEM++ with non-diagonal elements of the analytical formula against conductivities k

number of triangles, N tested to approximate the first order PT for the sphere in BEM++ accurate with the analytical solution at three decimal places for the cases of low and high conductivity as well as when the conductivity is equal to 1.5. Here, running the code in Matlab with the same N to approximate the first order PT at the given conductivity will cause the machine to run out of memory.

Conductivity, k	N
1×10^{-6}	124928
1.5	61952
10000	247808

Table 2: N needed for each k

5 Conclusions

During this study, we have compared our two previous methods to approximate the first order PT in more details than our previous study. We provide extensive results to show that our method through BEM++ provides better convergence to the approximated first order PT specifically for ellipsoids at specified conductivity. We then believe that our method in BEM++ should become a priority in our future investigation about the PT in electric and electromagnetic applications of inverse problems.

Acknowledgments

References

[1] H. Ammari and H. Kang, *Polarization and Moment Tensors with Applications to Inverse Problems and Effective Medium Theory*, Springer, New York, 2007.

[2] L. A. Marsh, C. Ktistis, A. Järvi, D. W. Armitage and A. J. Peyton. Three-dimensional object location and inversion of the magnetic polarizability tensor at a single frequency using a walk-through metal detector. *Measurement Science and Technology*, 24(4):045102, 2013.

[3] Taufiq K. A. K. and W. R. B. Lionheart. Do electro-sensing fish use the first order polarization tensor for object characterization? *In 100 years of Electrical Imaging*, Presses des Mines, Paris, 2012.

[4] Taufiq K. A. K. and W. R. B. Lionheart. Some properties of the first order polarization tensor for 3-D domains. *MATEMATIKA UTM*, 29(1):1–18, 2013.

[5] Taufiq K. A. K. and W. R. B. Lionheart. Computing the first order polarization tensor : Welcome BEM++! *Menemui Matematik*, 35(2):15–20, 2013.

[6] W. Śmigaj, S. Arridge, T. Betcke, J. Phillips and M. Schweiger. Solving boundary integral problems with BEM++, *ACM Trans. Math. Software,* 41(2):6:1–6:40, 2015.

[7] A. Adler, R. Gaburro and W. Lionheart, Electrical Impedance Tomography, in O. Scherzer (ed) *Handbook of Mathematical Methods in Imaging*, Springer-Verlag, USA, 2011.

[8] D. S. Holder (ed), *Electrical Impedance Tomography : Methods, History and Applications*, Institute of Physics Publishing, London, 2005.

[9] S. Rjasanow and O. Steinbach, *The Fast Solution of Boundary Integral Equations*, Springer, New York, 2007.

[10] J. Schöberl. NETGEN An advancing front 2D/3D-mesh generator based on abstract rules. *Computing and Visualization in Science*, 1(1):41–52, 1997.