

Solution of Fuzzy Reliability Allocation Optimization Problems

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Abstract: *In this work, the reliability allocation optimization problems in fuzzy environment have been developed and their result have also discussed. The numerical solutions of crisp reliability optimization problems and have been compared and the fuzzy solution and its effectiveness have also been presented and discussed. The penalty function method is developed and mixed with Nelder and Mend's algorithm of direct optimization problem's solution have been used together to solve this nonlinear programming problem.*

Keywords: *Fuzzy numbers, nonlinear programming (NLPP), reliability allocation optimization problems, Fuzzy decision making, penalty function method.*

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1 Introduction

Fuzzy reliability used in modeling problems in many applications including engineering [1, 2, 3, 4], biology [5, 6], real life problems and mathematical models [7, 8, 9] etc. Reliability and design engineers must translate overall system performance, including reliability, into component performance including reliability. The process of assigning reliability requirements for individual component to attain specified system reliability is called reliability allocation. So, we discuss reliability allocation optimization model in design system. Reliability Allocation deals with the setting of reliability goals for individual subsystems such that a specified reliability goal is met and the hardware and software subsystem goals are well balanced among themselves.

The objective of this allocation is to use the reliability model to assign reliability to the subsystems so as to achieve the specified reliability goal for the system. In addition, reliability allocation problems may appear in many real life applications, including, a software development approach [10], and engineering [11]. In this paper fuzzy allocation optimization problem is classified as NLPP as in goal programming model [11, 18]. This NLPP is feasible to solve a typical ill-structured reliability problem with a vague reliability objective as well as fuzzy flexible constraints [21]. This can be done by formulating the crisp problem into fuzzy environment using the

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properties of fuzzy set theory perspective by using fuzzy decision making as mentioned in [12, 13] for reliability series system in geometric programming approach. However, the fuzzy nonlinear programming problem is not just an alternative or even a superior way of analysing a given problem, it's useful in solving problems in which difficult or impossible to use due to the inherent qualitative imprecise or subjective nature of the problem formulation or to have an accurate solution or to increase system reliability as close to one. In section 2 we introduce some important definitions that are useful in our problem. In section 3, we state the general nonlinear programming problem in fuzzy environment by transforming the crisp problem into the fuzzy problem. Section 4, we modify and develop the regular penalty function method in order to solve fuzzy NLPP combined with Nelder and Mend's. Finally, in section 5 we introduce the fuzzy allocation optimization problem and solved by our proposed method in numerical example.

2 Preliminaries

2.1 Reliability Function: $R(t)$ (Survival Function) [14]:

The probability that a system (component) does not fail in the interval $[0, t]$ can be expressed as follows:

$$R(t) = \text{pr}(T > t), \text{ for } t > 0, \quad (1)$$

where pr is the probability and T is a random variable for failure time of his component

2.2 Unreliability Function; $Q(t)$ [14]:

The probability that the system fails with the time interval $(0, t)$ can be expressed as follows:

$$Q(t) = \text{pr}(\tau \leq t) = \int_0^t f(\tau) d\tau, \quad (2)$$

Therefore, $R(t) + Q(t) = 1$, $R(t) = 1 - Q(t)$.

2.3 Component (Subsystem) [15]:

The component (subsystem) is the basic element that controls the functioning or a system. The reliability of the component $p_i(t)$ is defined as the probability that the component functions well until time t , and the unreliability of the component $q_i(t)$ can be expressed as: $q_i(t) = 1 - p_i(t)$, for $i = 1, 2, \dots, n$.

2.4 Mean Time between Failures (MTBF) [16]:

The (MTBF) is a concept, which is frequently used in reliability work. It is defined to be the average or expected lifetime of an item.

Then, from the definition of a mean or expected value,

$$\text{MTBF} = \int_0^{\infty} t f(t) dt. \quad (3)$$

Alternatively:

$$MTBF = \int_0^{\infty} [1 - Q(t)] dt.$$

We most commonly express the MTBF in terms of the reliability function, namely:

$$MTBF = \int_0^{\infty} R(t) dt.$$

2.5 Fuzzy Set [17]:

If x is a collection of objects denoted generally by X , then a fuzzy set \tilde{A} in X is a set of order pairs: $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$, where $\mu_{\tilde{A}}(x) : x \rightarrow [0, 1]$ is called the membership function or grade of membership (also degree of compatibility or degree of truth) of x in \tilde{A} which maps x to the membership range M (when M contains only the two points 0 and 1), \tilde{A} is a nonfuzzy and $\mu_{\tilde{A}}(x)$ is identical to the characteristic function of crisp set. The range of membership function is a subset of the non-negative real numbers whose supremum is finite. Elements with a zero degree of membership are normally not listed.

2.6 Fuzzy Numbers with Linear Member Ship Function [18]:

The function $L: X \rightarrow [0, 1]$ is a function with two parameters defined as:

$$L(x; \alpha, \beta) = \begin{cases} 1, & \text{if } x < \alpha \\ \frac{\alpha + \beta - x}{\beta}, & \text{if } \alpha \leq x \leq \alpha + \beta \\ 0, & \text{if } x > \beta \end{cases}$$

where L is called the trapezoidal linear membership function.

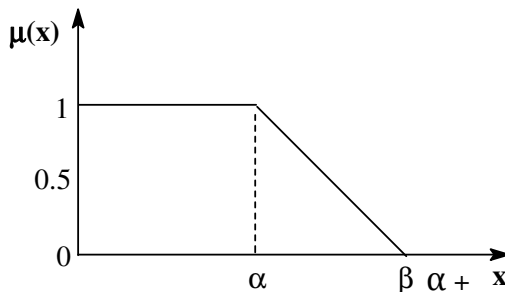


Figure 1: L-Function.

2.7 Fuzzy Decision Making [19]:

Assume that we are given a fuzzy goal (fuzzy objective function) \tilde{G} and fuzzy constraints \tilde{C} in a space of alternatives X . The \tilde{G} and \tilde{C} combine to form a decision, \tilde{D} , which is a fuzzy set resulting from intersection of \tilde{G} and \tilde{C} . In symbols, $\tilde{D} \cap \tilde{C}$ is, correspondingly, the membership function of \tilde{D} can be defined as:

$\mu_{\tilde{D}} = \text{Min} \{ \mu_{\tilde{G}}, \mu_{\tilde{C}} \}$. More generally, suppose that we have n goals $\tilde{G}_1, \tilde{G}_2, \dots, \tilde{G}_n$ and m constraints $\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_m$. The, the resultant decision is defined as:

$\tilde{D} = \tilde{G}_1 \cap \tilde{G}_2 \cap \dots \cap \tilde{G}_n \cap \tilde{C}_1 \cap \tilde{C}_2 \cap \dots \cap \tilde{C}_m$ and correspondingly:

$$\begin{aligned} \mu_{\tilde{D}} &= \text{Min} \{ \min \{ \mu_{\tilde{G}_1}, \mu_{\tilde{G}_2}, \dots, \mu_{\tilde{G}_n} \}, \min \{ \mu_{\tilde{C}_1}, \mu_{\tilde{C}_2}, \dots, \mu_{\tilde{C}_m} \} \} \\ &= \text{Min} \{ \mu_{\tilde{G}_1}, \mu_{\tilde{G}_2}, \dots, \mu_{\tilde{G}_n}, \mu_{\tilde{C}_1}, \mu_{\tilde{C}_2}, \dots, \mu_{\tilde{C}_m} \} = \text{Min} \{ \mu_{\tilde{G}_j}, \mu_{\tilde{C}_i} \} \end{aligned}$$

for $j = 1, 2, \dots, n$ and $i = 1, 2, \dots, m$.

2.8 Maximum Decision Maker [17]:

If the decision-maker wants to have “crisp” decision proposal, it seems appropriate to suggest to him the divided which has the highest degree of membership in the fuzzy set “decision”. Let us call this the maximizing decision, defined by:

$$X_{\max} = \text{Max}_x M_{\tilde{D}}(x) = \text{Max}_x \text{Min} \{ \mu_{\tilde{D}_j}(x), \mu_{\tilde{C}_i}(x) \},$$

where \tilde{D}_j and \tilde{C}_i are in the definition (2.7) for $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$.

3 Fuzzy Nonlinear Programming Problem

In this section we discuss the optimization problem with nonlinear fuzzy objective function and fuzzy flexible constraints. Consider the following NLPP

$$\text{Min/Max } f(x)$$

Subject to:

$$g_i(x) \geq (\leq) b_i, \quad i = 1, 2, \dots, m. \quad (4)$$

For all $x \in R^n$ and $x \geq 0$. Now, the fuzzy version for problem (4) is as follows:

$$\text{Fuzzy Min/Max } f(x)$$

Subject to:

$$g_i(x) \tilde{\geq} (\tilde{\leq}) b_i, \quad i = 1, 2, \dots, m. \quad (5)$$

where $x \in \mathbb{R}^n$ and $x \geq 0$. In problem (4), the tilde sign denotes a fuzzy satisfaction of the constraints. The sign $(\lesssim) \gtrsim$ denotes that $g_i(x) \leq (\geq) b_i$ can be satisfied to degree smaller than 1, these constraints are called flexible constraints. The fuzzy max (min) corresponds to achieving the highest (lowest) possible aspiration level for the general $f(x)$. This problem can be solved by using the properties of fuzzy decision making and maximize decision as follows:

1. Fuzzify the objective function. This is done by calculating the lower and the upper bounds of the optimal values. The bounds of optimal values z_λ and z_u are obtained by solving the standard crisp nonlinear programming problem as follows:

$$z_1 = \text{Min/Max } f(x),$$

subject to:

$$g_i(x) \leq (\geq) b_i, \quad i=1, 2, \dots, m, \tag{6}$$

for all $x \in \mathbb{R}^n$ and $x \geq 0$.

$$z_2 = \text{Min/Max } f(x),$$

subject to:

$$g_i(x) \leq (\geq) b_i + p_i, \quad i=1, 2, \dots, m, \tag{7}$$

where the objective function take the values between z_1 and z_2 . Let $z_\lambda = \min(z_1, z_2)$ and $z_u = \max(z_1, z_2)$, z_λ and z_u is called the lower and upper bounds of the optimal values, receptively.

Let \tilde{M} be the fuzzy set representing the objective function $f(x)$ such that $\tilde{M} = \{(x, \mu_{\tilde{M}}(x)) : x \in \mathbb{R}^n\}$, where:

$$\mu_{\tilde{M}}(x) = \begin{cases} 1, & \text{if } z_u < f(x) \\ \frac{f(x) - z_\ell}{z_u - z_\ell}, & \text{if } z_\ell \leq f(x) \leq z_u \\ 0, & \text{if } f(x) < z_\ell \end{cases}$$

This represents the satisfaction of the aspiration level of the objective. Note that, p_i is a vector of relaxation and can be found by falsifying B_i (denoted by \tilde{b}_i) by using the definition of L-function of the membership function as follows:

$$\tilde{b}_i = \{(x, \mu_{\tilde{b}_i}(x)) \mid x \in \mathbb{R}\},$$

where

$$\mu_{\tilde{b}_i}(x) = \begin{cases} 1, & \text{if } x < b_i \\ \frac{b_i + p_i - x}{p_i}, & \text{if } b_i \leq x \leq b_i + p_i \\ 0, & \text{if } x \geq b_i + p_i \end{cases}$$

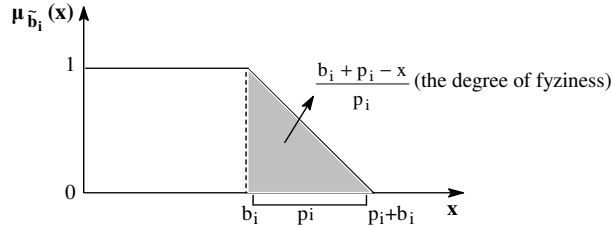


Figure 2: $\mu_{\tilde{b}_i}(x)$ membership function

2. Now, fuzzify the constraint $g_i(x)$, $i = 1, 2, \dots, m$. Let \tilde{C}_i be the fuzzy set for i -th constraints, such that $\tilde{C}_i = \{(x, \mu_{\tilde{C}_i}(x)) \mid x \in \mathbb{R}^n\}$, where:

$$\mu_{\tilde{C}_i}(x) = \begin{cases} 1, & \text{if } g_i(x) < b_i \\ \frac{b_i + p_i - g_i(x)}{p_i}, & \text{if } b_i \leq g_i(x) \leq b_i + p_i \\ 0, & \text{if } g_i(x) > b_i + p_i \end{cases}$$

Using the definition of fuzzy decision making. Let \tilde{D} be the fuzzy decision set, where: $\tilde{D} = \tilde{M} \cap \tilde{C}_i$, $i = 1, 2, \dots, m$. Therefore, $\tilde{D} = \tilde{M} \cap \tilde{C}_1 \cap \tilde{C}_2 \cap \dots \cap \tilde{C}_m$ and $\tilde{D} = \{(x, \mu_{\tilde{D}}(x)) : x \in \mathbb{R}^n\}$, where $\mu_{\tilde{D}}(x) = \text{Min}\{\mu_{\tilde{M}}(x), \min\{\mu_{\tilde{C}_1}(x), \mu_{\tilde{C}_2}(x), \dots, \mu_{\tilde{C}_m}(x)\}\}$. Let:

$$\lambda = \text{Min}\{\mu_{\tilde{M}}(x), \min\{\mu_{\tilde{C}_1}(x), \mu_{\tilde{C}_2}(x), \dots, \mu_{\tilde{C}_m}(x)\}\}$$

So, we have the optimal decision: $x^* = \text{Max } \lambda$, $x^* \in \mathbb{R}^n$

The problem (7) become to the following crisp NLPP:

$$\begin{aligned} & \text{Max } \lambda, \\ & \text{subject to:} \\ & \bar{g}_1 : \lambda - \mu_{\tilde{M}}(x) \leq 0 \\ & \bar{g}_2 : \lambda - \mu_{\tilde{C}_1}(x) \leq 0 \\ & \vdots \\ & \bar{g}_m : \lambda - \mu_{\tilde{C}_{m-1}}(x) \leq 0 \\ & \bar{g}_{m+1} : \lambda - \mu_{\tilde{C}_m}(x) \leq 0 \end{aligned} \tag{8}$$

where $0 \leq \lambda \leq 1$, $x \geq 0$ and $x \in \mathbb{R}^n$. This is equivalent to the problem:

$$\begin{aligned}
 & \text{Min } -\lambda, \\
 & \text{subject to:} \\
 & \bar{g}_1 : \left(\frac{f(x) - z_\ell}{z_u - z_\ell} \right) - \lambda \geq 0 \\
 & \bar{g}_2 : \left(\frac{b_1 + p_1 - g_1(x)}{p_1} \right) - \lambda \geq 0 \\
 & \vdots \\
 & \bar{g}_{m+1} : \left(\frac{b_m + p_m - g_m(x)}{p_m} \right) - \lambda \geq 0
 \end{aligned} \tag{9}$$

where $0 \leq \lambda \leq 1$, $x \geq 0$ and $x \in \mathbb{R}^n$.

4 Penalty Function Method for NLPP

There is survival methods used in fuzzy reliability problems, such as a novel approach for solving unconstrained model mechanical structure [22], also the numerical integration algorithm in the fuzzy general strength model [23]. The penalty method [24] belongs to the first

$$\begin{aligned}
 & \text{(A) Min/Max } f(x) \\
 & \text{Subject to:} \\
 & \quad g_i(x) \geq (\leq) 0, \\
 & \quad i = 1, 2, \dots, m \text{ and } x \geq 0, \forall x \in \mathbb{R}^n
 \end{aligned} \tag{10}$$

$$\begin{aligned}
 & \text{(B) Min/Max } f(x) \\
 & \text{Subject to:} \\
 & \quad g_i(x) = 0, \\
 & \quad i = 1, 2, \dots, m \text{ and } x \geq 0, \forall x \in \mathbb{R}^n
 \end{aligned} \tag{11}$$

To construct the unconstrained problems, so-called penalty terms are added to the objective function which penalizes $f(x)$ whenever the feasibility region is left. A factor σ_k controls the degree of penalizing $f(x)$.

Proceeding from a sequence $\{\sigma_k\}$ with $\sigma_k \longrightarrow \infty$ for $k = 0, 1, \dots$, penalty function can be defined by [26]:

$$1. \text{ Min/Max } \varphi(x, \sigma) = f(x) + \frac{1}{2} \sigma_k \sum_{i=1}^m (\min(0, g_i(x)))^2, \tag{12}$$

for problem (10).

$$2. \text{ Min/Max } \varphi(x, \sigma) = f(x) + \frac{1}{2} \sigma_k \sum_{i=1}^m (g_i(x))^2, \quad (13)$$

for problem (11)

The unconstrained nonlinear programming problems are solved by any standard technique, e.g., Nelder and Mead [24, 25] method combined with a line search. However, the line search must be performed quite accurately due to this step, narrow valleys created by the penalty terms, respectively. The technique of solving a sequence of minimization (maximization) problems of by using a penalty function method is as follows:

1. Choose a sequence $\{\sigma_k\} \rightarrow \infty$.
2. For each σ_k finding a local minimizer (maximizer) $x(\sigma_k)$ say, $\min_x (\max_x) \varphi(x, \sigma_k)$. By any steeple optimization method.
3. Stop when the penalty terms $\frac{1}{2} \sigma_{ok}$, or $\frac{1}{2} \sigma_k \sum_{i=1}^m (g_i(x))^2$ is zero and the constraints satisfies the solution at once when the penalty terms is zero.
4. The convergence of the solution of (NLPP) is using the penalty function method and its properties can be found in [25].
5. Use this formula (12-13) in Nelder and Mead's [24] algorithm for the direct solution of the optimization problem.

5 Fuzzy Reliability Allocation Optimization Problems

Now, consider a system consisting of n -components. Goal reliability is sought of this system. The objective is to allocate reliability for all or some of the components of that system. In order to meet that goal with minimum cost, the problem is formulated as a NLPP [27] as follows:

$$\begin{aligned} \text{Min. } C_s &= \sum_{i=1}^n k_i (R_i) \\ \text{subject to:} & \\ R_i, \text{ min} &\leq R_i \leq R_i, \text{ max,} \\ R_s, \text{ min} &\leq R_s \leq R_s, \text{ max,} \end{aligned} \quad (14)$$

where:

C_s : Cost function,

K_i : Cost of the component (subsystem my),

- R_s: System reliability,
- R_i: Reliability of component (subsystem i),
- R_i min: Minimum reliability of the component i,
- R_i max: Maximum reliability of the component i,
- R_s max: The upper limit of system reliability,
- R_s min: The lower limit of system reliable or system reliability goal,
- C_{sAF}: Cost function after fuzzy,
- R_{sAF}: System reliability after fuzzy.

We will solve the problem in crisp, and then some fuzzy environments will be developed to generalize the solution. Consider the life support system a space capsule problem [28] of the reliability of this system:

$$R_s = 1 - x_3((1 - x_1)(1 - x_4)) - (1 - x_3)[1 - x_2(1 - (1 - x_1)(1 - x_4))]^2$$

The cost function C_s is:

$$C_s = 2 \sum_{i=1}^4 k_i x_i^{\alpha_i}$$

with k₁ = k₂ = k₃ = 100, k₄ = 150 ad α_i = 1, for all i. The problem is to select x_i to minimize C_s, subject to:

$$0.5 = x_{i,\min} \leq x_i \leq x_{i,\max}, i = 1, 2, 3, 4,$$

$$0.9 \leq R_{s,\min} \leq R_s \leq R_{s,\max} = 1,$$

where x_i is the reliability of each component (i = 1, 2, 3, 4). One can write the NLPP problem as follows:

$$\text{Min } C_s = [100(x_1 + x_2 + x_3) + 150x_4]$$

Subject to:

$$g_1: x_1 \geq 0.5 \quad g_2: x_2 \geq 0.5 \quad g_3: x_3 \geq 0.5 \quad g_4: x_4 \geq 0.5 \quad g_5: x_1 \leq 1 \tag{15}$$

$$g_6: x_2 \leq 1 \quad g_7: x_3 \leq 1 \quad g_8: x_4 \leq 1 \quad g_9: R_s \geq 0.9 \quad g_{10}: R_s \leq 1$$

The solution of (14) using the method in section 4 has been found as follows:

At σ = 25×10⁷, the penalty term equal to zero,

$$\text{And } X_1^* = 0.5079, X_2^* = 0.8293, X_3^* = 0.5, X_4^* = 0.5,$$

$$C_s^* = 517.7838, R_s^* = 0.90011656, \text{ with}$$

$$g_1^* = 9.699 \times 10^{-3}, g_2^* = 3.293 \times 10^{-1}, g_3^* = 0,$$

$$g_4^* = 0, g_5^* = 4.903 \times 10^{-1}, g_6^* = -1.707 \times 10^{-1},$$

$$g_7^* = -0.5, g_8^* = -0.5, g_9^* = 1.165 \times 10^{-5}, g_{10}^* = -9.998 \times 10^{-2}.$$

The following table shows the results of problem (14)

Table 1: Solution of problem (14) by Penalty function method

<i>Present solution</i>	
x_1	0.509700
x_2	0.829300
x_3	0.500000
x_4	0.500000
R_s	0.900116
C_s	517.7838

Now, the fuzzier version of problem (14) is as follows:

$$\tilde{\text{Min}} C_s = [100(x_1 + x_2 + x_3) + 150x_4]$$

Subject to:

$$\begin{aligned} g_1: x_1 \gtrsim 0.5 \quad g_2: x_2 \gtrsim 0.5 \quad g_3: x_3 \gtrsim 0.5 \quad g_4: x_4 \gtrsim 0.5 \quad g_5: x_1 \lesssim 1 \\ g_6: x_2 \lesssim 1 \quad g_7: x_3 \lesssim 1 \quad g_8: x_4 \leq 1 \quad g_9: R_s \gtrsim 0.9 \quad g_{10}: R_s \lesssim 1. \end{aligned} \quad (16)$$

1. Let $\tilde{\mathbf{m}}$ be the fuzzy set of the objective function C_s , such that $\tilde{\mathbf{m}} = \{(x, \mu_{\tilde{\mathbf{m}}}(x)) \mid x \in R\}$, with:

$$\mu_{\tilde{\mathbf{m}}}(x) = \begin{cases} 1, & \text{if } z_u < C_s \\ \frac{C_s - z_\ell}{z_u - z_\ell}, & \text{if } z_\ell \leq C_s \leq z_u \\ 0, & \text{if } z_\ell > C_s \end{cases}$$

where $z_u = \max(z_1, z_2)$ and $z_\ell = \min(z_1, z_2)$ and $z_1 = C_s$ have the solution of crisp problem (3.19). To find z_2 , we have $b_1 = b_2 = b_3 = b_4 = 0.5$ and

$b_9 = 0.9$. Therefore $\tilde{0.5} = \{(x, \mu_{\tilde{0.5}}(x)) \mid x \in R\}$, with:

$$\mu_{\tilde{0.5}}(x) = \begin{cases} 1, & \text{if } z_u < 0.5 \\ \frac{0.7 - x}{0.2}, & \text{if } 0.5 \leq C_s \leq 0.7 \\ 0, & \text{if } z_\ell > 0.7 \end{cases}$$

And $\tilde{0.9} = \{(x, \mu_{\tilde{0.9}}(x)) \mid x \in R\}$, with:

$$\mu_{0.9}(x) = \begin{cases} 1, & \text{if } z_u < 0.9 \\ \frac{0.9 - x}{0.05}, & \text{if } 0.9 \leq C_s \leq 0.95 \\ 0, & \text{if } z_\ell > 0.95 \end{cases}$$

Hence $p_1 = p_2 = p_3 = p_4 = 0.3$ and $p_9 = 0.05$. Therefore:

$$z_2 = \text{Min } 2[100(x_1 + x_2 + x_3) + 150x_4],$$

subject to:

$$\begin{aligned} g_1: x_1 \geq 0.7, g_2: x_2 \geq 0.7, g_3: x_3 \geq 0.7, g_4: x_4 \geq 0.7, g_5: x_1 \leq 1, \\ g_6: x_2 \leq 1, g_7: x_3 \leq 1, g_8: x_4 \leq 1, g_9: R_s \geq 0.95, g_{10}: R_s \leq 1. \end{aligned} \tag{17}$$

The solution of (14) using the method in section 4 has been found as follows:

At $\sigma = 5 \times 10^7$, the penalty term equal to zero, and

$$X_1^* = 0.7009, X_2^* = 0.7000, X_3^* = 0.7000, X_4^* = 0.7000,$$

$$C_s^* = 63.1780, R_s^* = 0.95487442,$$

$$\text{With } g_1^* = 9 \times 10^{-4}, g_2^* = 0, g_3^* = 0, g_4^* = 0, g_5^* = -2.991 \times 10^{-2},$$

$$g_6^* = -3 \times 10^{-1}, g_7^* = -3 \times 10^{-1}, g_8^* = -3 \times 10^{-1}, g_9^* = 4.874 \times 10^{-3},$$

$$g_{01}^* = -4.512 \times 10^{-2}. \text{ Hence:}$$

$$\mu_{\tilde{m}}(x) = \begin{cases} 1, & \text{if } 630.1780 < C_s \\ \frac{C_s - 517.7838}{630.1780 - 517.7838}, & \text{if } 517.7838 \leq C_s \leq 630.1780 \\ 0, & \text{if } 517.7838 > C_s \end{cases}$$

Where $C_s = 2[100(x_1 + x_2 + x_3) + 150x_4]$.

2. Let \tilde{C}_1 , be the fuzzy sets for the constraint g_1 , such that:

$\tilde{C}_1 = \{(x, \mu_{\tilde{C}_1}(x)) \mid x \in \mathbb{R}\}$, with:

$$\mu_{\tilde{C}_1}(x) = \begin{cases} 1, & \text{if } x_1 < 0.5 \\ \frac{0.7 - x_1}{0.2}, & \text{if } 0.5 \leq x_1 \leq 0.7 \\ 0, & \text{if } x_1 > 0.7 \end{cases}$$

Let \tilde{C}_2 , be the fuzzy sets for the constraint g_2 , such that

$\tilde{C}_2 = \{(x, \mu_{\tilde{C}_2}(x)) \mid x \in \mathbb{R}\}$, with:

$$\mu_{\tilde{C}_2}(x) = \begin{cases} 1, & \text{if } x_2 < 0.5 \\ \frac{0.7 - x_2}{0.2}, & \text{if } 0.5 \leq x_2 \leq 0.7 \\ 0, & \text{if } x_2 > 0.7 \end{cases}$$

Let \tilde{C}_3 , be the fuzzy sets for the constraint g_3 , such that

$\tilde{C}_3 = \{(x, \mu_{\tilde{C}_3}(x)) \mid x \in \mathbb{R}\}$, with:

$$\mu_{\tilde{C}_3}(x) = \begin{cases} 1, & \text{if } x_3 < 0.5 \\ \frac{0.7 - x_3}{0.2}, & \text{if } 0.5 \leq x_3 \leq 0.7 \\ 0, & \text{if } x_3 > 0.7 \end{cases}$$

Let \tilde{C}_4 , be the fuzzy sets for the constraint g_4 , such that

$\tilde{C}_4 = \{(x, \mu_{\tilde{C}_4}(x)) \mid x \in \mathbb{R}\}$, with:

$$\mu_{\tilde{C}_4}(x) = \begin{cases} 1, & \text{if } x_4 < 0.5 \\ \frac{0.7 - x_4}{0.2}, & \text{if } 0.5 \leq x_4 \leq 0.7 \\ 0, & \text{if } x_4 > 0.7 \end{cases}$$

Finally, let \tilde{C}_9 be the fuzzy set for the constraint g_9 , such that

$\tilde{C}_9 = \{(x, \mu_{\tilde{C}_9}(x)) \mid x \in \mathbb{R}\}$, with:

$$\mu_{\tilde{C}_9}(x) = \begin{cases} 1, & \text{if } R_s < 0.9 \\ \frac{0.95 - R_s}{0.05}, & \text{if } 0.9 \leq R_s \leq 0.95 \\ 0, & \text{if } R_s > 0.95 \end{cases}$$

It should be noted that the following constraints g_5, g_6, g_7, g_8 , and g_{10} are satisfied completely and there is no fuzziness can be added. So, we suggest to take their characteristic functions on behalf its fuzziness to get better results and make the study of the problem in the fuzzy environments well order, and as follows:

$$\mu_{\bar{g}_i}(x) = \begin{cases} 1, & \text{if } x_{i-4} \leq 1 \\ 0, & \text{otherwise} \end{cases}, i=5, 6, 7, 8$$

and

$$\mu_{\bar{g}_{10}}(x) = \begin{cases} 1, & \text{if } R_s \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

3- The following NLPP (18) has been developed into NLPP (19) as follows:

Min $-\lambda$,

subject to:

$$\bar{g}_1: \frac{C_s - 517.7838}{630.1780 - 517.7838} - \lambda \geq 0,$$

$$\bar{g}_2: \frac{0.7 - x_1}{0.2} - \lambda \geq 0,$$

$$\bar{g}_3: \frac{0.7 - x_2}{0.2} - \lambda \geq 0,$$

$$\bar{g}_4: \frac{0.7 - x_3}{0.2} - \lambda \geq 0,$$

$$\bar{g}_5: \frac{0.7 - x_4}{0.2} - \lambda \geq 0,$$

$$\bar{g}_6: x_1 \leq 1,$$

(18)

$$\bar{g}_7: x_2 \leq 1,$$

$$\bar{g}_8: x_3 \leq 1,$$

$$\bar{g}_9: x_4 \leq 1,$$

$$\bar{g}_{10}: \frac{0.95 - R_s}{0.05} - \lambda \geq 0,$$

$$\bar{g}_{11}: R_s \leq 1,$$

where $0 \leq \lambda \leq 1$. The solution of (18) has the solution of unconstrained NLPP:

$$\text{Min } \varphi(x_1, x_2, x_3, x_4, \sigma) = -\lambda + \frac{1}{2} \sigma \sum_{i=1}^{10} (\text{Min}(0, \bar{g}_i))^2$$

(19)

where \bar{g}_i is defined in problem (18). The result of problem (18) using the method in section 4.

Table 2: Results of problem (18) by penalty function method

σ	x_1	x_2	x_3	x_4	λ
10	0.8	0.75	0.7	0.65	0.5
100	0.6210	0.6210	0.6208	0.6223	0.4082
1000	0.6224	0.6240	0.6216	0.6236	0.3847
1×10^4	0.6237	0.6239	0.6196	0.6238	0.3810
1×10^5	0.6234	0.6238	0.6198	0.6239	0.3808
25×10^4	0.6233	0.6239	0.6198	0.6239	0.3806
5×10^5	0.6234	0.6238	0.6199	0.6239	0.3806
75×10^4	0.6228	0.6238	0.6203	0.6239	0.3805
1×10^6	0.6228	0.6238	0.6203	0.6239	0.3805
25×10^5	0.6229	0.6238	0.6202	0.6239	0.3805
5×10^6	0.6231	0.6239	0.6200	0.6239	0.3806
75×10^5	0.6291	0.6242	0.6198	0.6242	0.3790
1×10^7	0.6213	0.6236	0.6235	0.6235	0.3821
25×10^6	0.6201	0.6230	0.6238	0.6239	0.3804
5×10^7	0.6203	0.6230	0.6239	0.6239	0.3805
75×10^6	0.6203	0.6237	0.6237	0.6238	0.3809

At $\sigma = 75 \times 10^6$, the penalty term equal to zero, and

$$X_1^* = 0.6203, X_2^* = 0.6237, X_3^* = 0.6237, X_4^* = 0.6238, \lambda^* = 0.3809,$$

$$C_{sAF} = 561.380, R_{sAF} = 0.90629238 \text{ with}$$

$$\bar{g}_1 = 6.986 \times 10^{-3}, \bar{g}_2 = 9.999 \times 10^{-4}, \bar{g}_3 = 5.999 \times 10^{-4}, \bar{g}_4 = 5.999 \times 10^{-4}, \bar{g}_5 = 5.999 \times 10^{-4}, \bar{g}_6 = 3.767 \times 10^{-1}, \bar{g}_7 = 3.763 \times 10^{-1}, \bar{g}_8 = 3.763 \times 10^{-1},$$

$$\bar{g}_9 = 3.762 \times 10^{-1}, \bar{g}_{10} = 4.932 \times 10^{-1} \text{ and } \bar{g}_{11} = 6.6 \times 10^{-3}.$$

Therefore, $z_\lambda \leq C_{sAF} \leq z_u$. Note that problem (18) which is good for generalizations of crisp problem.

6 Conclusions

In this work, the fuzzy solution of reliability allocation optimization problems is presented. Furthermore, it is proposed that the results solution of fuzzy optimization is a generalization of the solution of the crisp optimization problem. In our work, the penalty function, has been competed to be zero, the numerical solution is very close to the exact solution, as the theory of the penalty function methods states.

References

- [1] K. Amit, P. Y. Shiv and K. Surendra. New approach for electric robot fuzzy reliability analysis. *International Journal of Performability Engineering*, 3(2):257–266, 2007.
- [2] B. Fabio, B. Franco and G. M. Pier. Fuzzy reliability analysis of concrete structures. *Computers and Structures*, 82:1033–1052, 2004.
- [3] N. Heathe and R. Chimka. Fuzzy reliability in conceptual design. *In annual IEEE Reliability and Maintainability Symposium*, pages 360–364, 2003.
- [4] B. Li, M. Zhu and K. Xu. A practical engineering method for fuzzy reliability analysis of mechanical structures. *Reliability Engineering and System Safety*, 67:311–315, 2000.
- [5] F. Huang, J. Xu, Y. Lou and J. Gu. Optimum design of fuzzy reliability based on genetic-ant colonies intelligent algorithm. *In IEEE International Symposium on Computational Intelligence and Design*, vol. 1, pages 141–145, 2008.
- [6] A. I. Pereguda and D. A. Timashov. A fuzzy reliability model for safety system-protected object complex. *Reliability, Theory and Applications*, 1(17):28–33, 2010.
- [7] S. Boyles, D. Y. Lin, V. Valsaraj, S. T. Waller. Fuzzy reliability assessment for traffic data. *Transportation Research Part C*, 2008.
- [8] Z. Li and K. C. Kapur. Models and measures for fuzzy reliability and relationship to multi-state reliability. *International Journal of Performability Engineering*, 7(3):241–250, 2011.
- [9] R. Viertl. On reliability estimation based on fuzzy lifetime data. *Forschungsbericht SM*, 139(5):1750–1755, 2007.
- [10] R. M Lyu. Optimal allocation of test resources for software reliability growth modeling in software development. *IEEE Transactions On Reliability*, 51(2):183–129, 2002.
- [11] Y. Alice, C. Chengbin and C. Eric. Reliability allocation problem in a series–parallel system. *Reliability Engineering and System Safety*, 90(1):55–61, 2005.
- [12] G. S. Mahapatra and P. K. Roy. Fuzzy Decision-Making on Reliability of Series System: Fuzzy Geometric Programming Approach. *Annals of Fuzzy Mathematics and Informatics*, 1 (1):107–118, 2011.
- [13] L. V. Utkin, S. V. Gurov and I. B. Shubinsky. A method to solve fuzzy reliability optimization problem. *Microelectronics and Reliability*, 35:171–181, 1995.
- [14] G. W. Dummer and R. C. Winton, *An Elementary Guide to Reliability*, Oxford, New York, 1986.
- [15] V. Bury. *Statistic Models in Applied Science*. John Wiley and Sons, New York, 1975.
- [16] N. Andrzejcs and R. Kevin R. *Reliability of Structures*. McGraw-Hill, New York, 2000.
- [17] H. J. Zimmerman. *H. J. Fuzzy Set Theory and Applications*. Kluwer-Nijhoff Publishing, U.S.A., 1985.

- [18] D. Driankov, H. Hellendoorn and M. Reinfrank. *An Introduction to fuzzy Control*. Springer Science and Business Media, 1996.
- [19] R. E. Bellman and L. A. Zadeh. Decision making in a fuzzy environment. *Management Science*, 17(4):141–164, 1970.
- [20] F. J. Ali and S. Amir. Solving nonlinear programming problem in fuzzy environment. *International Journal of Contemporary Mathematical Sciences*, 7(4):159–170, 2012.
- [21] S. Park. Fuzzy apportionment of system reliability. *IEEE Transactions on Reliability*, 36(4):129–132, 1987.
- [22] B. Li, M. Zhu and K. Xu. A practical engineering method for fuzzy reliability analysis of mechanical structures. *Reliability Engineering and System Safety*, 67:311–315, 2000.
- [23] J. Qimi and C. Chun-Hsien. A numerical algorithm of fuzzy reliability. *Reliability Engineering and System Safety*, 80:299–307, 2003.
- [24] S. F. Shu-Kai and Z. Erwie. Simulation optimization using an enhanced nelder-mead simplex search algorithm. In *Proceedings of the Fifth Asia Pacific Industrial Engineering and Management Systems Conference*, pages 12–15, 2004.
- [25] M. Zhiqing, H. Qiyang and D. Chuangyin. A penalty function algorithm with objective parameters for nonlinear mathematical programming. *Journal of Industrial and Management Optimization*, 5(3):585–601, 2009.
- [26] R. Fletcher. *Directional Methods of Optimization*. John Wiley and Sons, 1987.
- [27] S. C. Rocco, A. J. Miller, O. Moren and N. Carrasquero. Cellular evolutionary approach applied to reliability optimization of complex system, *IEEE Transactions Reliability*, 29(3):210–215, 2000.
- [28] D. Kusum and D. Dipti, Reliability optimization of complex systems through CSOMGA. *Journal of Information and Computing Science*, 4(3):163–172, 2009