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A Combination of Power and Exponential Series for the Development of Hybrid Block Integrators for the Solution of First Order Ordinary Differential Equations

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ABSTRACT

The research article presented the derivation and implementation of some Block Hybrid integrators at step sizes $k = 3$ and $k = 4$ respectively for the solution of first order ordinary differential equation via power and exponential series as basic functions. The derived methods were found to be zero-stable, consistent and convergent. Also, for the values of $\alpha \neq 0$ present in the exponential series were tested with various value of numbers within the number system, the schemes obtained remain the same throughout. The newly derived methods displayed their superiority when tested on some real-life problems and non-linear differential equations used in the literatures.

Keywords: Hybrid, Block integrators, combination of basis function, power and exponential series

1 INTRODUCTION

The techniques for the derivation of Linear Multistep Methods for the solution of initial value problems in Ordinary Differential Equations (ODEs) has been discussed extensively in various literatures over the years and these includes,([1], [2],[3], [4], [5],[6],[7],[8],[9] and [10]) to mention a few. All of them used a single Basis function in their work such as, Power series, Chebyshev, Hermite, Laguerre, Lagendre polynomials, Trigonometric and Exponential Functions. A remarkable progress in numerical methods for approximating solutions of initial value problems (IVPs) in (ODEs) has received considerable attention in recent decades and many researchers have shown interest in constructing more efficient methods. In this research paper two bases were combined to obtain some highly efficient methods to handle problems on ODEs effectively.

1.1 Background of the Study

In many areas of applications of pure and applied sciences where ODEs emerge, some of such equations were stiff or non- linear in nature, the solution of these classes of problems often pose some level of difficulties that resulted in fewer successes of some analytical methods to handle them hence there is need to develop efficient and adequate numerical tools to handle such mathematical models. During the past half-century, the growth in power and availability of digital computer has led to an increasing use of realistic mathematical models in Science and Economics, Engineering, and Numerical analysis of increasing sophistication were needed to solve these more detailed mathematical models of the world, see [11].

The Hybridization of Linear Multistep Method has circumvented the Dahlquist Barrier theorem because of the high orders obtained with these methods. Our newly approach has provided some effective and efficient methods with maximum order $p > k + 2$ without necessarily increasing the step length of the method, since the higher the order of Linear Multistep Method the better the rate of convergence of the results.

Definition 1.

This is an infinite series of the form of:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots + \quad (1)$$

and

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c)^1 + a_2(x-c)^2 + a_3(x-c)^3 \ldots \quad (2)$$

Then equation (2) is a Maclaurin series. If the center c is zero then, it becomes power series**.**

Definition 2 (Exponential series).

This is a function in which an independent variable appears as an exponent.

$$e^{\alpha x} = \sum_{n=0}^{\infty} \frac{\alpha^n x^n}{n!} = 1 + \alpha x + \frac{\alpha^2 x^2}{2!} + \frac{\alpha^3 x^3}{3!} + \frac{\alpha^4 x^4}{4!} + \cdots \quad (3)$$

Theorem 1 (Dahlquist Order Barrier for LMM). A zero-stable Linear Multi-step Method of step number k can have order exceeding $k + 1$ when k is odd or exceeding $k + 2$ when k is even (see [8]).

2 METHODOLOGY

Consider the first order ordinary differential equation of the form

$$y' = f(x, y), \qquad y(x_0) = y_0 \text{ for } a \le x \le b \quad (4)$$

Assumed our approximate solution as:

$$y(x) = \sum_{n=0}^{t+c-2} a_n x^n + a_{t+c-1} \sum_{n=0}^{t+c-1} \frac{\alpha^n x^n}{n!} \quad (5)$$

The first derivatives of (5) is given as:

$$y'(x) = \sum_{n=1}^{t+c-2} n a_n x^{n-1} + a_{t+c-1} \sum_{n=1}^{t+c-1} \frac{\alpha^n x^n}{(n-1)!} \quad (6)$$

where i and c are interpolation and collocation points, while the degree of the polynomial is $(t + c - 1)$. For the consideration of the block integrator methods of First order ordinary differential equations at $k = 3$ and $k = 4$, respectively.

The derivation is as follows:

2.1 Derivation of LMM at $k = 3$

Equation (5) is interpolated at $y(x_i) = y_{n+i},\ i = (0,\ \frac{1}{2})$, and equation (6) is collocated at $f(x_i) = f_{n+i},\ i = \left(0\ ,\left(\frac{1}{2}\right),3\right)$

Our polynomial equation is of the form

$$AX = B \quad (7)$$

where $A = [a_0, a_1, a_2, a_3, a_4, a_5 a_6, a_7]^T$, $\quad B = \left[y_n, y_{n+\frac{1}{2}}, f_n, f_{n+\frac{1}{2}}, f_{n+1} f_{n+\frac{3}{2}}, f_{n+2}, f_{n+\frac{5}{2}}, f_{n+3}\right]^T$

$$X = \left[1, x, x^2, x^3, x^4, x^5, x^6, \left(1 + \alpha x + \frac{1}{2}\alpha^2 x^2 + \frac{1}{6}\alpha^3 x^3 + \frac{1}{24}\alpha^4 x^4 + \frac{1}{120}\alpha^5 x^5 + \frac{1}{720}\alpha^6 x^6\right)\right]$$

Specifically Equations (5) and (6) give non-linear system of equations which of the form

$$a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5 + a_6 x_n^6 + a_7(1 + \alpha x_n + \frac{1}{2}\alpha^2 x_n^2 + \frac{1}{6}\alpha^3 x_n^3 +$$
$$\frac{1}{24}\alpha^4 x_n^4 + \frac{1}{120}\alpha^5 x_n^5 + \frac{1}{720}\alpha^6 x_n^6 + \frac{1}{5040}\alpha^7 x_n^7) = y_n$$

$$a_0 + a_1 x_{n+\frac{1}{2}} + a_2 x_{n+\frac{1}{2}}^2 + a_3 x_{n+\frac{1}{2}}^3 + a_4 x_{n+\frac{1}{2}}^4 + a_5 x_{n+\frac{1}{2}}^5 + a_6 x_{n+\frac{1}{2}}^6 + a_7(1 + \alpha x_{n+\frac{1}{2}} + \frac{1}{2}\alpha^2 x_{n+\frac{1}{2}}^2 +$$
$$\frac{1}{6}\alpha^3 x_{n+\frac{1}{2}}^3 + \frac{1}{24}\alpha^4 x_{n+\frac{1}{2}}^4 + \frac{1}{120}\alpha^5 x_{n+\frac{1}{2}}^5 + \frac{1}{720}\alpha^6 x_{n+\frac{1}{2}}^6 + \frac{1}{5040}\alpha^7 x_{n+\frac{1}{2}}^7) = y_{n+\frac{1}{2}}$$

$$a_1 + 2a_2 x_n + 3a_3 x_n^2 + 4a_4 x_n^3 + 5a_5 x_n^4 + 6a_6 x_n^5 + a_7(\alpha + \alpha^2 x_n + \frac{1}{2}\alpha^3 x_n^2 + \frac{1}{6}\alpha^4 x_n^3 +$$

$$\frac{1}{24}\alpha^5x_n^4+\frac{1}{120}\alpha^6x_n^5+\frac{1}{720}\alpha^7x_n^6)=f_n$$

$$a_1+2a_2x_{n+\frac{1}{2}}+3a_3x^2_{n+\frac{1}{2}}+4a_4x^3_{n+\frac{1}{2}}+5a_5x^4_{n+\frac{1}{2}}+6a_6x^5_{n+\frac{1}{2}}+a_7(\alpha+\alpha^2x_{n+\frac{1}{2}}+\frac{1}{2}\alpha^3x^2_{n+\frac{1}{2}}$$

$$+\frac{1}{6}\alpha^4x^3_{n+\frac{1}{2}}+\frac{1}{24}\alpha^5x^4_{n+\frac{1}{2}}+\frac{1}{120}\alpha^6x^5_{n+\frac{1}{2}}+\frac{1}{720}\alpha^7x^6_{n+\frac{1}{2}})=f_{n+\frac{1}{2}}$$

$$a_1+2a_2x_{n+1}+3a_3x^2_{n+1}+4a_4x^3_{n+1}+5a_5x^4_{n+1}+6a_6x^5_{n+1}+a_7(\alpha+\alpha^2x_{n+1}+\frac{1}{2}\alpha^3x^2_{n+1}+$$

$$\frac{1}{6}\alpha^4x^3_{n+1}+\frac{1}{24}\alpha^5x^4_{n+1}+\frac{1}{120}\alpha^6x^5_{n+1}+\frac{1}{720}\alpha^7x^6_{n+1})=f_{n+1}$$

$$a_1+2a_2x_{n+\frac{3}{2}}+3a_3x^2_{n+\frac{3}{2}}+4a_4x^3_{n+\frac{3}{2}}+5a_5x^4_{n+\frac{3}{2}}+6a_6x^5_{n+\frac{3}{2}}+a_7(\alpha+\alpha^2x_{n+\frac{3}{2}}+\frac{1}{2}\alpha^3x^2_{n+\frac{3}{2}}$$

$$+\frac{1}{6}\alpha^4x^3_{n+\frac{3}{2}}+\frac{1}{24}\alpha^5x^4_{n+\frac{3}{2}}+\frac{1}{120}\alpha^6x^5_{n+\frac{3}{2}}+\frac{1}{720}\alpha^7x^6_{n+\frac{3}{2}})=f_{n+\frac{3}{2}}$$

$$a_1+2a_2x_{n+2}+3a_3x^2_{n+2}+4a_4x^3_{n+2}+5a_5x^4_{n+2}+6a_6x^5_{n+2}+a_7(\alpha+\alpha^2x_{n+2}+\frac{1}{2}\alpha^3x^2_{n+2}+$$

$$\frac{1}{6}\alpha^4x^3_{n+2}+\frac{1}{24}\alpha^5x^4_{n+2}+\frac{1}{120}\alpha^6x^5_{n+2}+\frac{1}{720}\alpha^7x^6_{n+2})=f_{n+2}$$

$$a_1+2a_2x_{n+\frac{5}{2}}+3a_3x^2_{n+\frac{5}{2}}+4a_4x^3_{n+\frac{5}{2}}+5a_5x^4_{n+\frac{5}{2}}+6a_6x^5_{n+\frac{5}{2}}+a_7(\alpha+\alpha^2x_{n+\frac{5}{2}}+\frac{1}{2}\alpha^3x^2_{n+\frac{5}{2}}+$$

$$\frac{1}{6}\alpha^4x^3_{n+\frac{5}{2}}+\frac{1}{24}\alpha^5x^4_{n+\frac{5}{2}}+\frac{1}{120}\alpha^6x^5_{n+\frac{5}{2}}+\frac{1}{720}\alpha^7x^6_{n+\frac{5}{2}})=f_{n+\frac{5}{2}}$$

$$a_1+2a_2x_{n+3}+3a_3x^2_{n+3}+4a_4x^3_{n+3}+5a_5x^4_{n+3}+6a_6x^5_{n+3}+a_7(\alpha+\alpha^2x_{n+3}+\frac{1}{2}\alpha^3x^2_{n+3}+$$

$$\frac{1}{6}\alpha^4x^3_{n+3}+\frac{1}{24}\alpha^5x^4_{n+3}+\frac{1}{120}\alpha^6x^5_{n+3}+\frac{1}{720}\alpha^7x^6_{n+3})=f_{n+3}$$

(8)

Re-arrange Equation (8) in matrix form:

$$
\begin{bmatrix}
1 & x_n^1 & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6, & \left(1+\alpha x_n+\frac{1}{2}\alpha^2x_n^2+\frac{1}{6}\alpha^3x_n^3+\frac{1}{24}\alpha^4x_n^4+\frac{1}{120}\alpha^5x_n^5+\frac{1}{720}\alpha^6x_n^6+\frac{1}{5040}\alpha^7x_n^7\right) \\
1 & x_{n+\frac{1}{2}}^1 & x_{n+\frac{1}{2}}^2 & x_{n+\frac{1}{2}}^3 & x_{n+\frac{1}{2}}^4 & x_{n+\frac{1}{2}}^5 & x_{n+\frac{1}{2}}^6, & \left(1+\alpha x_{n+\frac{1}{2}}+\frac{1}{2}\alpha^2x_{n+\frac{1}{2}}^2+\frac{1}{6}\alpha^3x_{n+\frac{1}{2}}^3+\frac{1}{24}\alpha^4x_{n+\frac{1}{2}}^4+\frac{1}{120}\alpha^5x_{n+\frac{1}{2}}^5+\frac{1}{720}\alpha^6x_{n+\frac{1}{2}}^6+\frac{1}{5040}\alpha^7x_{n+\frac{1}{2}}^7\right) \\
0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & \left(\alpha+\alpha^2x_n+\frac{1}{2}\alpha^3x_n^2+\frac{1}{6}\alpha^4x_n^3+\frac{1}{24}\alpha^5x_n^4+\frac{1}{120}\alpha^6x_n^5+\frac{1}{720}\alpha^7x_n^6\right) \\
0 & 1 & 2x_{n+\frac{1}{2}} & 3x_{n+\frac{1}{2}}^2 & 4x_{n+\frac{1}{2}}^3 & 5x_{n+\frac{1}{2}}^4 & 6x_{n+\frac{1}{2}}^5 & \left(\alpha+\alpha^2x_{n+\frac{1}{2}}+\frac{1}{2}\alpha^3x_{n+\frac{1}{2}}^2+\frac{1}{6}\alpha^4x_{n+\frac{1}{2}}^3+\frac{1}{24}\alpha^5x_{n+\frac{1}{2}}^4+\frac{1}{120}\alpha^6x_{n+\frac{1}{2}}^5+\frac{1}{720}\alpha^7x_{n+\frac{1}{2}}^6\right) \\
0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & \left(\alpha+\alpha^2x_{n+1}+\frac{1}{2}\alpha^3x_{n+1}^2+\frac{1}{6}\alpha^4x_{n+1}^3+\frac{1}{24}\alpha^5x_{n+1}^4+\frac{1}{120}\alpha^6x_{n+1}^5+\frac{1}{720}\alpha^7x_{n+1}^6\right) \\
0 & 1 & 2x_{n+\frac{3}{2}} & 3x_{n+\frac{3}{2}}^2 & 4x_{n+\frac{3}{2}}^3 & 5x_{n+\frac{3}{2}}^4 & 6x_{n+\frac{3}{2}}^5 & \left(\alpha+\alpha^2x_{n+\frac{3}{2}}+\frac{1}{2}\alpha^3x_{n+\frac{3}{2}}^2+\frac{1}{6}\alpha^4x_{n+\frac{3}{2}}^3+\frac{1}{24}\alpha^5x_{n+\frac{3}{2}}^4+\frac{1}{120}\alpha^6x_{n+\frac{3}{2}}^5+\frac{1}{720}\alpha^7x_{n+\frac{3}{2}}^6\right) \\
0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 & \left(\alpha+\alpha^2x_{n+2}+\frac{1}{2}\alpha^3x_{n+2}^2+\frac{1}{6}\alpha^4x_{n+2}^3+\frac{1}{24}\alpha^5x_{n+2}^4+\frac{1}{120}\alpha^6x_{n+2}^5+\frac{1}{720}\alpha^7x_{n+2}^6\right) \\
0 & 1 & 2x_{n+\frac{5}{2}} & 3x_{n+\frac{5}{2}}^2 & 4x_{n+\frac{5}{2}}^3 & 5x_{n+\frac{5}{2}}^4 & 6x_{n+\frac{5}{2}}^5 & \left(\alpha+\alpha^2x_{n+\frac{5}{2}}+\frac{1}{2}\alpha^3x_{n+\frac{5}{2}}^2+\frac{1}{6}\alpha^4x_{n+\frac{5}{2}}^3+\frac{1}{24}\alpha^5x_{n+\frac{5}{2}}^4+\frac{1}{120}\alpha^6x_{n+\frac{5}{2}}^5+\frac{1}{720}\alpha^7x_{n+\frac{5}{2}}^6\right) \\
0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 & 6x_{n+3}^5 & \left(\alpha+\alpha^2x_{n+3}+\frac{1}{2}\alpha^3x_{n+3}^2+\frac{1}{6}\alpha^4x_{n+3}^3+\frac{1}{24}\alpha^5x_{n+3}^4+\frac{1}{120}\alpha^6x_{n+3}^5+\frac{1}{720}\alpha^7x_{n+3}^6\right)
\end{bmatrix}
$$

$$
*\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+\frac{1}{2}} \\ f_n \\ f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \end{bmatrix} \tag{9}
$$

Using Maple17 Mathematical software to determine the values of $\alpha's$ in equation (9) and substituted in equation (5) to form our continuous formula which of the form

$$y(x) = \alpha_n(x)y_n + \alpha_{n+\frac{1}{2}}(x)y_{n+\frac{1}{2}} + h\left[\beta_n(x)f_n+\beta_{n+\frac{1}{2}}(x)f_{n+\frac{1}{2}}+\beta_{n+1}(x)f_{n+1} + \beta_{n+\frac{3}{2}}(x)f_{n+\frac{3}{2}}\right.$$

$$\left.+ \beta_{n+2}(x)f_{n+2} + \beta_{n+\frac{5}{2}}(x)f_{n+\frac{5}{2}} + \beta_{n+3}(x)f_{n+3}\right] \quad (10)$$

where $\alpha_{n+j}(x)$ and $\beta_{n+j}(x), j = 0, \left(\frac{1}{2}\right), 3$ are continuous functions to be determined.

Evaluating equation (10) at $x = x_{n+i}\ , i = \left(\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\right)$ to obtain our newly discrete hybrid block method. However, it worth to note that after testing with number system for values of $\alpha's$ in the exponential series equation (9) but the schemes obtained remain the same throughout.

$$y_{n+\frac{1}{2}} - y_n = \frac{19087}{120960}hf_n + \frac{2713}{5040}hf_{n+\frac{1}{2}} - \frac{15487}{40320}hf_{n+1} + \frac{293}{945}hf_{n+\frac{3}{2}} - \frac{6737}{40320}hf_{n+2} + \frac{263}{5040}hf_{n+\frac{5}{2}} - \frac{863}{120960}hf_{n+3}$$

$$y_{n+1} - y_n = \frac{1139}{7560}hf_n + \frac{47}{63}hf_{n+\frac{1}{2}} + \frac{11}{2520}hf_{n+1} + \frac{166}{945}hf_{n+\frac{3}{2}} - \frac{269}{2520}hf_{n+2} + \frac{11}{315}hf_{n+\frac{5}{2}} - \frac{37}{7560}hf_{n+3}$$

$$y_{n+\frac{3}{2}} - y_n = \frac{137}{896}hf_n + \frac{81}{112}hf_{n+\frac{1}{2}} + \frac{1161}{4480}hf_{n+1} + \frac{17}{35}hf_{n+\frac{3}{2}} - \frac{729}{4480}hf_{n+2} + \frac{27}{560}hf_{n+\frac{5}{2}} - \frac{29}{4480}hf_{n+3}$$

$$y_{n+2} - y_n = \frac{143}{945}hf_n + \frac{232}{315}hf_{n+\frac{1}{2}} + \frac{64}{315}hf_{n+1} + \frac{752}{945}hf_{n+\frac{3}{2}} + \frac{29}{315}hf_{n+2} + \frac{8}{315}hf_{n+\frac{5}{2}} - \frac{4}{945}hf_{n+3}$$

$$y_{n+\frac{5}{2}} - y_n = \frac{3715}{24192}hf_n + \frac{725}{1008}hf_{n+\frac{1}{2}} + \frac{2125}{8064}hf_{n+1} + \frac{125}{189}hf_{n+\frac{3}{2}} + \frac{3875}{8064}hf_{n+2} + \frac{235}{1008}hf_{n+\frac{5}{2}} - \frac{275}{24192}hf_{n+3}$$

$$y_{n+3} - y_n = \frac{41}{280}hf_n + \frac{27}{35}hf_{n+\frac{1}{2}} + \frac{27}{280}hf_{n+1} + \frac{34}{35}hf_{n+\frac{3}{2}} + \frac{27}{280}hf_{n+2} + \frac{27}{35}hf_{n+\frac{5}{2}} + \frac{41}{280}hf_{n+3}$$

(11)

The above equation (11) is the Hybrid Block method at $k = 3$, can be represented in a matrix equation form as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \\ y_{n+\frac{5}{2}} \\ y_{n+3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_{n-\frac{5}{2}} \\ y_{n-2} \\ y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix}$$

$$+\begin{bmatrix} \frac{2713}{5040} & -\frac{15487}{40320} & \frac{293}{945} & -\frac{6737}{40320} & \frac{263}{5040} & -\frac{863}{120960} \\ \frac{47}{63} & \frac{11}{2520} & \frac{166}{945} & -\frac{269}{2520} & \frac{11}{315} & -\frac{37}{7560} \\ \frac{81}{112} & \frac{1161}{4480} & \frac{17}{35} & -\frac{729}{4480} & \frac{27}{560} & -\frac{29}{4480} \\ \frac{232}{315} & \frac{64}{315} & \frac{752}{945} & \frac{29}{315} & \frac{8}{315} & -\frac{4}{945} \\ \frac{725}{1008} & \frac{2125}{8064} & \frac{125}{189} & \frac{3875}{8064} & \frac{235}{1008} & -\frac{275}{24192} \\ \frac{27}{35} & \frac{27}{280} & \frac{34}{35} & \frac{27}{280} & \frac{27}{35} & \frac{41}{280} \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{19087}{120960} \\ 0 & 0 & 0 & 0 & 0 & \frac{1139}{7560} \\ 0 & 0 & 0 & 0 & 0 & \frac{137}{896} \\ 0 & 0 & 0 & 0 & 0 & \frac{143}{945} \\ 0 & 0 & 0 & 0 & 0 & \frac{3715}{24192} \\ 0 & 0 & 0 & 0 & 0 & \frac{41}{280} \end{bmatrix} \begin{bmatrix} f_{n-\frac{5}{2}} \\ f_{n-2} \\ f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{bmatrix} \tag{12}$$

2.2 Derivation of LMM at $k = 4$

For $k = 4$ the equation (5) is interpolated at $y(x_i) = y_{n+i},\ i = (0)$, and equation (6) will be collocate at $f(x_i) = f_{n+i},\ i = \left(\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4\right),\quad if\ \ t = 1, c = 9\ and\ s = 1 + 9 - 1 = 9$, where t and c is the interpolation and collocation points while s is the degree of the polynomials, which give rise to a system of non-linear equations of the form.

$AX = B$

where

$A = [a_0, a_1, a_2, a_3, a_4, a_5 a_6, a_7, a_8]^T$,$B = \left[y_n, f_{n+\frac{1}{2}}, f_{n+1} f_{n+\frac{3}{2}}, f_{n+2}, f_{n+\frac{5}{2}}, f_{n+3} f_{\frac{7}{2}}, f_4\right]^T$

$X = \Big[1, x, x^2, x^3, x^4, x^5, x^6, x^7, \Big(1 + \alpha x + \frac{1}{2}\alpha^2 x^2 + \frac{1}{6}\alpha^3 x^3 + \frac{1}{24}\alpha^4 x^4 + \frac{1}{120}\alpha^5 x^5 + \frac{1}{720}\alpha^6 x^6 + \frac{1}{5040}\alpha^7 x^7\Big)\Big]$

Specifically Equations (5) and (6) will give:

$a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5 + a_6 x_n^6 + a_7 x_n^7 + a_8(1 + \alpha x_n + \frac{1}{2}\alpha^2 x_n^2 + \frac{1}{6}\alpha^3 x_n^3 + \frac{1}{24}\alpha^4 x_n^4 + \frac{1}{120}\alpha^5 x_n^5 + \frac{1}{720}\alpha^6 x_n^6 + \frac{1}{5040}\alpha^7 x_n^7) = y_n$

$a_0 + a_1 x_{n+\frac{1}{2}} + a_2 x_{n+\frac{1}{2}}^2 + a_3 x_{n+\frac{1}{2}}^3 + a_4 x_{n+\frac{1}{2}}^4 + a_5 x_{n+\frac{1}{2}}^5 + a_6 x_{n+\frac{1}{2}}^6 + a_7 x_{n+\frac{1}{2}}^7 + a_8(1 + \alpha x_{n+\frac{1}{2}} + \frac{1}{2}\alpha^2 x_{n+\frac{1}{2}}^2 + \frac{1}{6}\alpha^3 x_{n+\frac{1}{2}}^3 + \frac{1}{24}\alpha^4 x_{n+\frac{1}{2}}^4 + \frac{1}{120}\alpha^5 x_{n+\frac{1}{2}}^5 + \frac{1}{720}\alpha^6 x_{n+\frac{1}{2}}^6 + \frac{1}{5040}\alpha^7 x_{n+\frac{1}{2}}^7) = y_{n+\frac{1}{2}}$

$a_1 + 2a_2 x_{n+\frac{1}{2}} + 3a_3 x_{n+\frac{1}{2}}^2 + 4a_4 x_{n+\frac{1}{2}}^3 + 5a_5 x_{n+\frac{1}{2}}^4 + 6a_6 x_{n+\frac{1}{2}}^5 + 7a_7 x_{n+\frac{1}{2}}^6 + a_8(\alpha + \alpha^2 x_{n+\frac{1}{2}} + \frac{1}{2}\alpha^3 x_{n+\frac{1}{2}}^2 + \frac{1}{6}\alpha^4 x_{n+\frac{1}{2}}^3 + \frac{1}{24}\alpha^5 x_{n+\frac{1}{2}}^4 + \frac{1}{120}\alpha^6 x_{n+\frac{1}{2}}^5 + \frac{1}{720}\alpha^7 x_{n+\frac{1}{2}}^6) = f_{n+\frac{1}{2}}$

$a_1 + 2a_2 x_{n+1} + 3a_3 x_{n+1}^2 + 4a_4 x_{n+1}^3 + 5a_5 x_{n+1}^4 + 6a_6 x_{n+1}^5 + 7a_7 x_{n+1}^6 + a_8(\alpha + \alpha^2 x_{n+1} + \frac{1}{2}\alpha^3 x_{n+1}^2 + \frac{1}{6}\alpha^4 x_{n+1}^3 + \frac{1}{24}\alpha^5 x_{n+1}^4 + \frac{1}{120}\alpha^6 x_{n+1}^5 + \frac{1}{720}\alpha^7 x_{n+1}^6) = f_{n+1}$

$$a_1 + 2a_2x_{n+\frac{3}{2}} + 3a_3x^2_{n+\frac{3}{2}} + 4a_4x^3_{n+\frac{3}{2}} + 5a_5x^4_{n+\frac{3}{2}} + 6a_6x^5_{n+\frac{3}{2}} + 7a_7x^6_{n+\frac{3}{2}} + a_8(\alpha + \alpha^2x_{n+\frac{3}{2}} + \frac{1}{2}\alpha^3x^2_{n+\frac{3}{2}} + \frac{1}{6}\alpha^4x^3_{n+\frac{3}{2}} + \frac{1}{24}\alpha^5x^4_{n+\frac{3}{2}} + \frac{1}{120}\alpha^6x^5_{n+\frac{3}{2}} + \frac{1}{720}\alpha^7x^6_{n+\frac{3}{2}}) = f_{n+\frac{3}{2}}$$

$$a_1 + 2a_2x_{n+2} + 3a_3x^2_{n+2} + 4a_4x^3_{n+2} + 5a_5x^4_{n+2} + 6a_6x^5_{n+2} + 7a_7x^6_{n+2} + a_8(\alpha + \alpha^2x_{n+2} + \frac{1}{2}\alpha^3x^2_{n+2} + \frac{1}{6}\alpha^4x^3_{n+2} + \frac{1}{24}\alpha^5x^4_{n+2} + \frac{1}{120}\alpha^6x^5_{n+2} + \frac{1}{720}\alpha^7x^6_{n+2}) = f_{n+2}$$

$$a_1 + 2a_2x_{n+\frac{5}{2}} + 3a_3x^2_{n+\frac{5}{2}} + 4a_4x^3_{n+\frac{5}{2}} + 5a_5x^4_{n+\frac{5}{2}} + 6a_6x^5_{n+\frac{5}{2}} + 7a_7x^6_{n} + a_8(\alpha + \alpha^2x_{n+\frac{5}{2}} + \frac{1}{2}\alpha^3x^2_{n+\frac{5}{2}} + \frac{1}{6}\alpha^4x^3_{n+\frac{5}{2}} + \frac{1}{24}\alpha^5x^4_{n+\frac{5}{2}} + \frac{1}{120}\alpha^6x^5_{n+\frac{5}{2}} + \frac{1}{720}\alpha^7x^6_{n+\frac{5}{2}}) = f_{n+\frac{5}{2}}$$

$$a_1 + 2a_2x_{n+3} + 3a_3x^2_{n+3} + 4a_4x^3_{n+3} + 5a_5x^4_{n+3} + 6a_6x^5_{n+3} + 7a_7x^6_{n} + a_8(\alpha + \alpha^2x_{n+3} + \frac{1}{2}\alpha^3x^2_{n+3} + \frac{1}{6}\alpha^4x^3_{n+3} + \frac{1}{24}\alpha^5x^4_{n+3} + \frac{1}{120}\alpha^6x^5_{n+3} + \frac{1}{720}\alpha^7x^6_{n+3}) = f_{n+3}$$

$$a_1 + 2a_2x_{n+\frac{7}{2}} + 3a_3x^2_{n+\frac{7}{2}} + 4a_4x^3_{n+\frac{7}{2}} + 5a_5x^4_{n+\frac{7}{2}} + 6a_6x^5_{n+\frac{7}{2}} + 7a_7x^6_{n+\frac{7}{2}} + a_8(\alpha + \alpha^2x_{n+\frac{7}{2}} + \frac{1}{2}\alpha^3x^2_{n+\frac{7}{2}} + \frac{1}{6}\alpha^4x^3_{n+\frac{7}{2}} + \frac{1}{24}\alpha^5x^4_{n+\frac{7}{2}} + \frac{1}{120}\alpha^6x^5_{n+\frac{7}{2}} + \frac{1}{720}\alpha^7x^6_{n+\frac{7}{2}}) = f_{n+\frac{7}{2}}$$

$$a_1 + 2a_2x_{n+4} + 3a_3x^2_{n+4} + 4a_4x^3_{n+4} + 5a_5x^4_{n+4} + 6a_6x^5_{n+4} + 7a_7x^6_{n+4} + a_8(\alpha + \alpha^2x_{n+4} + \frac{1}{2}\alpha^3x^2_{n+4} + \frac{1}{6}\alpha^4x^3_{n+4} + \frac{1}{24}\alpha^5x^4_{n+4} + \frac{1}{120}\alpha^6x^5_{n+4} + \frac{1}{720}\alpha^7x^6_{n+4}) = f_{n+4} \quad (13)$$

Following the same analysis as of Block method at $k = 3$ to obtain the continuous formulation of the form

$$y(x) = \alpha_n(x)y_n + \alpha_{n+\frac{1}{2}}(x)y_{n+\frac{1}{2}} + h\Big[\beta_n(x)f_n + \beta_{n+\frac{1}{2}}(x)f_{n+\frac{1}{2}} + \beta_{n+1}(x)f_{n+1} + \beta_{n+\frac{3}{2}}(x)f_{n+\frac{3}{2}} + \beta_{n+2}(x)f_{n+2} + \beta_{n+\frac{5}{2}}(x)f_{n+\frac{5}{2}} + \beta_{n+3}(x)f_{n+3} + \beta_{n+\frac{7}{2}}(x)f_{n+\frac{7}{2}} + \beta_{n+4}(x)f_{n+4}\Big] \quad (14)$$

Similarly, Maple 17 Mathematical software was used to determine the continuous functions of $a_{n+j}(x)$ and $+\beta_{n+j}(x)$ in equation (14) and which then evaluated at $x = x_{n+i}$, $i = \left(\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4\right)$ to obtain our discrete block method as follows:

$$y_{n+\frac{1}{2}} - y_n = \frac{16083}{8960}hf_{n+\frac{1}{2}} - \frac{1152169}{241920}hf_{n+1} + \frac{242653}{26880}hf_{n+\frac{3}{2}} - \frac{296053}{26880}hf_{n+2} + \frac{2102243}{241920}hf_{n+\frac{5}{2}} - \frac{115747}{26880}hf_{n+3} + \frac{32863}{120960}hf_{n+\frac{7}{2}} - \frac{5257}{34560}hf_{n+4}$$

$$y_{n+1}-y_n=\frac{368}{189}hf_{n+\frac{1}{2}}-\frac{703}{168}hf_{n+1}+\frac{179}{21}hf_{n+\frac{3}{2}}-\frac{79417}{7560}hf_{n+2}+\frac{874}{105}hf_{n+\frac{5}{2}}-\frac{3473}{840}hf_{n+3}+\frac{1111}{945}hf_{n+\frac{7}{2}}-\frac{41}{280}hf_{n+4}$$

$$y_{n+\frac{3}{2}}-y_n=\frac{497}{256}hf_{n+\frac{1}{2}}-\frac{35723}{8960}hf_{n+1}+\frac{80127}{8960}hf_{n+\frac{3}{2}}-\frac{95783}{8960}hf_{n+2}+\frac{75577}{8960}hf_{n+\frac{5}{2}}-\frac{37473}{8960}hf_{n+3}+\frac{2129}{1792}hf_{n+\frac{7}{2}}-\frac{265}{1792}hf_{n+4}$$

$$y_{n+2}-y_n=\frac{68}{35}hf_{n+\frac{1}{2}}-\frac{3784}{945}hf_{n+1}+\frac{964}{105}hf_{n+\frac{3}{2}}-\frac{1087}{105}hf_{n+2}+\frac{7892}{945}hf_{n+\frac{5}{2}}-\frac{436}{105}hf_{n+3}+\frac{124}{105}hf_{n+\frac{7}{2}}-\frac{139}{945}hf_{n+4}$$

$$y_{n+\frac{5}{2}}-y_n==\frac{93965}{48384}hf_{n+\frac{1}{2}}-\frac{21485}{5376}hf_{n+1}+\frac{49145}{5376}hf_{n+\frac{3}{2}}-\frac{487225}{48384}hf_{n+2}+\frac{46415}{5376}hf_{n+\frac{5}{2}}-\frac{22535}{5376}hf_{n+3}+\frac{57515}{48384}hf_{n+\frac{7}{2}}-\frac{265}{1792}hf_{n+4}$$

$$y_{n+3}-y_n==\frac{68}{35}hf_{n+\frac{1}{2}}-\frac{1121}{280}hf_{n+1}+\frac{321}{35}hf_{n+\frac{3}{2}}-\frac{2843}{280}hf_{n+2}+\frac{314}{35}hf_{n+\frac{5}{2}}-\frac{1107}{280}hf_{n+3}+\frac{41}{35}hf_{n+\frac{7}{2}}-\frac{41}{280}hf_{n+4}$$

$$y_{n+\frac{7}{2}}-y_n=\frac{497}{256}hf_{n+\frac{1}{2}}-\frac{27587}{6912}hf_{n+1}+\frac{7007}{768}hf_{n+\frac{3}{2}}-\frac{38563}{3840}hf_{n+2}+\frac{303653}{34560}hf_{n+\frac{5}{2}}-\frac{13573}{3840}hf_{n+3}+\frac{5257}{3840}hf_{n+\frac{7}{2}}-\frac{5257}{34560}hf_{n+4}$$

$$y_{n+4}-y_n=\frac{368}{189}hf_{n+\frac{1}{2}}-\frac{424}{105}hf_{n+1}+\frac{976}{105}hf_{n+\frac{3}{2}}-\frac{1087}{105}hf_{n+2}+\frac{9836}{945}hf_{n+\frac{5}{2}}-\frac{976}{105}hf_{n+3}+\frac{368}{189}hf_{n+\frac{7}{2}} \quad (15)$$

The above equation (15) at $k=4$, can be represented in matrix equation form as follows:

$$\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+\frac{1}{2}}\\y_{n+1}\\y_{n+\frac{3}{2}}\\y_{n+2}\\y_{n+\frac{5}{2}}\\y_{n+3}\\y_{n+\frac{7}{2}}\\y_{n+4}\end{bmatrix}=\begin{bmatrix}0&0&0&0&0&0&0&-1\\0&0&0&0&0&0&0&-1\\0&0&0&0&0&0&0&-1\\0&0&0&0&0&0&0&-1\\0&0&0&0&0&0&0&-1\\0&0&0&0&0&0&0&-1\\0&0&0&0&0&0&0&-1\\0&0&0&0&0&0&0&-1\end{bmatrix}\begin{bmatrix}y_{n-\frac{7}{2}}\\y_{n-3}\\y_{n-\frac{5}{2}}\\y_{n-2}\\y_{n-\frac{3}{2}}\\y_{n-1}\\y_{n-\frac{1}{2}}\\y_n\end{bmatrix}$$

$$
+\begin{bmatrix}
\frac{16083}{8960} & -\frac{1152169}{241920} & \frac{242653}{26880} & -\frac{296053}{26880} & \frac{2102243}{241920} & -\frac{115747}{26880} & \frac{32863}{26880} & -\frac{5257}{34560} \\
\frac{368}{189} & -\frac{703}{168} & \frac{179}{21} & -\frac{79417}{7560} & \frac{874}{105} & -\frac{3473}{840} & \frac{1111}{945} & -\frac{41}{280} \\
\frac{497}{256} & -\frac{35723}{8960} & \frac{80127}{8960} & -\frac{95783}{8960} & \frac{75577}{8960} & -\frac{37473}{8960} & \frac{2129}{1792} & -\frac{265}{1792} \\
\frac{68}{35} & -\frac{3784}{945} & \frac{964}{105} & -\frac{1087}{105} & \frac{7892}{945} & -\frac{436}{105} & \frac{124}{105} & -\frac{139}{945} \\
\frac{93965}{48384} & -\frac{21485}{5376} & \frac{49145}{5376} & -\frac{487225}{48384} & \frac{46415}{5376} & -\frac{22535}{5376} & \frac{57515}{48384} & -\frac{265}{1792} \\
\frac{68}{35} & -\frac{1121}{280} & \frac{321}{35} & -\frac{2843}{280} & \frac{314}{35} & -\frac{1107}{280} & \frac{41}{35} & -\frac{41}{280} \\
\frac{497}{256} & -\frac{27587}{6912} & \frac{7007}{768} & -\frac{38563}{3840} & \frac{303653}{34560} & -\frac{13573}{3840} & \frac{5257}{3840} & -\frac{5257}{34560} \\
\frac{368}{189} & -\frac{424}{105} & \frac{976}{105} & -\frac{9836}{945} & \frac{976}{105} & -\frac{424}{105} & \frac{368}{189} & 0
\end{bmatrix}
\begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \\ f_{n+\frac{7}{2}} \\ f_{n+4} \end{bmatrix} +
$$

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} f_{n-\frac{7}{2}} \\ f_{n-3} \\ f_{n-\frac{5}{2}} \\ f_{n-2} \\ f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{bmatrix} \quad (16)
$$

3 ANALYSIS OF BASIC PROPERTIES OF THE NEWLY HYBRID BLOCK INTEGRATOR METHODS

3.1 Order and Error constant at methods $k = 3$ and $k = 4$

Given linear differential operator:

$L[y(x);\ h] = \sum_{j=0}^{k}[(\alpha_j\, y(x + jh) - h\beta_j y'(x + jh)] = 0$

where $y(x)$ is an arbitrary function, continuously differentiable on an interval $[a, b]$.

The Taylor's expansion about the point x, gives

$L[y(x);\ h] = C_0 y(x) + C_1 hy'(x) + C_2 hy''(x) + C_3 hy'''(x) + C_4 hy^{iv}(x) + \cdots +$

We obtained the coefficients of h as

$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_k$

$C_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots + k\alpha_k - (\beta_0 + \beta_1 + \cdots + \beta_k$

$$\vdots$$

$$C_q = \frac{1}{q!}(\alpha_1 + 2^q\alpha_2 + 3^q\alpha_3 + \cdots + k^q\alpha_k)\frac{1}{(q-1)!}(\beta_1 + 2^{q-1}\beta_2 + 3^{q-1}\beta_3 + \cdots k^{q-1}\beta_k)$$

$$q = 2,3, \dots .. \tag{17}$$

such that, $C_0 = C_1 = \cdots C_P = 0 \quad and \quad C_{P+1} \neq 0$ see ([12], [13])

Using formula (17) for the method (12) of $k = 3$, $C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 =$

$C_7 = 0$ but $C_8 \neq 0$

Implies that the order and the error constant are given by $p = [7,7,7,7,7,7]^T$ *and* $C_{P+1} = C_8 = \left[\frac{275}{6193152}, \frac{1}{30240}, \frac{9}{229376}, \frac{1}{3240}, \frac{275}{6193152}, -\frac{9}{716800}\right]^T$ respectively.

Similarly, using formula (17) for the method (16) at $k = 4$, the orders and the error constants were obtained as $p = [8,8,8,8,8,8,8,8]^T$ and $C_{p+1} = C_9 = \left[\frac{1070017}{1857945600}, \frac{32377}{58060800}, \frac{12881}{22937600}, \frac{4063}{7257600}, \frac{41705}{74317824}, \frac{401}{716800}, \frac{149527}{265420800}, \frac{989}{1814400}\right]^T$

3.2 Zero Stability of the Methods

A block method of k=3 and k=4, (12) and (16) respectively are said to be zero stable if the roots $\left\|rA^{(0)} - A^{(1)}\right\| = 0$, we have

$$A^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\rho(r) = \det\left[rA^{(0)} - A^{(1)}\right] = \det\left[r\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right] = 0$$

$$= det\begin{bmatrix} r & 0 & 0 & 0 & 0 & -1 \\ 0 & r & 0 & 0 & 0 & -1 \\ 0 & 0 & r & 0 & 0 & -1 \\ 0 & 0 & 0 & r & 0 & -1 \\ 0 & 0 & 0 & 0 & r & -1 \\ 0 & 0 & 0 & 0 & 0 & r-1 \end{bmatrix} = r^7[r-1] = 0$$

This implies that $r_1 = r_2 = r_3 = r_4 = r_5 = r_6 = 0$ *and* $r_7 = 1$ is zero stable.

Similarly, for (16) at $k = 4$. is also zero stable.

3.3 Consistence

A numerical method is said to be consistent if the order of LMM is

p$\geq$ 1, *ie* the order of the scheme must be greater or equals to 1.

The condition was satisfied for both block methods, since orders are uniform order 7 and 8. Hence it is consistent.

3.4 Convergence

Zero stability and consistence are the necessary conditions for convergence of LMM. [8], hence both Block methods were convergent.

4 NUMERICAL EXPERIMENTS

Problem1. SIR Model (see [5])

The SIR model is an epidemiological model that computes the theoretical number of people infected with a contagious illness in a closed population over time. The name of this class of models derives from the fact that they involve coupled of equations relating the number of susceptible people S(t), number of people infected I(t) and the number of people who have recovered R(t). It is given by the following three (3) coupled equations:

$$\frac{dS}{dt} = \mu(1 - S) \qquad \text{(a)}$$

$$\frac{dI}{dt} = -\mu I - \gamma I + \beta IS \qquad \text{(b)}$$

$$\frac{dR}{dt} = -\mu R + \gamma I \qquad \text{(c)}$$

where μ, γ and β are positive parameters.

Define y to be

$y = S + I + R$ (d)

and adding the solution of (a), (b) and (c), we obtain the following evolution equation for y,

$y' = \mu(1 - y)$ (e)

Taking $\mu = 0.5$ with initial condition $y(0) = 0.5$ (for a particular closed population) we obtain,

$y'(t) = 0.5(1 - y), \quad y(0) = 0.5$

whose exact solution is,

$y(t) = 1 - 0.5e^{-0.5t}$

Problem 2. Mixing problems

A 1500-gallon tank initially contains 600 gallons of water with 5lbs of salt dissolved in it. Water enters the tank at a rate of 9gal/hr and the water entering the tank has a salt concentration of $\frac{1}{5}(1 + cosx)$lbs/gal. If a well-mixed solution leaves the tank at a rate of 6gal/hr, how much salt is in the tank when it overflows? See ([13])

Formulation:

So the IVP for this situation is,

$y' = 9\left(\frac{1}{5}(1 + cosx)\right) - 6\left(\frac{y(x)}{600+3x}\right), y(0) = 5$ or

$y' = \frac{9}{5}(1 + cosx) - \left(\frac{2y(x)}{200+x}\right), \qquad y(0) = 5$

The exact solution is

$y(x) = \frac{9}{5}\left(\frac{1}{3}(200 + x) + sinx + \frac{2cosx}{200+x} - \frac{2sinx}{(200+x)^2}\right) - \frac{4600720}{(200+x)^2}$

Problem 3

Solve the IVP

$y' = xe^{3x} - 2y, \qquad y(0) = 0$

The exact solution is

$y(x) = \frac{1}{5}xe^{3x} - \frac{1}{25}e^{3x} + \frac{1}{25}e^{-2x}$

Table 1: Performance of New Hybrid Block Integrator at k=3 for Problem 1

x	Exact solution	New Hybrid Block integrator method at K=3	Error in Ref [5] Order six Block integrator K=5	Error in New Hybrid integrator method at K=3
0.1	0.524385287749643	0.524385287749646	5.574×10^{-12}	3.000×10^{-15}
0.2	0.547581290982020	0.547581290982020	3.946×10^{-12}	–
0.3	0.569646011787471	0.569646011787471	8.183×10^{-12}	–
0.4	0.590634623461009	0.590634623461011	3.436×10^{-11}	2.000×10^{-15}
0.5	0.610599608464298	0.610599608464296	1.929×10^{-10}	2.000×10^{-15}
0.6	0.629590889659141	0.629590889659141	1.879×10^{-10}	–
0.7	0.647655955140644	0.647655955140645	1.776×10^{-10}	1.000×10^{-15}
0.8	0.664839976982180	0.664839976982180	1.724×10^{-10}	–
0.9	0.681185924189114	0.681185924189114	1.847×10^{-10}	–
1.0	0.696734670143684	0.696734670143686	3.005×10^{-10}	2.000×10^{-15}

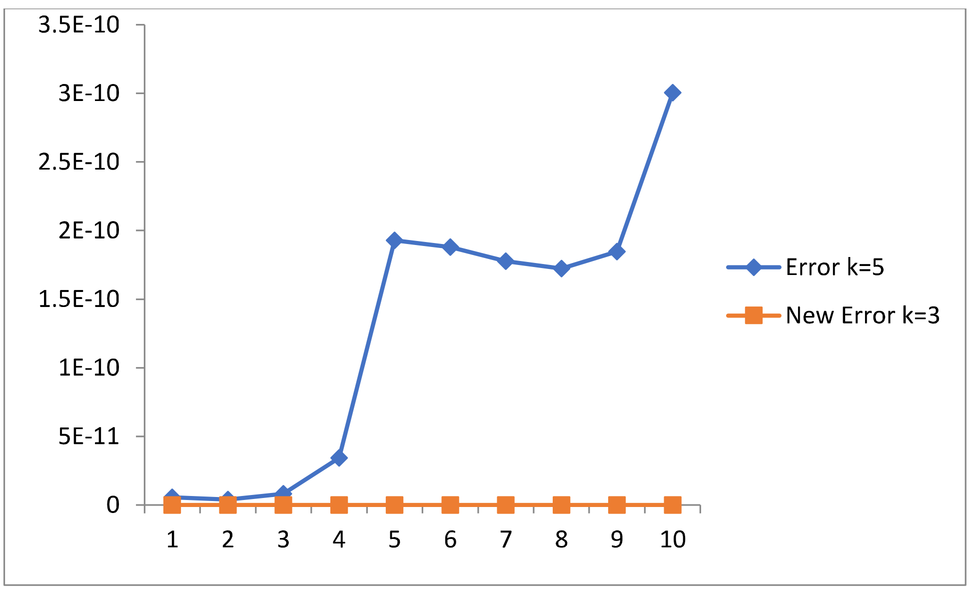

Figure 1: Error graph of Problem between Ref [5] at $k = 5$ and New Hybrid Integrator at k=3

Table2: Approximate solutions on Test Problem two (2) Mixing problem

x	Exact solution	At K=3 Values	At K=4 Values	Error of K=3	Error of K=4
0.1	5.354524092320860	5.35452409232080	5.354524092319910	6.0×10^{-14}	9.5×10^{-13}
0.2	5.706901931222570	5.706901931222080	5.706901931221200	4.9×10^{-13}	1.37×10^{-12}
0.3	6.055359334472330	6.055359334472180	6.055359334471210	1.5×10^{-13}	1.12×10^{-12}
0.4	6.398159569553200	6.398159569552600	6.398159569551910	6.0×10^{-13}	1.29×10^{-12}
0.5	6.733620708232820	6.733620708232250	6.733620708230830	5.7×10^{-13}	1.99×10^{-12}
0.6	7.060132433607410	7.060132433607230	7.060132433605500	1.8×10^{-13}	1.91×10^{-12}
0.7	7.376172131694110	7.376172131693260	7.37617213169160	8.5×10^{-13}	2.15×10^{-12}
0.8	7.680320106810110	7.680320106810050	7.680320106808730	6.0×10^{-14}	1.38×10^{-12}
0.9	7.971273768733360	7.971273768733290	7.971273768731030	7.0×10^{-14}	2.33×10^{-12}
1.0	8.247860649884670	8.247860649883960	8.247860649882210	7.1×10^{-13}	2.46×10^{-12}

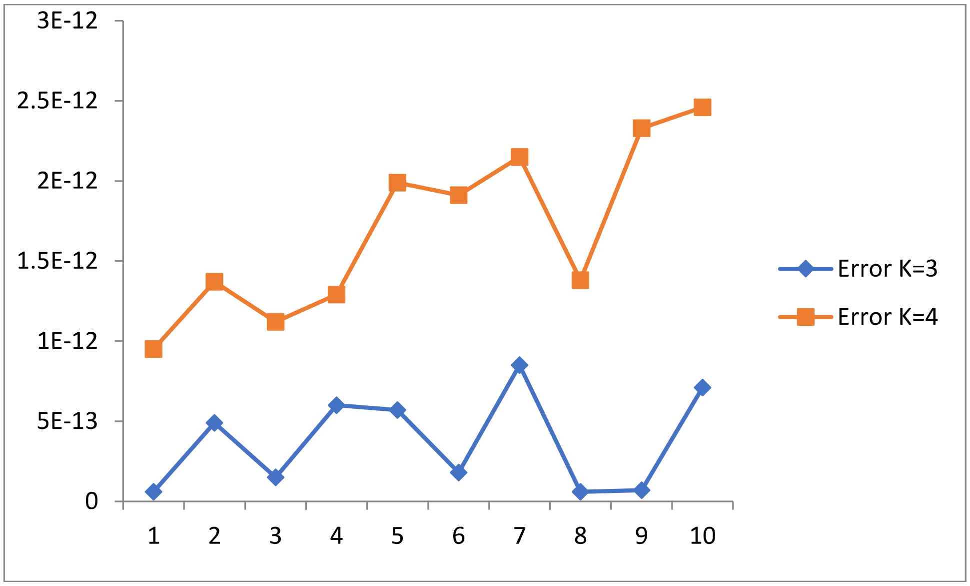

Figure 2: Error graph between K=3 and K=4 of New Hybrid Integrator Method

Table 3: Approximate solutions on problem 3

x	Exact Solution	Approximation in Ref [1], 4-step Optimal Order Scheme	New Block Integrator Method at K=3	New Block Integrator Method at K=4
0.1	0.005752053971	0.0057518112	0.005752052639	0.005752044314
0.2	0.026812801841	0.0268122411	0.0268128007921	0.026812793897
0.3	0.071144527666	0.0711435312	0.0711445287688	0.071144521176
0.4	0.150777835474	0.1507845599	0.150777832725	0.150777830527
0.5	0.283616521867	0.2836371233	0.283616519730	0.283616481466
0.6	0.496019565629	0.4960597062	0.496019569246	0.496019532413
0.7	0.826480869814	0.8265583436	0.826480862870	0.826480842668
0.8	1.330857026396	1.3309723576	1.330857021019	1.330857005553
0.9	2.089774397011	2.0899659578	2.089774407153	2.089774331648
1.0	3.219099319039	3.2193643909	3.219099308038	3.219099268188

Table 4. Absolute Error of New methods $k = 3$ and $k = 4$ of Problem 3

Error [1], 4-step Optimal Order Scheme	**Error in New Hybrid integrator method of K=3**	**Error in New Hybrid integrator method of K=4**
2.427×10^{-7}	1.332×10^{-9}	9.657×10^{-9}
5.607×10^{-7}	1.049×10^{-9}	7.943×10^{-9}
9.964×10^{-7}	1.101×10^{-9}	6.490×10^{-9}
6.724×10^{-6}	2.748×10^{-9}	4.946×10^{-9}
2.060×10^{-5}	2.136×10^{-9}	4.040×10^{-9}
4.014×10^{-5}	3.617×10^{-9}	3.321×10^{-8}
7.747×10^{-5}	6.943×10^{-9}	2.714×10^{-8}
1.153×10^{-4}	5.377×10^{-9}	2.084×10^{-8}
1.915×10^{-4}	1.014×10^{-8}	6.536×10^{-8}
2.6507×10^{-4}	1.100×10^{-8}	5.085×10^{-8}

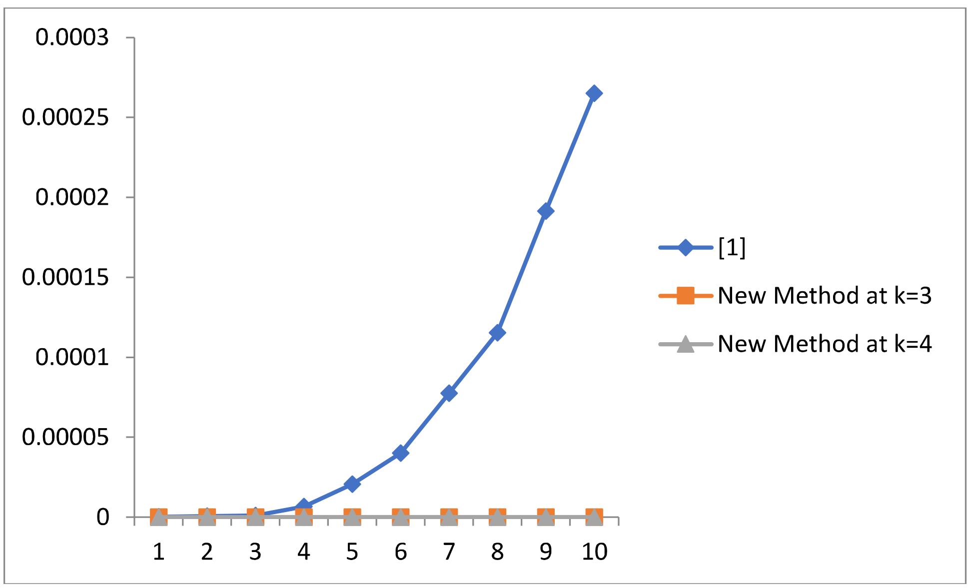

Figure 3: Error graph between [1] and New Hybrid Method k=3 and k=4 of Problem 3

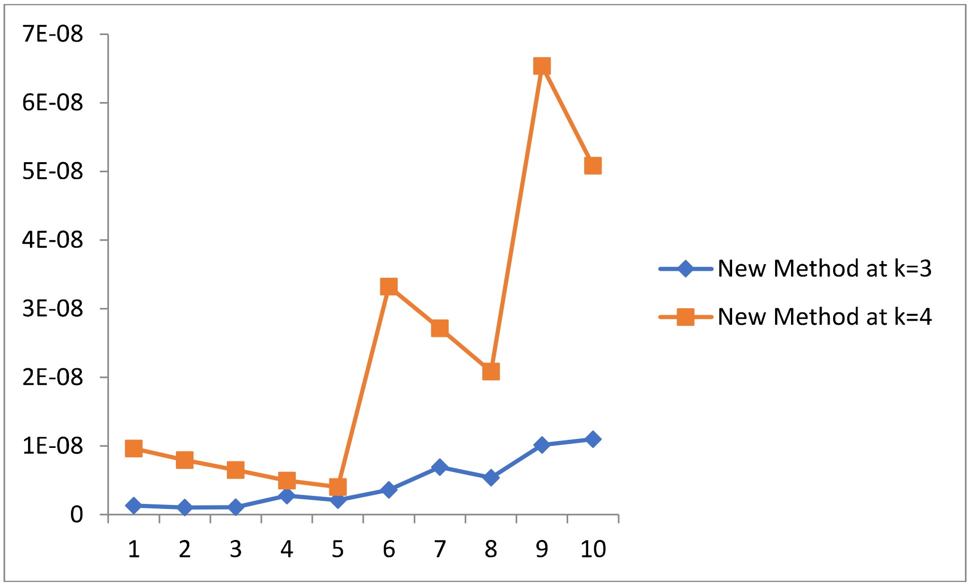

Figure 4: Error graph between New Method k=3 and New Hybrid Method k=4 of Problem 3

5 DISCUSSION OF THE RESULTS

Table 1 display the result on SIR model, the new method of $k = 3$ of uniform order 7.

performed excellently well when compared with [5], of method K=5 with order 6 (see Figure 1).

Table 2 shows the result of mixing problem models with methods $k = 3$ and $k = 4$. The method at $k = 3$ displays its superiority over the method at $k = 4$ despite of it's of uniform order 8 (see Figure 2).

Table 3 shows the numerical computation to example 3 with methods [1], $k = 3$ and $k = 4$. The method at $k = 3$ also displays its superiority over the method at $k = 4$ despite of it's of uniform order 8 (see Figures 3 and 4). Our interest in this research is based on the accuracy of the solutions of the methods rather than computational time

6 CONCLUSION

We conclude that the block integrator method, utilizing a combination of basic functions, is computationally reliable for solving both linear and nonlinear first-order ordinary differential equations. Additionally, increasing the step size does not necessarily ensure the convergence of the results, as demonstrated in Figures 2 and 3.

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