Applied Mathematics and Computational Intelligence
Volume 12, No. 4, Nov 2023 [76-93]

An Investigation of Stability, Controllability and Observability of a Three Degree of Freedom Translational Mechanical System Using State Space Approach

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Received: 14 Nov 2022
Accepted: 12 Sept 2023

ABSTRACT

State-space modelling approach which is essentially time-domain developed in the late 1960s is a new approach to the analysis and design of complex control systems. Several researches have carried out modelling and analysis of mechanical systems with a number of degrees of freedom of movement using state space approach among which is the work of Sivak and Darina [33] who modelled a system with two degrees of freedom. In this paper, we provide an extension of the work of Sivak and Darina [33] to model and analyse a three degree of freedom translational mechanical system using state-space approach. The system was first presented in equivalent free body diagrams, then Newton's second law of motion was used to derive its equations of motion. The state-space formulation in the controllable canonical form obtained from the time-domain differential equations is adopted and the Laplace transform method was used in the analysis to determine the poles (natural frequencies) of the system using two numerical examples in MATLAB software. Software was also used to determine the controllability and observability matrices of the system. The stability, controllability and observability of the system were then discussed from the poles, controllability matrix and observability matrix respectively. The results show the system was found to be stable, controllable and observable suggesting that a state feedback control design for the system is possible.

Keywords: *controllability, mechanical system, observability, stability, state-space.*

1 INTRODUCTION

To design a control system that will behave in a desirable manner, we need a way to predict the behaviour of the quantities of interest over time, specifically how they change in response to different inputs. Mathematical models are most oftenly used to predict future behaviour and control system design methodologies are based on such models [1]. [2] reported that understanding control theory requires engineers to be well versed in basic mathematical concepts and skills, such as solving differential equations and problems in Laplace transformation. The role of control theory according to [3] is to help us gain insight on how and why feedback control systems work and how to systematically deal with various design and analysis issues. Specifically, the following issues are of both practical importance and theoretical interest:

1. Stability and stability margins of closed-loop systems.
2. How fast and smooth the error between the output and the set point is driven to zero?
3. How well the control system handles external disturbances coming from the surrounding environment, sensor noises and internal dynamic changes.

Therefore, mathematical models of physical processes are the foundations of control theory. The existing analysis and synthesis tools are all based on certain types of mathematical descriptions of the systems to be controlled [4]. There has been increased interest in the application of active control to processes which involve mechanical systems such as load alleviation, active flutter control, fatigue reduction and ride control. For this purpose according to [5], the control system engineer requires a mathematical model of the plant to be controlled in order to design a system which will accomplish the desired objectives. According to [6], such differential equations may be obtained by using physical laws governing a particular system. For example, Newtons Law for mechanical systems and Kirchhoffs Law for electrical systems.

In engineering and science, the Laplace transform is used for solving problems of time-invariant systems such as electrical circuits, harmonics, oscillations, mechanical system, control theory and optical devices [7]. The Laplace transforms the time domain into frequency domain (the inputs and output function of complex angular frequency in radians per unit time). Based on specifications, Laplace transform simplifies the process of analysing the behaviour of a dynamic or synthesizing a new system. However, a modern complex system may have many inputs and many outputs (MIMO) which may be interrelated in a complicated manner. To analyse such systems using the Laplace, we would rely on the principle of superposition to create a system of simultaneous Laplace equations for each of the output and input. For such systems, this classical approach doesn't simplify the situation, because the systems of equations need to be transformed into the frequency domain first, manipulated, and then transformed back into the time domain [8].

[9] also pointed out that, one of the drawbacks of using Laplace transforms to solve ordinary differential equations with a forcing term is its lack of generality where each new forcing function requires a repetition of the entire process. According to [10], the disadvantage of the Laplace transform method was that as the system grew in complexity including nonlinear, time-varying, high system order and multiple inputs and multiple outputs (MIMO) systems, the method either became too difficult or lost its applicability.

Modern control theory which is a new approach to the analysis and design of complex control systems has been developed around 1960 [6]. This approach is based on the concept of 'state' where multiple first-order differential equations are analyzed in vector form to account for system with MIMO without adding much unnecessary complexity. In a state-space system, the internal state of the system is explicitly accounted for an equation known as the State Equation. The system output is given in terms of a combination of the current system state and the current system input, through the output equation. These two equations form a system of equations known collectively as state space equations:

$$\dot{\mathbf{X}} = A\mathbf{X} + B\mathbf{U}$$
$$\mathbf{Y} = C\mathbf{X} + D\mathbf{U} \tag{1}$$

Where $\mathbf{X}, \mathbf{U}, \mathbf{Y}$ are the state, input and output vectors respectively. A is the state matrix and D is the direct transmission matrix. If $D = 0$, this implies that there is no direct connection between the input $\mathbf{U}$ and the output $\mathbf{Y}$ [11].

The B and C in the model are matrices allowing the model to readily handle multiple inputs, multiple outputs systems as well as being able to handle time-varying and nonlinear systems. Also, because the system is represented compactly by matrices, it is easily manipulated by computers [12]. Thus, the modern state-space design is a comprehensive term referring to modelling and control of complex systems. This approach maximizes computational accuracy, efficiency, and programming convenience within a general format that may include linear, nonlinear, discrete and time-varying representations [13]. [14] pointed out that the dynamic behaviour of any system can fully be determined by state variables and further recommends that, instead of using other methods to solve differential equations, the state equations can yield a great deal of information about a system even when they are not solved explicitly. [15] outlined some advantages of state-variable representation to dynamical systems as follows:

(i) It provides systematic analysis and synthesis of higher-order systems without truncation of system dynamics.
(ii) It is a convenient tool for MIMO systems.
(iii) It is a uniform platform for representing time-invariant systems, time-varying systems, linear systems as well as nonlinear systems.
(iv) It can describe the dynamics in almost all systems (mechanical systems, electrical systems, biological systems, economical systems, social systems etc).

In view of the great efficiency and flexibility of the state-space approach and the substantial computational advantages available [13], the possibility of employing this kind of realization for multivariable control systems studies is obviously worth doing. Studies on state-space modeling are receiving considerable attention from many researchers because of its applications in many fields of engineering.

[13] stated that state space methods are used extensively in single-dimensional scalar and multivariable system studies. Mechanical engineering systems with two or more parts with significantly different levels of energy dissipation are encountered frequently in dynamical designs [16]. Following the recommendations given in recent works of literature [17], a state-space representation is suitable for analysis and control system design. [18] and [19] pointed out that modeling of MIMO mechanical systems has evolved from simple mass-spring models to relatively complicated models that include relatively high degrees of freedom. According to [20] and [21], MIMO systems modeling has been a topic of much research activity for the last few decades.

The primary goal of using state-space representation for modeling mechanical systems is to predict the internal and external forces during a time interval [22], [23], [24], and [25]. [26] revealed that the inclusion of multi-degree of freedom in a mechanical system model normally leads to more precise results, but also much effort is needed to figure out the process and what is happening with internal states in the system structure. The methodology described in the work of [17] for the production of simple models of mechanical systems revealed that state-space representation gives a suitable and compact way to model and analyze systems with more than one input and more than one output. [27] applied Laplace transform to find the eigen solution of the thermoelastic interaction in an unbounded medium with a spherical cavity using a state-space approach. [28] developed an algorithm using a state-space formulation for the equation of motion of a mechanical system to identify its state transition matrix from measured multiple-input/multiple-output frequency response functions. This work showed that it is possible to derive the natural frequencies and mode shapes of a mechanical system using state-space approach. [29] calculated the eigenvalues and eigenvectors of a spinning spacecraft containing elastic parts using a new concept in the form of a state-vector.

[30] developed a state-space formulation for controllability and observability testing using a single degree of freedom forced vibrating system and found that it is completely observable if the rank of the model matrix is equal to n (n = order of the system). The combination of state space and Laplace transform techniques has been found effective in the work of [31] in determining feedback gains of a simple distributed parameter flexible system. [32] developed a linear theory from the generalized Laplace transform method and study controllability and observability in terms of the Gramian and various rank conditions. This work proved the Kalman controllability and observability rank conditions.

In a study conducted by [10], a state-space approach was used to model, identify and control a two degree of freedom (single-input, multiple-output (SIMO)) rotational mechanical system. The system was shown to be stable, having one negative real pole and another pole at the origin. It was however investigated that the model was uncontrollable and unobservable. The reason was that the 4th order model controllability and observability matrices were rank deficient and that the shaft that connects the two masses is not flexible enough to exhibit a spring-like behavior. [33] worked on a translational system with two degrees of freedom (SIMO) in a controllable canonical state-space representation and found that solving the system numerically by using state-space approach gave accurate results.

In the above studies, state-space method is worthy to be noted for its high efficiency and practicability regarding the vibration and control of mechanical systems. The application of state-space approach is simple and unique for cases of multi-degree of freedom systems.

The reviewed works of literature of interest to this study are the works of [10] and [33]. In 2009, Anderson modelled and controlled a two degree of freedom rotational mechanical system using a state-space approach. This model was found to be rank deficient because the stiffness of the system was not flexible to exhibit spring-like behavior. Hence, the system was reduced and controlled as a single degree of freedom. Anderson further suggested that if his work is repeated on a similar rectilinear (translational) system where the two masses are connected via true springs, then the model would have full rank and the two masses could be controlled independently. This similar translational system was explored in the work of [33] who dealt with a two degree of freedom. The work successfully carried out the state-space formulation and solved numerically to simulate the motion of the system using MATLAB. However, investigation of the system's stability, controllability, and observability was not carried out in that work. Therefore this work will deal with a three degree of freedom translational mechanical system and in addition to the work of [33], the work will investigate the stability, controllability, and observability of the system.

2 DERIVING THE SYSTEM'S EQUATIONS OF MOTION

According to [34], the equation of motion of a system with n-degrees of freedom is paired with normal second-order ordinary differential equations. The equilibrium equation of a multiple-degree of freedom system can be shown as [35]:

$$[M]\ddot{\mathbf{X}}+[B]\dot{\mathbf{X}}+[K]\mathbf{X}=\mathbf{F}(t) \tag{2}$$

Where [M], [B], [K] are the mass, damper, and stiffness matrices respectively while $\ddot{\mathbf{X}}, \dot{\mathbf{X}}, \mathbf{X}, \mathbf{F}$ are vectors of accelerations, velocities, displacements, and input of the system respectively.

The system as shown in Figure 1 below has bodies of masses M_1, M_2, M_3 with stiffnesses K_1, K_2 and dampers B_1, B_2. The masses are not connected to any rigid frame and the system performs linear motion in the direction of springs and dampers axes. The weights of the springs are not considered. A step input force is used for the model excitation and the respective masses perform linear forced oscillating motion.

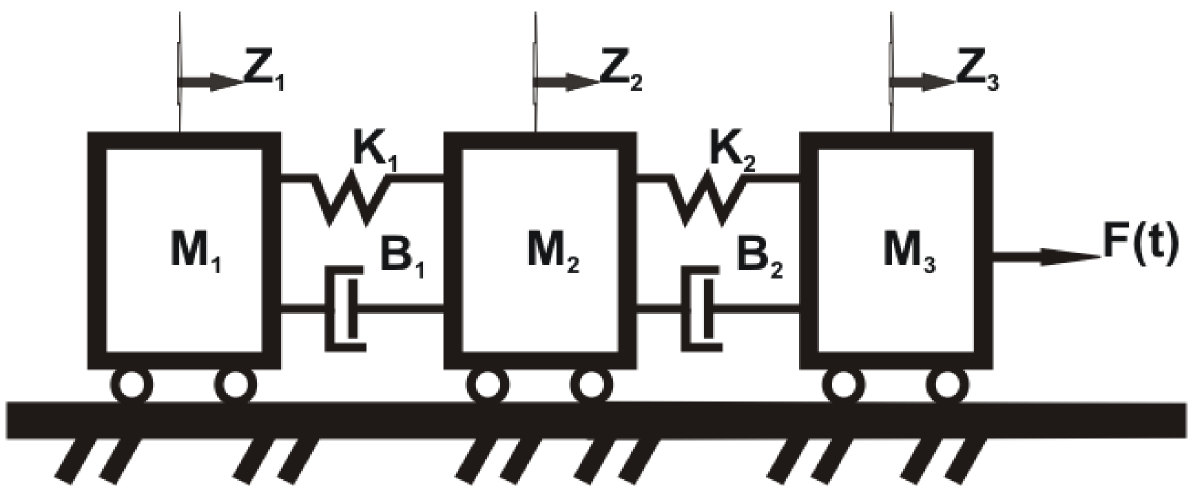

Figure 1: Original damped three degrees of freedom system model

The free-body diagrams of the masses, with the assumed positive directions for their displacements, velocities, and accelerations describing the coordinates $Z_1(t), Z_2(t), Z_3(t)$ measured from their respective equilibrium positions are indicated in Figure 2. The application of Newton's second law of motion gives the equations of motion of the masses M_1, M_2, M_3 respectively as follows:

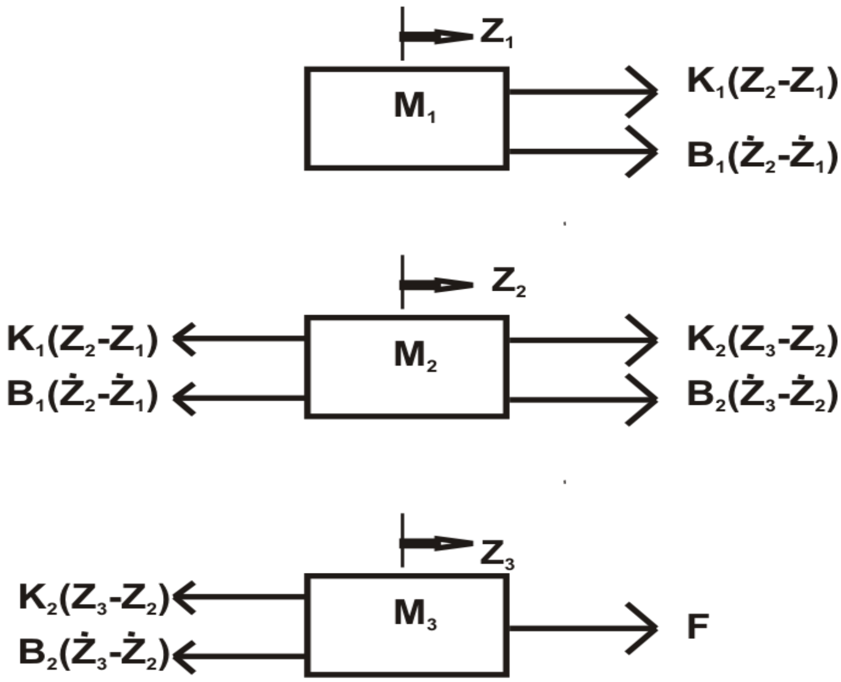

Figure 2: Three free-body diagrams for M_1, M_2, M_3

$$M_1\ddot{Z}_1 + B_1\dot{Z}_1 - B_1\dot{Z}_2 + K_1Z_1 - K_1Z_2 = 0 \tag{3}$$

$$\begin{aligned} M_2\ddot{Z}_2 - B_1\dot{Z}_1 + (B_1 + B_2)\dot{Z}_2 - B_2\dot{Z}_3 \\ - K_1Z_1 + (K_1 + K_2)Z_2 - K_2Z_3 = 0 \end{aligned} \tag{4}$$

$$M_3\ddot{Z}_3 - B_2\dot{Z}_2 + B_2\dot{Z}_3 - K_2Z_2 + K_2Z_3 = F_3 \tag{5}$$

The three equations above are second-order differential equations that require knowledge of the initial states of position and velocity for all the three degrees of freedom in order to solve for the transient response.

3 STATE SPACE MODEL

Solving (3), (4), and (5) for the highest derivatives, $\ddot{Z}_1$, $\ddot{Z}_2$ and $\ddot{Z}_3$:

$$\ddot{Z}_1 = -\frac{K_1Z_1}{M_1} - \frac{B_1\dot{Z}_1}{M_1} + \frac{K_1Z_2}{M_1} + \frac{B_1\dot{Z}_2}{M_1} \tag{6}$$

$$\ddot{Z}_2 = \frac{K_1Z_1}{M_2} + \frac{B_1\dot{Z}_1}{M_2} - \frac{(K_1 + K_2)Z_2}{M_2} - \frac{(B_1 + B_2)\dot{Z}_2}{M_2} + \frac{K_2Z_3}{M_2} + \frac{B_2\dot{Z}_3}{M_2} \tag{7}$$

$$\ddot{Z}_3 = \frac{K_2 Z_2}{M_3} + \frac{B_1 \dot{Z}_2}{M_3} - \frac{K_2 Z_3}{M_3} - \frac{B_2 \dot{Z}_3}{M_3} + \frac{F}{M_3} \tag{8}$$

Changing the notation from Z to X, we define our state variables as:

$$X_1 = Z_1 \quad \textit{position} \quad \textit{of} \quad M_1 \tag{9}$$

$$X_2 = \dot{Z}_1 \quad \textit{velocity} \quad \textit{of} \quad M_1 \tag{10}$$

$$X_3 = Z_2 \quad \textit{position} \quad \textit{of} \quad M_2 \tag{11}$$

$$X_4 = \dot{Z}_2 \quad \textit{velocity} \quad \textit{of} \quad M_2 \tag{12}$$

$$X_5 = Z_3 \quad \textit{position} \quad \textit{of} \quad M_3 \tag{13}$$

$$X_6 = \dot{Z}_3 \quad \textit{velocity} \quad \textit{of} \quad M_3 \tag{14}$$

Thus, the three second-order differential equations above are converted to six first-order differential equations. Observing the relationship between the position states and velocity states and combining in matrix form, we have the state equation as:

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \\ \dot{X}_5 \\ \dot{X}_6 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{-K_1}{M_1} & \frac{-B_1}{M_1} & \frac{K_1}{M_1} & \frac{B_1}{M_1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{K_1}{M_2} & \frac{B_1}{M_2} & \frac{-(K_1+K_2)}{M_2} & \frac{-(B_1+B_2)}{M_2} & \frac{K_2}{M_2} & \frac{B_2}{M_2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{K_2}{M_3} & \frac{B_2}{M_3} & \frac{-K_2}{M_3} & \frac{-B_2}{M_3} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{F}{M_3} \end{pmatrix} (1) \tag{15}$$

Written in the standard form as

$$\dot{\mathbf{X}} = A\mathbf{X} + B\mathbf{U} \tag{16}$$

and the output of the system as

$$\mathbf{Y} = C\mathbf{X} + D\mathbf{U} \tag{17}$$

With $D = 0$ as stated in [11], the output becomes

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{pmatrix} \tag{18}$$

3.1 State Space System Block Diagram

Consider the block diagram as shown in Figure 3.

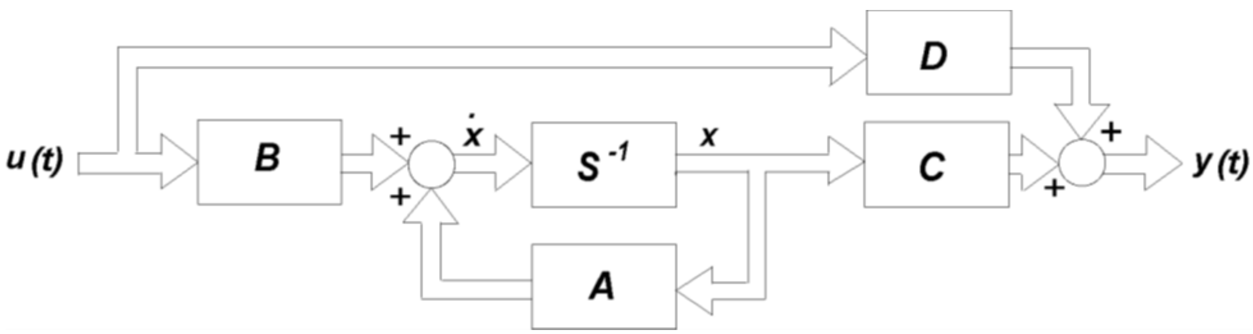

Figure 3: Vector Block Diagram Described by the State-Space System Dynamic

The scalar input $u(t)$ is fed into both the input matrix B and the direct transmission matrix D. The output of the input matrix is an $n \times 1$ vector, where n is the number of states. The direct transmission matrix is a scalar, and its output is fed into a summing junction to be added to the output of the C matrix. The output of the B matrix is added to the feedback terms coming from the system matrix A and is fed into an integrator block, "S^{-1}" with $n \times n$ identity matrix. The output $y(t)$ is obtained by the summation of the results from C and D matrices.

4 STABILITY, CONTROLLABILITY, AND OBSERVABILITY

In this section, three fundamental properties of the system will be discussed from the state space point of view as each has an impact on the control strategy of the system which are stability, controllability and observability [10]. Stability is a key to understand the dynamic behavior of a system. Meanwhile the controllability and observability is important to determine whether the system is well suited for a pole placement control strategy.

4.1 Stability

According to [6], a system is stable if the output eventually comes back to its equilibrium state when the system is subjected to an initial condition. It is unstable if the output diverges without bound from its equilibrium state when the system is subjected to an initial condition. The most important problem in linear control systems concerns stability, that is, under what conditions will a system become unstable. If it is unstable, how should we stabilize the system. According to [36], the time response of a control system consists of two parts: the transient response and the steady-state response. Transient response is the component of the of the system due to initial conditions while steady state response represent the output due to the forcing function and it describes the manner in which the system output behaves as time (t) approaches infinity [37]. Thus, the system response $\mathbf{Y}(t)$ may be written as:

$$\mathbf{Y}(t) = \mathbf{Y}_{tr}(t) + \mathbf{Y}_{ss}(t) \tag{19}$$

Where $\mathbf{Y}_{tr}(t)$ and $\mathbf{Y}_{ss}(t)$ are the transient response and the steady state response respectively. For system stability, this transient response eventually reaches equilibrium.

4.1.1 Condition for Stability

A control system is stable if and only if all closed-loop poles (eigenvalues) lie in the left-half S-plane or on the imaginary S-plane.

4.1.2 The System's Eigenvalues (Poles)

As discussed in [38], any successful control strategy must take into consideration the effect of the control forces associated with the independent elastic degree of freedom as well as the effect of the selection of the reference conditions and the associated mode shapes on the design of the control system. Thus, the system frequencies that define the control bandwidth, stability, and response characteristics need to be identified. One of basic analyses of a dynamic system is to solve for its eigenvalues (natural frequencies) which show the frequencies where the system will amplify inputs and basic characteristic of the system. In this system, we set the forcing function to zero and write the homogeneous state equation:

$$\dot{\mathbf{X}} = A\mathbf{X}, \qquad \mathbf{X}(0) = 0 \tag{20}$$

By taking the Laplace transform of the above equation, the system's response function can be expressed as an eigenvalue problem [35]. Taking the Laplace transform of the first derivative for the homogeneous state equation above, we have:

$$\begin{aligned} \ell[\dot{\mathbf{X}}] &= S\mathbf{X}(S) - \mathbf{X}(0) \\ &= A\mathbf{X}(S) \end{aligned} \tag{21}$$

Thus, we have:

$$S\mathbf{X}(S) - \mathbf{X}(0) = A\mathbf{X}(S) \tag{22}$$

Assuming the zero initial conditions and solving for $\mathbf{X}(S)$:

$$\begin{aligned} S\mathbf{X}(S) &= A\mathbf{X}(S) \\ (S\mathbf{I} - A)\mathbf{X}(S) &= 0 \end{aligned} \tag{23}$$

Where the complex variable S is given by S = σ+iω, such that σ is the real part and iω is the imaginary part with $i^2 = -1$ and **I** is an n×n identity matrix. Equation (23) is the eigenvalue problem in Laplace transform form where the determinant of the term (S**I**−A) has to equal to be zero in order to have nontrivial solution.

$$|(S\mathbf{I} - A)| = 0 \tag{24}$$

Taking the system matrix and inserting it in equation (24), we have:

$$|(S\mathbf{I} - A)| = \left| \begin{pmatrix} S & -1 & 0 & 0 & 0 & 0 \\ \frac{K_1}{M_1} & \frac{S+B_1}{M_1} & \frac{-K_1}{M_1} & \frac{-B_1}{M_1} & 0 & 0 \\ 0 & 0 & S & -1 & 0 & 0 \\ \frac{-K_1}{M_2} & \frac{-B_1}{M_2} & \frac{(K_1+K_2)}{M_2} & \frac{S+(B_1+B_2)}{M_2} & \frac{-K_2}{M_2} & \frac{-B_2}{M_2} \\ 0 & 0 & 0 & 0 & S & -1 \\ 0 & 0 & \frac{-K_2}{M_3} & \frac{-B_2}{M_3} & \frac{K_2}{M_3} & \frac{S+B_2}{M_3} \end{pmatrix} \right| = 0 \tag{25}$$

For the three degree of freedom system matrix, taking the closed-form determinant is complicated so we used MATLAB’s ’eig’ function to solve the eigenvalue problem numerically. We consider 2 cases in this study which are:

Case 1: $M_1 = M_2 = M_3 = 1kg$, $K_1 = K_2 = 1N/m$ and proportional damping of $B_1 = B_2 = 0.1Ns/m$.

Case 2: $M_1 = 1000kg$, $M_2 = 750kg$, $M_3 = 500kg$, $K_1 = 1750N/m$, $K_2 = 3500N/m$ and non-proportional damping of $B_1 = 140Ns/m$, $B_2 = 70Ns/m$.

The resulting eigenvalues for each case are tabulated in Tables 1 and 2 respectively and plotted in complex plane as shown in Figures 4 and 5.

Table 1: Damped 3-degree-of-freedon system's eigenvalues, S_J with proportional damping B_1=B_2=0.1Ns/m

S_j	$\sigma_j + i\omega_j$
S_1	$-0.0000 + i0.0000$
S_2	$-0.0000 - i0.0000$
S_3	$-0.0500 + i0.9987$
S_4	$-0.0500 - i0.9987$
S_5	$-0.1500 + i1.7255$
S_6	$-0.1500 - i1.7255$

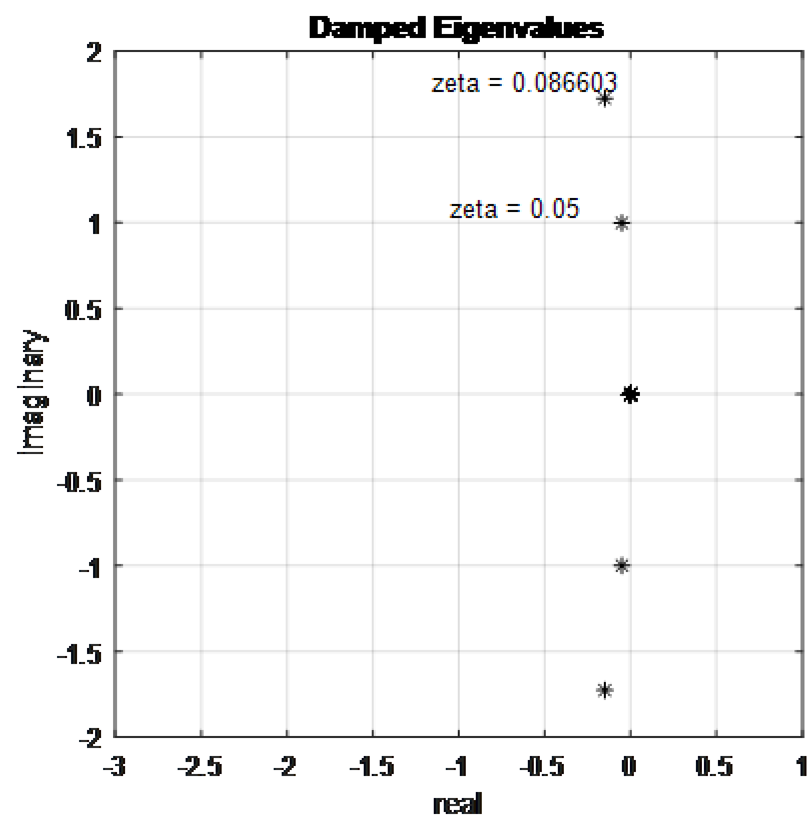

Figure 4: Plot of the damped eigenvalues with B1 = B2 = 0.1

Note that the two eigenvalues which correspond to each of the three modes are complex conjugate of each other and their real parts are all negatives.

Table 2: Damped 3-degree-of-freedon system's eigenvalues, S_J with non-proportional damping B_1=140Ns/m, B_2=70Ns/m

S_j	$\sigma_j + i\omega_j$
S_1	$-0.0000 + i0.0000$
S_2	$-0.0000 - i0.0000$
S_3	$-0.1036 + i1.6858$
S_4	$-0.1036 - i1.6858$
S_5	$-0.1764 + i3.5850$
S_6	$-0.1764 - i3.5850$

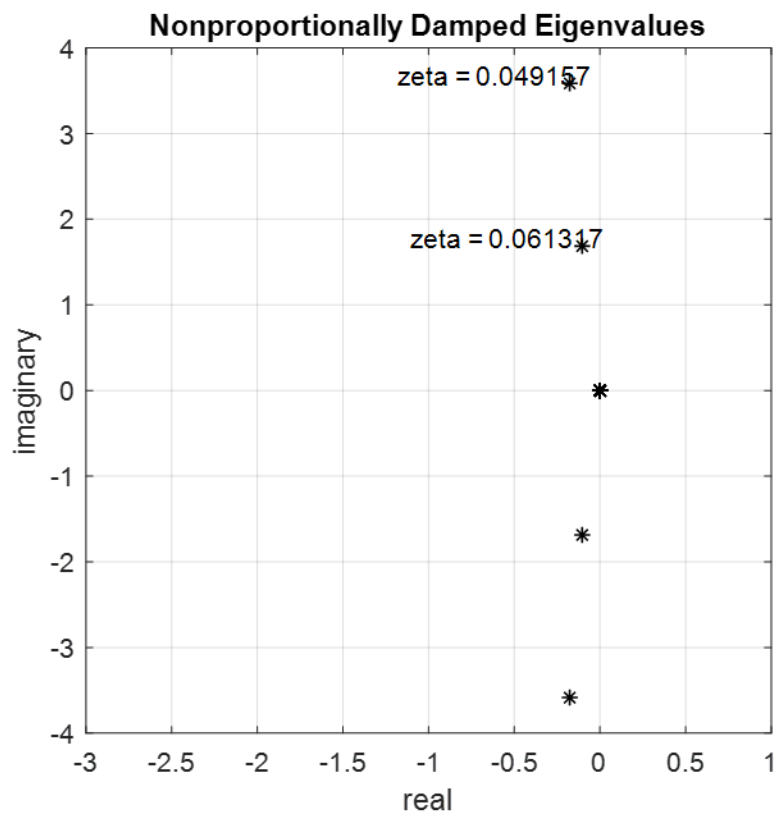

Figure 5: Plot of the damped eigenvalues with B1=140, B2=70

Note that the two eigenvalues which correspond to each of the three modes are complex conjugate of each other and their real parts are all negatives.

4.2 Controllability and Observability

A system is said to be controllable at time t_0 if it is possible by means of an unconstrained control vector to transfer the system from any initial state $X(t_0)$ to any other state in a finite interval of time.

Meanwhile, a system is observable at time t_0 if it is possible to determine $X(t_0)$ from the observation of the output over a finite interval of time [39]. The concept of controllability and observability introduced by Kalman in 1960 plays an important role in the design of control systems in state space. In fact, [40] reported that the conditions of controllability and observability may govern the existence of a complete solution to the control system design problem. As pointed out in [41], various important system properties, such as the existence of an optimal control under a criterion like H_2 and H_∞ norm, possibilities of stabilizing a plant and/or locating its poles to a desirable area, convergences of a state estimation procedure are closely related to the controllability and/or observability of the plant at hand.

4.2.1 Conditions for Controllability and Observability

Consider the state-space model:

$$\begin{aligned} \dot{X} &= AX + BU \\ Y &= CX + DU \end{aligned} \tag{26}$$

[31] proved the Kalman controllability and observability rank conditions and found out that the system given by the state space model above is completely state controllable if the vectors $\mathbf{B}, \mathbf{AB}, \mathbf{A}^2\mathbf{B}, \ldots\ldots, \mathbf{A}^{n-1}\mathbf{B}$ are linearly independent or the $n \times n$ matrix $[\mathbf{B} \vdots \mathbf{AB} \vdots \mathbf{A}^2\mathbf{B} \vdots \ldots\ldots \vdots \mathbf{A}^{n-1}\mathbf{B}]$ is of rank n. This is called the controllability matrix and is denoted by C_m

Similarly, the system described by the state space model is completely observable if the vectors $\mathbf{C}, \mathbf{AC}, \mathbf{A}^2\mathbf{C}, \ldots\ldots, \mathbf{A}^{n-1}\mathbf{C}$ are linearly independent or the $n \times n$ matrix $[\mathbf{C} \vdots \mathbf{AC} \vdots \mathbf{A}^2\mathbf{C} \vdots \ldots\ldots \vdots \mathbf{A}^{n-1}\mathbf{C}]$ is of rank n. This is called the observability matrix and is denoted by O_m.

4.2.2 Controllability and Observability Matrices of the System

MATLAB was used to calculate the controllability and observability matrices C_{m1}, O_{m1} and C_{m2}, O_{m2} respectively for the two numerical cases considered in subsection 4.1.2. The results are given below:

$$C_{m1} = \begin{bmatrix} 0 & 0 & 0 & 0.0100 & 0.1960 & 0.8813 \\ 0 & 0 & 0.0100 & 0.1960 & 0.8813 & -1.1484 \\ 0 & 0 & 0.1000 & 0.9700 & -0.5910 & -2.7327 \\ 0 & 0.1000 & 0.9700 & -0.5910 & -2.7327 & 2.5928 \\ 0 & 1 & -0.1000 & -0.9800 & 0.3950 & 1.8514 \\ 1 & -0.1000 & -0.9800 & 0.3950 & 1.8514 & -1.4444 \end{bmatrix} \tag{27}$$

$Rank = 6$

$$O_{m1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -0.1000 & 1 & 0.1000 & 0 & 0 \\ 1 & 0.1000 & -2 & -0.2000 & 1 & 0.1000 \\ 0 & 0 & 1 & 0.1000 & -1 & -0.1000 \\ 0.2000 & -0.9800 & -0.3000 & 0.9700 & 0.1000 & 0.0100 \\ -0.3000 & 0.9700 & 0.6000 & -1.9400 & -0.3000 & 0.9700 \\ 0.1000 & 0.0100 & -0.3000 & 0.9700 & 0.2000 & -0.9800 \\ 1.9500 & 0.3950 & -2.9100 & -0.5910 & 0.9600 & 0.1960 \\ -2.9100 & -0.5910 & 5.8200 & 1.1820 & -2.9100 & -0.5910 \\ 0.9600 & 0.1960 & -2.9100 & -0.5910 & 1.9500 & 0.3950 \\ -0.9860 & 1.8514 & 1.7730 & -2.7327 & -0.7870 & 0.8813 \\ 1.7730 & -2.7327 & -3.5460 & 5.4654 & 1.7730 & -2.7372 \\ -0.7870 & 0.8813 & 1.7730 & -2.7327 & -0.9860 & 1.8514 \end{bmatrix} \tag{28}$$

$Rank = 6$

$$C_{m2} = \begin{bmatrix} 0 & 0 & 0 & 0.0183 & 1.1331 & 10.5096 \\ 0 & 0 & 0.0183 & 1.1331 & 10.5096 & -23.8654 \\ 0 & 0 & 0.1307 & 6.4785 & -4.5503 & -88.9153 \\ 0 & 0.1307 & 6.4785 & -4.5503 & -88.9153 & 93.1168 \\ 0 & 1.4000 & -0.1960 & -9.7543 & 4.5593 & 112.3537 \\ 1.4000 & -0.1960 & -9.7543 & 4.5593 & 112.3537 & -91.9444 \end{bmatrix} \tag{29}$$

$$Rank = 6$$

$$O_{m2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1.7500 & -0.1400 & 1.7500 & 0.1400 & 0 & 0 \\ 2.3333 & 0.1867 & -7 & -0.2800 & 4.6667 & 0.0933 \\ 0 & 0 & 7 & 1.1400 & -7 & -1.1400 \\ 0.5717 & -1.7042 & -1.2250 & 1.6912 & 0.6533 & 0.0131 \\ -0.9800 & 2.2549 & 2.9400 & -6.8824 & -1.9600 & 4.6275 \\ 0.3267 & 0.0261 & -1.9600 & 6.9412 & 1.6333 & -6.9673 \\ 6.9286 & 1.1260 & -14.7294 & -1.9353 & 7.8008 & 0.8093 \\ -20.0051 & -2.5804 & 84.5152 & 5.8306 & -64.5101 & -3.2502 \\ 16.1504 & 1.6187 & -97.3140 & -4.8753 & 81.1636 & 3.2566 \\ -6.4861 & 6.4097 & 21.1830 & -13.9166 & -14.6969 & 7.5069 \\ 18.1205 & -18.5554 & -68.0814 & 82.0663 & 49.9609 & -63.5109 \\ -14.2084 & 15.0137 & 59.7561 & -95.2664 & -45.5477 & 80.2526 \end{bmatrix} \tag{30}$$

$$Rank = 6$$

4.3 Discussion and Interpretation of Results

The real parts of the eigenvalues (poles) of the system as tabulated in Tables 1 and 2 are all negatives. The plots of these eigenvalues presented in Figures 4 and 5 also show that the damped system has six complex eigenvalues, two at the origin and the remaining four all lying with their real parts on the left-half of the iω axis. This is a marginally stable system. The matrices C_{m1}, O_{m1} in equations (27) and (28) and the matrices C_{m2}, O_{m2} in equations (29) and (30) respectively represent controllability and observability matrices in each case. Both C_{m1} and C_{m2} have six (6) linearly independent vectors. This shows that C_{m1} and C_{m2} are of full ranks, suggesting that the system is controllable. Similarly, to observability, both O_{m1} and O_{m2} have six (6) linearly independent vectors. This shows that O_{m1} and O_{m2} are of full ranks, suggesting that it is observable.

5 CONCLUSION

In this paper, a translational mechanical system with three degrees of freedom was model and analyzed using state-space approach. The system was first presented in equivalent free body diagrams, then Newton's second law of motion was used to derive its equations of motion and subsequently presented in state-space form using the controllable canonical representation. The natural frequencies were calculated and finally the stability, controllability, and observability of the system were discussed. Interpretations of the eigenvalues, controllability, and observability matrices show that the system is stable, controllable, and observable which suggests that a state feedback control design for the system is possible.

The study identified one of the system's most important properties (eigenvalues) for various masses, stiffnesses, and dampers which will help in solving problems commonly encountered by mechanical engineers (for example, resonance problem). With the knowledge of these properties, the control engineer will be able to adjust the dynamics of the system in order to achieve stability and balance in the system structure. The study helps us to understand some complex general concepts about control systems, in explaining controllability, and observability which will help in further research.

However, in the control design process, the mathematical model used and the real plant are different. Ideally, outputs should follow the inputs but the presence of uncertainties, disturbance signals, and sensor noises makes the process vulnerable leading to errors in the system performance. Therefore, a study can be carried out on the system "steady state error" in order to determine how well, the system tracks the desired trajectory. Further more, in order to obtain the final control solution of our system, with the knowledge of the system controllability and observability, state feedback and observer designs should be carried out.

ACKNOWLEDGEMENT

The authors are grateful to AMCI for accepting this paper and also to the anonymous reviewers for the constructive feedback provided to improve the paper.

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