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On Eccentric Graphs of Unique Eccentric Point Graphs and Diameter Maximal Graphs

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Abstract: *The eccentricity $e(u)$ of a point or a node u of a graph G is the maximum distance of u to any other point of G. A point v is an eccentric point of u if the distance from u to v equals $e(u)$. A graph G is called an unique eccentric point (u.e.p) graph if each point of G has a unique eccentric point. On the other hand, the eccentric graph G_e of a graph G is defined as a graph having the same set of points as G with two points u and v being adjacent in G_e if and only if either u is an eccentric point of v in G or v is an eccentric point of u in G. In this paper we obtain some properties of eccentric graphs of certain u.e.p graphs and diameter maximal graphs.*

Keywords: *Eccentricity; Eccentric graph; Unique eccentric point graph; Diameter maximal graph.*

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1 Introduction

The concept of a graph, studied intensively in graph theory, a well-known branch of Discrete Mathematics, has served as a discrete mathematical model for several real-life situations. An undirected graph or simply a graph $G = (V, E)$, with a finite set V of points (also called vertices or nodes) and a finite set E of edges has also found many applications [1] in a variety of situations, due to the usefulness of graphs in modeling binary relations among objects. In particular the notion of distance between two points in a graph and allied notions have been well studied [2].

For notions and notations related to graphs, we refer to [1, 2, 3]. If a graph $G = (V, E)$, we also write V as $V(G)$ and E as $E(G)$, if we need to specify the graph G. Also throughout this paper, we consider only simple undirected graphs in the sense that for any two distinct points of G there is at the most one edge joining them and there is no edge joining a node with itself. Given a graph G, the distance $d_G(u, v)$ or $d(u, v)$ between any two points u and v in G is defined as the length of the shortest path between u and v. The eccentricity $e(v)$ of a point v in G is defined as $e(v) = max\{d(v, u)| \, u \in V(G)\}$. For two points u, v in G, the point u is an eccentric point of v if the distance from v to u is equal to $e(v)$. The set of all eccentric points of v in G is denoted by $E_G(v)$ or simply by $E(v)$, if G is understood. The set of all eccentric points of G is

$$EP(G) = \bigcup_{v \in V(G)} E(v).$$

A graph G is called a unique eccentric point ($u.e.p$) graph [4] if the number $|E(u)|$ of elements of $E(u)$ is one i.e. $|E(u)| = 1$, for every point $u \in V(G)$. The radius $r(G)$ and the diameter $diam(G)$ of a graph G are respectively defined as $r(G) = min\{e(u)| \, u \in V(G)\}$ and $diam(G) = max\{e(u)| \, u \in V(G)\}$. A point u is a peripheral point of G if $e(u) = diam(G)$. The set of all peripheral points of G is denoted by $P(G)$. A graph G is a self-centered graph if $r(G) = diam(G)$. A graph G is said to be a diameter maximal graph (also called an upper diameter critical graph), if $diam(G + e) < diam(G)$, for every $e \in E(\overline{G})$, where $\overline{G}$ is the complement of G.

The eccentric graph G_e [5] of a graph G, is a graph with the same set of points as that of G and two points u and v in G_e are adjacent if and only if either u is an eccentric point of v or v is an eccentric point of u in G. As an illustration, a graph G with six nodes labelled $1, 2, \cdots, 6$ and its eccentric graph G_e are shown in Fig. 1.

We note that we are considering here the notion of eccentric graph of a given graph as in [5]. On the other hand in [6], the concept of eccentric graph is considered in a different sense by defining a given graph G itself as an eccentric graph, if every point of G is an eccentric point of some other point of G. It is to be pointed out that this notion of an eccentric graph is different from the concept of eccentric graph of a given graph. Here we obtain certain properties of the eccentric graphs of $u.e.p$ graphs, in particular, of diameters two and three. We also obtain certain properties of the eccentric graphs of diameter maximal graphs.

We need also the following well known notions [2]. A graph G is complete if there is an edge between every pair of distinct points. A complete graph on n points is denoted by K_n. A double star (Fig. 2) is a graph obtained by joining m (≥ 1) pendant edges to an end point of K_2 and another n (≥ 1) pendant edges to the other end point. If $m = n = 1$, then the double star becomes a path on four points. In a graph $G = (V, E)$, a subset $S \subseteq V$ is called a dominating set if each point $u \in V - S$ has a neighbor in S. The domination number $\gamma(G)$

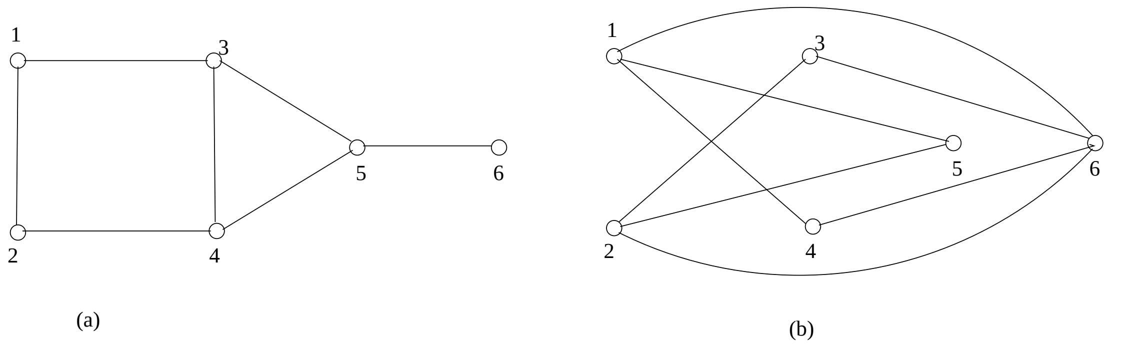

Figure 1: (a) A Graph G and (b) its eccentric graph G_e

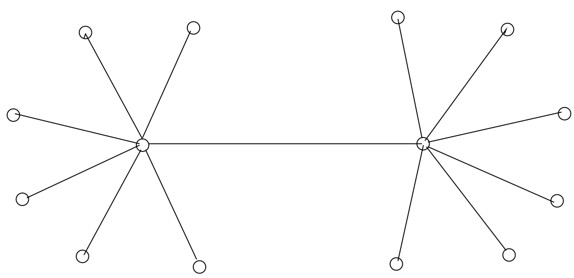

Figure 2: A double star

is the minimum size of a dominating set in G. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then the union of G_1 and G_2 is defined as the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The cartesian product graph $G_1 \times G_2$ has $V_1 \times V_2$ as its point set. Two points (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \times G_2$ if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. For three or more graphs $G_1, G_2, G_3, \cdots, G_n$, the sequential join $G_1 + G_2 + G_3 + \cdots + G_n$ is the graph $(G_1 + G_2) \cup (G_2 + G_3) \cup \cdots \cup (G_{n-1} + G_n)$.

We need the following known results in the sequel.

Lemma 1.1 *i*) [2] The eccentric graph of a self-centered *u.e.p* graph on $2n$ $(n \geq 1)$ points is a union of n copies of K_2.
ii) [4] For every *u.e.p* graph G, the number $|P(G)|$ of peripheral points is even.

Lemma 1.2 ([2], page 37) Let G be a diameter maximal graph.
i) If G is connected then it has a unique pair of eccentric peripheral points and for some $d-1$ positive integers $a_i, 2 \leq i \leq d$, G has the form $K_1 + K_{a_2} + K_{a_3} + \cdots + K_{a_d} + K_1$
ii) If G is disconnected, then $G = K_m \cup K_n$.
iii) If G has an odd diameter then G is a *u.e.p* graph.

Lemma 1.3 [4] If G is a *u.e.p* graph with diameter three, then G is either a self-centered graph or a diameter maximal graph.

Lemma 1.4 [7] Let G and H be simple connected graphs. Then $P(G \times H) = P(G) \times P(H)$.

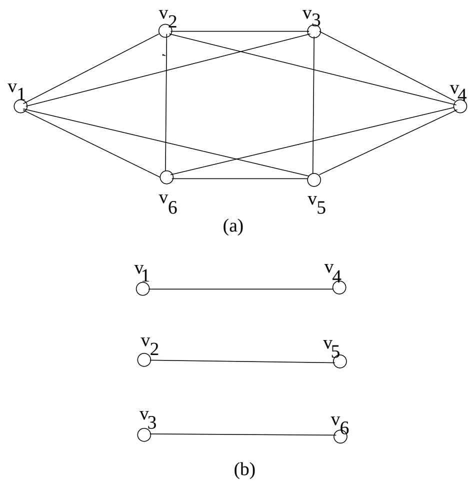

Figure 3: (a) A self-centered $u.e.p$ graph G and (b) its eccentric graph G_e

2 Eccentric graph of a $u.e.p$ Graph

In this section, we obtain certain properties of the eccentric graph of a given $u.e.p$ graph, in particular, of diameter two or three.

Theorem 2.1 If a graph G on $2n$ $(n \geq 1)$ points is a $u.e.p$ graph of diameter two, then the eccentric graph G_e is a union of n copies of K_2.

Proof. Let $G = (V, E)$ be a $u.e.p$ graph of diameter two. Then for u in $V(G)$, $e(u) = 1$ or $e(u) = 2$. If $e(u) = 1$ for some point u, then G can only be K_2 as G is a $u.e.p$ graph. But this is not possible as G has diameter two. Hence $e(u) = 2$, for all $u \in V(G)$. That is, G is self centered. Thus, it follows from Lemma 1.1(i), that G_e is a union of n copies of K_2.

Example 2.1 In the graph G in Fig. 3(a) every point has eccentricity $= 2$ so that $diam(G) = r(G) = 2$ and so the graph G is self-centered. The points v_i and v_{i+3} for $1 \leq i \leq 3$ are eccentric points of each other and thus each point has an unique eccentric point so that G is an $u.e.p$ graph. The eccentric graph G_e of G in Fig. 3(a) is shown in Fig. 3(b) and G_e is union of three copies of K_2.

Theorem 2.2 Let G be a non-self centered $u.e.p$ graph having the properties: i) $P(G) = EP(G)$, ii) $|P(G)| \geq 2$, and iii) property P : for every $u \in P(G)$, there exists at least one point $v \in V(G) - P(G)$ such that $E(v) = \{u\}$. Then the eccentric graph G_e is a union of double stars. In particular, G_e is a double star if $|P(G)| = 2$.

Proof. Let $G = (V, E)$ be a non-self centered $u.e.p$ graph having the properties i) to iii) stated in the Theorem. By Lemma 1.1(ii), $|P(G)|$ is even. Let $P(G) = \{u_o, u_1, u_2, \cdots, u_{2k-1}\}$, $k \geq 1$. Then we have $E(u_i) = \{u_j\}$ if and only if $E(u_j) = \{u_i\}$ for all $u_i, u_j \in P(G)$. Since G is a $u.e.p$ graph with $P(G) = EP(G)$, every point v in $V(G) = V$ (including the points of $P(G)$), has only one eccentric point u_i in $P(G)$; that is, for some $i, 0 \leq i \leq 2k-1$, $E(v) = \{u_i\}$. This implies that the eccentric graph G_e has the same set of points as G and every point of G_e is adjacent to only one point of G_e, which as a point of G is in $P(G)$. Note that the property iii) in the hypothesis of the Theorem ensures that if xy is an edge of G_e with x and y being points of $P(G)$, at least one point of G_e (which is a point of $V(G) - P(G)$) is adjacent to x and likewise for y. Thus G_e is a union of double stars. If

$|P(G)| = 2$, then $P(G) = \{u_0, u_1\}$, and hence, G_e is just a double star.

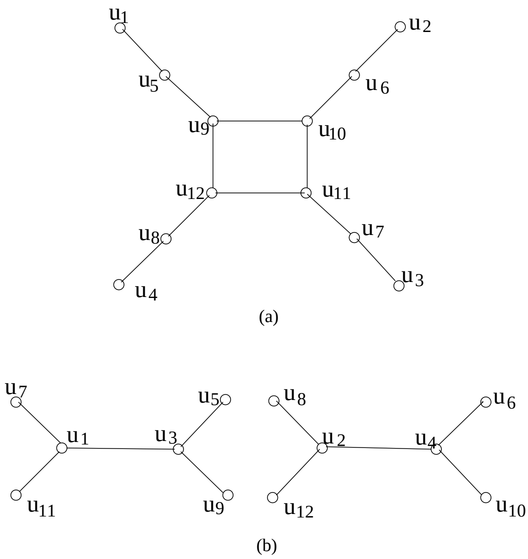

Figure 4: (a) A non self-centered *u.e.p* graph G and (b) its eccentric graph G_e

Example 2.2 A non self-centered *u.e.p* graph G is shown in Figure 4(a). The eccentricities of the vertices are as follows: $e(u_i) = 6$, for $i = 1, \cdots, 4$; $e(u_i) = 5$, for $i = 5, \cdots, 8$ and $e(u_i) = 4$, for $i = 9, \cdots, 12$. Hence $diam(G) = 6$; $radius(G) = 4$. The set of peripheral points is $P(G) = \{u_i | 1 \leq i \leq 4\}$ so that $|P(G)| \geq 2$. The points u_i and u_{i+2} for $i = 1, 2$ are eccentric points of each other. For $i = 1, 2$, the point u_i is an eccentric point of both u_{i+6} and u_{i+10}. For $i = 3, 4$, the point u_i is an eccentric point of both u_{i+2} and u_{i+6}. Hence the set of eccentric points of G is $EP(G) = \{u_i | 1 \leq i \leq 4\}$ so that $EP(G) = P(G)$. Also every peripheral point is an eccentric point of at least one non-peripheral point and hence G satisfies the hypotheses of Theorem 2.2. The eccentric graph G_e of the graph G in Fig. $4(a)$ is shown in Fig. $4(b)$ and it is union of two double stars.

Remark 2.1 We believe that the hypothesis $iii)$ of Theorem 2.2 can be relaxed but we have no proof of this. In other words, we conjecture that in a non self-centered *u.e.p* graph with two or more peripheral points such that these are the only eccentric points of the graph, every peripheral point is an eccentric point of at least one non-peripheral point.

Theorem 2.3 If G and H are two simple undirected graphs, then a point (u, v) in $V(G \times H)$ is an eccentric point of some point (x, y) in $V(G \times H)$ if and only if $u \in E_G(x)$ and $v \in E_H(y)$.

Proof. If (u, v) in $V(G \times H)$ is an eccentric point of some point (x, y) in $V(G \times H)$, then $d_{G\times H}((x, y), (u, v)) = e_{G\times H}((x, y))$. Since $d_{G\times H}((x, y), (u, v)) = d_G(x, u) + d_H(y, v)$ for all $x, u \in V(G)$ and $y, v \in V(H)$ and $e_{G\times H}((x, y)) = e_G(x)+e_H(y)$, we have $d_{G\times H}((x, y), (u, v))$ $= e_G(x) + e_H(y)$. Hence we obtain $d_G(x, u) + d_H(y, v) = e_G(x) + e_H(y)$. This implies that $d_G(x, u) = e_G(x)$ and $d_H(y, v) = e_H(y)$. For, if $d_G(x, u) < e_G(x)$, then $d_H(y, v) > e_H(y)$ which is not possible and the argument is similar for $d_H(y, v) < e_H(y)$. Hence $u \in E_G(x)$ and $v \in E_H(y)$

Conversely, suppose that $u \in E_G(x)$ and $v \in E_H(y)$ for $u, x \in V(G)$ and $v, y \in V(H)$. Then $d_{G\times H}((x, y), (u, v)) =d_G(x, u) + d_H(y, v) = e_G(x) + e_H(y) = e_{G\times H}((x, y))$. Therefore, (u, v)

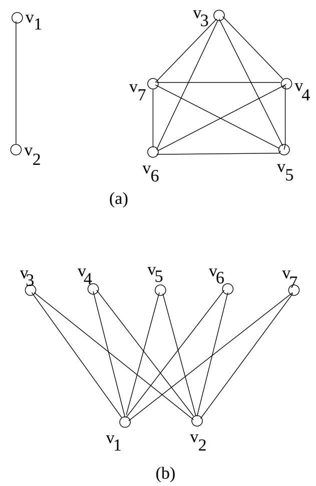

Figure 5: (a) A disconnected diameter maximal graph G and (b) its eccentric graph G_e

is an eccentric point of (x, y).

Corollary 2.1 If G and H are two simple undirected graphs, then $EP(G \times H) = EP(G) \times EP(H)$

Theorem 2.4 If G and H are non-self centered $u.e.p$ graphs with $P(G) = EP(G)$ and $P(H) = EP(H)$ and if both G and H satisfy property P in Theorem 2.2, then the eccentric graph of $G \times H$ is a union of double stars.

Proof. If G and H are non-self centered $u.e.p$ graphs with $P(G) = EP(G)$ and $P(H) = EP(H)$, then [4] $G \times H$ is a non-self centered $u.e.p$ graph. By Lemma 1.4 and Corollary 2.1, we have $P(G \times H) = EP(G \times H)$. Therefore, the result follows from Theorem 2.2, on noting that property P will be satisfied for $G \times H$, by Theorem 2.3.

3 Eccentric Graph of a Diameter Maximal Graph

In this section, we obtain certain properties of the eccentric graph of a diameter maximal graph.

Theorem 3.1 *i)* The eccentric graph G_e of a disconnected diameter maximal graph is a complete bipartite graph.
ii) The eccentric graph G_e of a connected diameter maximal graph with odd diameter is a double star.

Proof. Let G be a disconnected diameter maximal graph. Then by Lemma 1.2(ii) , $G = K_m \cup K_n$. Let the points of K_m be $u_i, 1 \leq i \leq m$ and the points of K_n be $v_i, 1 \leq i \leq n$. Now, clearly, the eccentric graph G_e of $K_m \cup K_n$ has the same points as that of $K_m \cup K_n$. Also, an edge of G_e joins u_i with v_j for every $i, j, 1 \leq i \leq m, 1 \leq j \leq n$ and hence G_e is a complete bipartite graph. (Fig. 5 illustrates this case.)

Let G be a connected diameter maximal graph with odd diameter d. Then by Lemma 1.2(i), for $d-1$ positive integers $a_i, 2 \leq i \leq d$, G has the form $G = K_1 = (\{u\}, \emptyset) + K_{a_2} + K_{a_3} + \cdots + K_{a_d} + K_1 = (\{v\}, \emptyset)$, where u, v are two points of G and $\emptyset$ denotes the empty set. This implies that G is a non-self centered graph. It follows from Lemma 1.2 (i) and (iii),

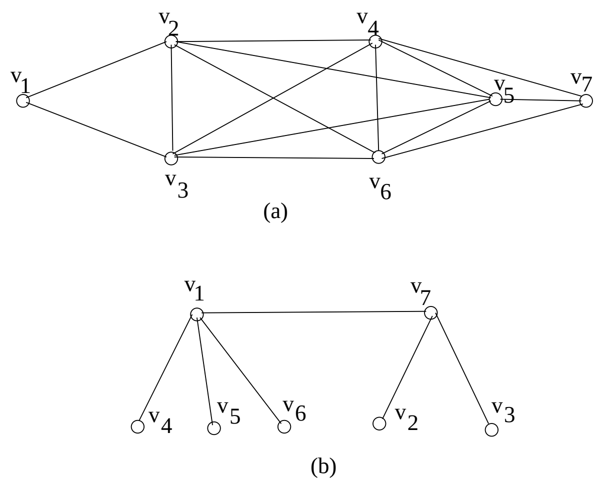

Figure 6: (a) A connected diameter maximal graph G with diameter 3 and (b) its eccentric graph G_e

that G is a non-self centered $u.e.p$ graph with $P(G) = EP(G) = \{u, v\}$. It is clear that G satisfies the hypothesis of Theorem 2.2. Hence, by Theorem 2.2, G_e is a double star. (Fig. 6 illustrates this case.)

Theorem 3.2 The eccentric graph of a $u.e.p$ graph of diameter three is either a union of copies of K_2 or a double star.

Proof. Let G be a $u.e.p$ graph of diameter three. Then by Lemma 1.3, G is either self centered or a diameter maximal graph.
If G is self centered then by Lemma 1.1(i) G_e is a union of copies of K_2.
If G is a diameter maximal graph then by Theorem 3.1, G_e is a double star.

Theorem 3.3 If G and H are connected diameter maximal graphs with odd diameters and if both G and H satisfy property P in Theorem 2.2, then the eccentric graph of $G \times H$ is a union of double stars.

Proof. If G and H are connected diameter maximal graphs with odd diameters, then [4] G and H are non-self centered $u.e.p$ graphs. By Lemma 1.2 (i), G and H, each has a unique pair of eccentric peripheral points so that $P(G) = EP(G)$ and $P(H) = EP(H)$ and $|P(G)| = 2 = |P(H)|$. It follows from Lemma 1.4 and Corollary 2.1 that $|P(G \times H)| = 4 = |EP(G \times H)|$. Consequently, $G \times H$ is a non self-centered $u.e.p$ graph with $P(G \times H) = EP(G \times H)$ and $|P(G \times H)| = 4$. Therefore, by Theorem 2.4 the eccentric graph of $G \times H$ is a union of two double stars.

We now obtain the domination number of a diameter maximal graph and its eccentric graph.

Theorem 3.4 The domination number of a diameter maximal graph with odd diameter d is $\lceil (d+1)/3 \rceil$.

Proof. Let G be a diameter maximal graph with odd diameter d. Then by Lemma 1.2(i), G has the form $G = G_o + G_1 + G_2 + \cdots + G_{d-1} + G_d$ where each $G_i = K_{a_i}, i = 1, 2, 3, 4, \cdots, d-1$ and $G_o = K_1 = G_d$. Now, let us consider some point u in G_1. It must be adjacent to all the points of G_o, G_1 and G_2. So any point of G_1 dominates every point of G_o, G_1 and G_2. Similarly, any point of G_4 dominates every point of G_3, G_4 and G_5. It follows that for every three consecutive cliques, any point of the middle clique is a dominating point. Therefore $\gamma(G) = \lceil (d+1)/3 \rceil$ since there are $d+1$ cliques.

Theorem 3.5 Let G be a diameter maximal graph with odd diameter. Then the domination number of the eccentric graph G_e is $\gamma(G_e) = 2$.

Proof. Let G be a diameter maximal graph with odd diameter. Then by Theorem 3.1, G_e is a double star with two central points u and v. Now the set $S = \{u, v\}$ is a minimal dominating set of G_e. Thus $\gamma(G_e) = 2$.

Remark 3.1 We note from Theorems 3.4 and 3.5 that the domination number of a given diameter maximal graph depends on its odd diameter whereas this number is a constant, namely, two for the corresponding eccentric graph.

Corollary 3.1 The domination number of the eccentric graph of $P_{2n}, n > 1$ is two.

Proof. The path $P_{2n}, n > 1$, is clearly a diameter maximal graph with odd diameter. Hence by Theorem 3.5, $\gamma((P_{2n})_e) = 2$.

4 Conclusions

In this paper we have obtained certain properties of eccentric graphs of *u.e.p* graphs, in particular of diameters two and three. We have also obtained properties of the eccentric graphs of diameter maximal graphs. A main problem that remains to be explored is to determine in general, classes of graphs for which eccentric graphs are *u.e.p* graphs. We mention that the complete graph K_2 and the path P_4 on four points satisfy this property. Also, the concept of eccentric connectivity index [8, 9] can also be investigated for graphs considered here. It will be of interest to examine properties of eccentric graphs of other classes of graphs.

Acknowledgments

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