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# Existence of asymptotically almost periodic solutions of integrodifferential equations

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**Abstract:** *In this paper, a class of semilinear integrodifferential equations of the form* $u''(t) + \alpha u'''(t) = \beta Au(t) + \gamma Au'(t) + f(t, u(t)) + \int_0^t g(t, s, u(s))ds, \quad t, s \geq 0$, *satisfying* $\alpha\beta < \gamma$ *with prescribed initial conditions are studied. Using certain strongly continuous families in operator theory and fixed point theory, we have established some sufficient conditions for the existence and uniqueness of an asymptotically almost periodic solutions.*

**Keywords:** *Existence, asymptotically almost periodic, regularized families of bounded operators, Fixed point theory, Strongly continuous families.*

**PACS:** *PACS1, PACS2, PACS3*

## 1 INTRODUCTION

The study of the asymptotic behavior of solutions of a differential equation is one of the most interesting themes of the qualitative theory of differential equations and for this reason has attracted interest of many researchers over the years. Bose and Gorain [3] studied a more realistic model of vibrations of elastic structure in which the stress is not simply proportional to the strain. As a result they shown that the dynamics of vibrations of elastic structures are governed by the following third order differential equation

$$\alpha u'''(t) + u''(t) - \beta \Delta u(t) - \gamma \Delta u'(t) = 0, \quad t \geq 0 \tag{1}$$

with suitable boundary and initial conditions. Several authors [4, 7, 8, 9] have discussed the boundary stabilization and obtained the explicit exponential energy decay rate for the solution of (1) subject to mixed boundary conditions. Andrade and Lizama [1] studied the existence of asymptotically almost periodic solutions for damped wave equations and the

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results have significance in the study of vibrations of flexible structures possessing internal material damping. The purpose of this paper is to prove the existence of asymptotically almost periodic mild solutions for an abstract semilinear equation of the form

$$u''(t) + \alpha u'''(t) = \beta Au(t) + \gamma Au'(t) + f(t, u(t)) + \int_0^t g(t, s, u(s))ds, \quad t, s \geq 0 \tag{2}$$

with appropriate initial conditions and where $\alpha, \beta, \gamma$ are positive constants satisfying $\alpha\beta < \gamma$. A surprising fact is that in order to get asymptotic behavior, some initial conditions should be forced to be zero. This leads to an unexpected property that is not present in the study of the same qualitative property for the Cauchy problem of order less that 3, see [2].

## 2 PRELIMINARIES

Let $\alpha, \beta, \gamma \in \mathcal{R}, \alpha \neq 0$ be given. We denote

$$k(t) = \frac{1}{\alpha}\int_0^t (t-s)e^{-\frac{s}{\alpha}}ds = -\alpha + t + \alpha e^{-\frac{t}{\alpha}}, \ t \in \mathcal{R}_+.$$

and

$$a(t) = \beta k(t) + \frac{\gamma}{\alpha}\int_0^t e^{-\frac{s}{\alpha}}ds = -(\alpha\beta - \gamma) + \beta t + (\alpha\beta - \gamma)e^{-\frac{t}{\alpha}}, \quad t \in \mathcal{R}_+.$$

In order to give a consistent definition of mild solution for equation (2) based on an operator theoretical approach, we introduce the following definition [11].

**Definition 2.1.** Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space $X$. We call A the generator of an $(\alpha, \beta, \gamma)$-regularized family $\{R(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ if the following conditions are satisfied:
(R1) $R(t)$ is strongly continuous on $\mathcal{R}_+$ and $R(0) = 0$;
(R2) $R(t)D(A) \subset D(A)$ and $AR(t)x = R(t)Ax$ for all $x \in D(A)$, $t \geq 0$;
(R3) The following equation holds:

$$R(t)x = k(t)x + \int_0^t a(t-s)R(s)Ax \ ds$$

for all $x \in D(A), t \geq 0$. In this case, $R(t)$ is called the $(\alpha, \beta, \gamma)$-regularized family generated by A.

**Proposition 2.2.** Let $R(t)$be an $(\alpha, \beta, \gamma)$-regularized family on X with generator A. Then the following holds:

(a) For all $x \in D(A)$ we have $R(.)x \in C^2(\mathcal{R}_+; X)$

(b) Let $x \in X$ and $t \geq 0$. Then $\int_0^t a(t-s)R(s)xds \in D(A)$ and

$$R(t)x = k(t)x + A\int_0^t a(t-s)R(s)xds.$$

Results on perturbation, approximation, asymptotic behavior, representation as well as ergodic type theorems for $(\alpha, \beta, \gamma)$- regularized families can also be deduced from the more general context of $(a, k)$ - regularized families([12 - 16]).

**Definition 2.3 [17]** A function $f : \mathcal{R} \to X$ is called almost periodic if $f$ is continuous, and for each $\epsilon > 0$ there exist $l(\epsilon) > 0$ such that for every interval of length $l(\epsilon)$ contains a number $\tau$ with the property that $||f(t+\tau) - f(t)|| \leq \epsilon$ for each $t \in R$. The number $\tau$ above is called an $\epsilon$ - translation number for $f$, and the collection of such functions will be denoted $AP(X)$.

**Definition 2.4.** A function $f : \mathcal{R} \times Y \to X$ is called almost periodic if $f$ is continuous, and for each $\epsilon > 0$ and any compact $K \subset Y$there exist $l(\epsilon) > 0$ such that every interval of length $l(\epsilon)$ contains a number $\tau$ with the property that $||f(t+\tau, x) - f(t, x)|| \leq \epsilon$ for all $t \in \mathcal{R}, x \in K$, and the collection of such functions will be denoted by $AP(\mathcal{R} \times Y, X)$.

**Lemma 2.5 [18]** Let $f \in AP(\mathcal{R} \times Y, X)$ and $h \in AP(Y)$ then the function $f(., h(.)) \in AP(X)$.

Let $C_0(\mathcal{R}_+, X)$ be the subspace of $BC(\mathcal{R}_+, X)$ such that $lim_{t\to\infty}||x(t)|| = 0$ and $C_0(\mathcal{R}_+ \times Y, X)$ denotes the space of all continuous function $h : \mathcal{R}_+ \times Y \to X$ such that $lim_{t\to\infty} h(t, x) = 0$ uniformly for $x$ in a compact subset of $Y$.

**Definition 2.6.** A continuous function $f : \mathcal{R}_+ \to X$(resp., $\mathcal{R}_+ \times Y \to X$) is called asymptotically almost periodic if it admits a decomposition $f = g + \phi$, where $g \in AP(X)$ (resp., $g \in AP(\mathcal{R} \times Y, X)$) and $\phi \in C_0(\mathcal{R}_+, X)$ (resp., $\phi \in C_0(\mathcal{R}_+ \times Y, X)$). Denote by $AAP(X)$ (resp., $AAP(\mathcal{R}_+ \times Y, X)$) the set of all such functions. We observe that $AAP(X)$ is a Banach space with sup norm.

**Lemma 2.7 [1]** Let $X$ and $Y$ two Banach spaces. Suppose that $f \in AAP(\mathcal{R}_+ \times Y; X)$ is uniformly continuous on any bounded subset $K \subset Y$, uniformly for $t \geq 0$. Then, $u \in AAP(Y)$ implies $f(., u(.)) \in AAP(X)$.

Let $h : \mathcal{R}_+ \to R$ be a continuous function such that $h(t) \geq 1$ for all $t \in \mathcal{R}_+$,and $h(t) \to \infty$ as $t \to \infty$. we consider the space

$$C_h(Z) = \{u \in C(\mathcal{R}_+, Z) : \lim_{t\to\infty} \frac{u(t)}{h(t)} = 0\}$$

endowed with norm $||u||_h = sup_{t\geq 0} \frac{||u(t)||}{h(t)}$.

**Lemma 2.8.[5]** A subset $K \subseteq C_h(X)$ is a relatively compact set if it verifies the following conditions:
(c-1) The set $\{K_b = u|_{[0,b]} : u \in K\}$ is relatively compact in $C([0, b]; X)$ for all $b \geq 0$.
(c-2) $lim_{t\to\infty} \frac{||u(t)||}{h(t)} = 0$ uniformly for all $u \in K$.

## 3 EXISTENCE OF SOLUTIONS

Let $\alpha, \beta, \gamma \in (0, \infty)$.Consider the linear equation

$$u''(t) + \alpha u'''(t) = \beta Au(t) + \gamma Au'(t) + f(t), \tag{3}$$

with initial condition $u(0) = x, u'(0) = y, u''(0) = z$,where A is the generator of a $(\alpha, \beta, \gamma)$-regularized family R(t).By a solution of (3.1) we understand a function $u \in C(\mathcal{R}_+; D(A)) \cap C^3(\mathcal{R}_+; X)$such that $u' \in C(\mathcal{R}_+; D(A))$.

The following result gives a complete description of the solutions for equation(3) in terms of $(\alpha, \beta, \gamma)$-regularized families. It corresponds to an extension of the standard variation of parameters formula for the second order Cauchy problem.

**Proposition 3.1.** Let R(t) be an $(\alpha,\beta,\gamma)$-regularized family on X with generator A. If $f \in L^1_{loc}(\mathcal{R}_+, D(A^2)), x \in D(A^3), y \in D(A^2)$ and $z \in D(A^2)$then $u(t)$given by

$$\begin{aligned} u(t) &= \alpha R''(t)x + R'(t)x - \gamma AR(t)x + \alpha R'(t)y + R(t)y + \alpha R(t)z \\ &+ \int_0^t R(t-s)f(s)ds, t \geq 0, \end{aligned}$$

is a solution of (3)

**Proof.** For all $i = 1, \ldots, 5$,we can write $R^{(i)}(t)\omega$ as follows:

$$R'(t)\omega = (1 - e^{-\frac{t}{\alpha}})\omega + \int_0^t \left[\beta + (\frac{\gamma}{\alpha} - \beta)e^{-\frac{1}{\alpha}(t-s)}\right] R(s)A\omega ds, \quad \omega \in D(A),$$

and we conclude from Proposition 2.3(b) that $R'(t)\omega \in D(A)$ for $\omega \in D(A)$.

$$R''(t)\omega = \frac{1}{\alpha}e^{-\frac{t}{\alpha}}\omega + \frac{\gamma}{\alpha}R(t)A\omega + \int_0^t (\frac{\beta}{\alpha} - \frac{\gamma}{\alpha^2})e^{-\frac{1}{\alpha}(t-s)}R(s)A\omega ds, \omega \in D(A),$$

and hence by (R2), we have $R''(t)\omega \in D(A)$ for $\omega \in D(A^2)$.

$$\begin{aligned} R'''(t)\omega &= - \frac{1}{\alpha^2}e^{-\frac{t}{\alpha}}\omega + \frac{\gamma}{\alpha}R'(t)A\omega + \frac{\beta}{\alpha}R(t)A\omega - \frac{\gamma}{\alpha^2}R(t)A\omega \\ &+ \int_0^t (\frac{\gamma}{\alpha^3} - \frac{\beta}{\alpha^2})e^{-\frac{1}{\alpha}(t-s)}R(s)A\omega ds, \ \omega \in D(A^2), \end{aligned}$$

and $R'''(t)\omega \in D(A)$ for $\omega \in D(A^2)$.

$$\begin{aligned} R^{(iv)}(t)\omega &= \frac{1}{\alpha^3}e^{-\frac{t}{\alpha}}\omega + \frac{\gamma}{\alpha}R''(t)A\omega + \frac{\beta}{\alpha}R'(t)A\omega - \frac{\gamma}{\alpha^2}R'(t)A\omega + \frac{\gamma}{\alpha^3}R(t)A\omega \\ &- \frac{\beta}{\alpha^2}R(t)A\omega + \int_0^t (\frac{\beta}{\alpha^3} - \frac{\gamma}{\alpha^4})e^{-\frac{1}{\alpha}(t-s)}R(s)A\omega ds, \omega \in D(A^2), \end{aligned}$$

and $R^{(iv)}(t)\omega \in D(A)$ for $\omega \in D(A^3)$.

$$\begin{aligned} R^{(v)}(t)\omega &= \frac{1}{\alpha^4}e^{-\frac{t}{\alpha}}\omega + \frac{\gamma}{\alpha}R'''(t)A\omega + \frac{\beta}{\alpha}R''(t)A\omega - \frac{\gamma}{\alpha^2}R''(t)A\omega + \frac{\gamma}{\alpha^3}R'(t)A\omega \\ &- \frac{\beta}{\alpha^2}R'(t)A\omega + \frac{\beta}{\alpha^3}R(t)A\omega - \frac{\gamma}{\alpha^4}R(t)A\omega \\ &+ \int_0^t (\frac{\gamma}{\alpha^5} - \frac{\beta}{\alpha^4})e^{-\frac{1}{\alpha}(t-s)}R(s)A\omega ds, \ \omega \ \in D(A^3), \end{aligned}$$

and $R^{(v)}(t)\omega \in D(A)$ for $\omega \in D(A^4)$. From the above, we deduce that if $x \in D(A^3), y \in D(A^2)$ *and* $z \in D(A^2)$ then $R(.)x \in C^5(\mathcal{R}_+, X)$, $R(.)y \in C^4(\mathcal{R}_+, X)$ *and* $R(.)z \in C^3(\mathcal{R}_+, X)$. Since $f \in L^1_{loc}(\mathcal{R}_+, D(A^2))$ we have

$$u'(t) = \alpha R'''(t)x + R''(t)x - \gamma R'(t)Ax + \alpha R''(t)y + R'(t)y + \alpha R'(t)z + \int_0^t R'(t-s)f(s)ds,$$

and hence $u'(t) \in D(A)$ for $x, y \in D(A^2)$ and $z \in D(A)$.

$$\begin{aligned} u''(t) = \alpha R^{(iv)}(t)x + R'''(t)x - \gamma R''(t)Ax + \alpha R'''(t)y + R''(t)y \\ + \alpha R''(t)z + \int_0^t R''(t-s)f(s)ds, \end{aligned}$$

and hence $u''(t) \in D(A)$ for $x \in D(A^3)$ and $y, z \in D(A^2)$.

$$u'''(t) = \alpha R^{(v)}(t)x + R^{(iv)}(t)x - \gamma R'''(t)Ax + \alpha R^{(iv)}(t)y + R'''(t)y$$
$$+\alpha R'''(t)z + \frac{1}{\alpha}f(t) + \int_0^t R'''(t-s)f(s)ds,$$

and hence $u \in C^3(\mathcal{R}_+, X)$. Using the fact that A is closed and the expressions for $R^{(i)}(t), i = 1, \ldots, 5$

$$\left[\alpha R^{(v)}(t) + R^{(iv)}(t)\right] x = [\gamma A R'''(t) + \beta A R''(t)]\, x,$$

$$\left[\alpha R^{(iv)}(t) + R'''(t)\right] y = [\gamma A R''(t) + \beta A R'(t)]\, y,$$

$$[\alpha R'''(t) + R''(t)]\, z = [\gamma A R'(t) + \beta A R(t)]\, z,$$

$$[\alpha R'''(t-s) + R''(t-s)]\, f(s) = [\gamma A R'(t-s) + \beta A R(t-s)]\, f(s), and$$

we conclude that $u(t)$ satisfy (3) with initial conditions $u(0) = x, u'(0) = y$ and $u''(0) = z$.

**Remark 3.2.** Observe that in the border case $\alpha = 0$ with $\gamma = 0$, the above theorem recover the variation of parameters formula for the second order Cauchy problem $u''(t) = Au(t) + f(t)$, so that the $(0, \beta, 0)$-regularized family $R(t)$ corresponds in this case to the sine family generated by A and $R'$(t) is the respective cosine family.

**Theorem 3.3 [6]** Let -B be a positive self adjoint operator on a Hilbert space H such that $\alpha\beta \le \gamma$. Then B is the generator of a bounded $(\alpha, \beta, \gamma)$-regularized family on H.

## 4 ASYMPTOTICALLY ALMOST PERIODIC SOLUTIONS

In this section we study the existence of asymptotically almost periodic solutions for the equation

$$u''(t) + \alpha u'''(t) = \beta A u(t) + \gamma A u'(t) + f(t),$$

with initial conditions $u(0) = x, u'(0) = y, u''(0) = z$, where A is the generator of a $(\alpha, \beta, \gamma)$-regularized family R(t). Assume that R(t) is differentiable. We introduce the following assumption.

**(A)** There exist constants $M > 0$ and $\omega > 0$ such that

$$||R'(t)|| + ||R(t)|| \le Me^{-\omega t}, t \ge 0.$$

Then we say that R(t) and $R'(t)$ are exponentially stable. The following result on regularity of the convolution under asymptotically almost periodic functions is one of the keys to obtain our results.

**Lemma 4.1.** Let $R(t)$ be an exponentially stable $(\alpha, \beta, \gamma)$-regularized family on X with generator A. If $f \in AAP(X)$then the function

$$F(t) = \int_0^t R(t-s)f(s)ds$$

belongs to AAP(X).

**Proof.** If $f = g + h$ with $g \in AP(X)$ and $h \in C_0(\mathcal{R}_+, X)$ then we have that $F(t) = G(t) + H(t)$, where $G(t) := \int_{-\infty}^{t} R(t-s)g(s)ds$ and

$H(t) := \int_0^t R(t-s)h(s)ds - \int_{-\infty}^{0} R(t-s)g(s)ds$

For $\epsilon > 0$,we take $l(\epsilon)$ involved in Definition 2.3, then for every interval of length $l(\epsilon)$ contains a number $\tau$ such that $||g(t+\tau) - g(t)|| \leq \epsilon$ for each $t \in \mathcal{R}$. The estimate

$$\begin{aligned} ||G(t+\tau) - G(t)|| &\leq \int_0^\infty ||R(s)||||g(t-s+\tau) - g(t-s)||ds \\ &\leq (M\int_0^\infty e^{-\omega s}ds)\epsilon. \end{aligned}$$

is responsible for the fact that $G \in AP(X)$. We claim that $||H(t)|| \to 0$ as $t \to \infty$. In fact, for each $\epsilon > 0$ there exists a $T > 0$ such that $||h(s)|| \leq \epsilon$ for all $s > T$. Then for all $t > 2T$ we deduce

$$\begin{aligned} ||H(t)|| &\leq \int_0^{\frac{t}{2}} Me^{-\omega(t-s)}||h(s)||ds + \int_{\frac{t}{2}}^{t} Me^{-\omega(t-s)}||h(s)||ds + \int_t^\infty Me^{-\omega s}||g(t-s)||ds \\ &\leq M(||h||_\infty + ||g||_\infty)\int_t^\infty e^{-\omega s}ds + \epsilon M\int_0^\infty e^{-\omega s}ds. \end{aligned}$$

Therefore, $lim_{t\to\infty}H(t) = 0$, that is, $H \in C_0(\mathcal{R}_+, X)$. This completes the proof.

Consider the initial value problem

$$u''(t) + \alpha u'''(t) = \beta Au(t) + \gamma Au'(t) + f(t, u(t)) + \int_0^t g(t, s, u(s))ds, \quad t, s \geq 0 \tag{4}$$

$$u(0) = 0, \quad u'(0) = y, \quad u''(0) = z,$$

where $\alpha, \beta, \gamma \in (0, \infty)$, $A$ is the generator of a $(\alpha, \beta, \gamma)$-regularized family $R(t)$ and $f : \mathcal{R}_+ \times X \to X$, $g : \mathcal{R}_+ \times \mathcal{R}_+ \times X \to X$ is a suitable function.

**Definition 4.2.** $R(t)$ be an $(\alpha, \beta, \gamma)$-generalized family on $X$ with generator $A$ is a continuous function $u : \mathcal{R}_+ \to X$ satisfying the integral equation

$$\begin{aligned} u(t) &= \alpha R'(t)y + R(t)y + \alpha R(t)z + \int_0^t R(t-s)f(s, u(s))ds \\ &\quad + \int_0^t R(t-s)\int_0^s g(s, \tau, u(\tau))d\tau ds, \quad s, \tau \geq 0 \end{aligned}$$

where $y, z \in X$ is called a mild solution to the equation (4).

Initially we study conditions to existence and uniqueness of a mild solution for (4) when the function $f, g$ are Lipschitz continuous.

**Theorem 4.3.** Let $R(t)$ be an $(\alpha, \beta, \gamma)$ regularized family on $X$ with generator $A$ that satisfies assumption (A). Let $f \in AAP(\mathcal{R}_+ \times X, X)$ and $g \in AAP(\mathcal{R}_+ \times \mathcal{R}_+ \times X, X)$and suppose that there exist an integrable bounded function $L : \mathcal{R}_+ \to \mathcal{R}_+$ such that

$$||f(t, x) - f(t, y)|| \leq L(t)||x - y||, \quad \forall x, y \in X, \quad t > 0 \tag{5}$$

and integrable bounded function $K : \mathcal{R}_+ \times \mathcal{R}_+ \to \mathcal{R}_+$ such that

$$||g(t,s,x) - g(t,s,y)|| \leq K(t,s)||x - y||, \quad \forall x, y \in X, \quad s, t > 0. \tag{6}$$

Then equation (4) has a unique asymptotically almost periodic mild solution.

**Proof:** Define the operator $\Lambda$ on the space AAP(X) by

$$\begin{aligned}\Lambda u(t) &= \alpha R'(t)y + R(t)y + \alpha R(t)z + \int_0^t R(t-s)f(s,u(s))ds \\ &+ \int_0^t R(t-s)\int_0^s g(s,\tau,u(\tau))d\tau ds, \quad s,\tau \geq 0. \end{aligned}\tag{7}$$

We show that $\Lambda u \in AAP(X)$. We observe that since $t \to \infty$ we have $||\alpha R'(t)y|| \leq \alpha||y||Me^{-\omega t} \to 0$, $||R(t)y|| \leq ||y||Me^{-\omega t} \to 0$, $||\alpha R'(t)z|| \leq \alpha||z||Me^{-\omega t} \to 0$, then $R(\cdot)y \in AAP(X)$, $\alpha R'(\cdot)z \in AAP(X)$ and $\alpha R'(\cdot)y \in AAP(X)$. It follows from Lemma 2.7 that the functions $s \to f(s,u(s))$ is AAP(X) and $(t,s) \to g(t,s,u(s))$ *is* $AAP(X)$. Then by Lemma 4.1

$$\int_0^t R(t-s)f(s,u(s))ds \in AAP(X)$$

and

$$\int_0^t R(t-s)\int_0^s g((s,\tau,u(\tau))d\tau)ds \in AAP(X).$$

Further, for $u_1, u_2 \in AAP(X)$, we have

$$\begin{aligned}&||\Lambda u_1(t) - \Lambda u_2(t)|| \\ &\quad\leq M\int_0^t e^{-\omega(t-s)}L(s)ds||u_1-u_2||_\infty + M\int_0^t e^{-\omega(t-s)}\int_0^s g(s,\tau,u(\tau)d\tau)ds||u_1-u_2||_\infty \\ &\quad\leq M\int_0^t L(s)ds||u_1-u_2||_\infty + M\int_0^t\int_0^s g(s,\tau,u(\tau)d\tau)ds||u_1-u_2||_\infty \\ &\quad\leq M\left[||L||_1 + ||K||_1\right]||u_1-u_2||_\infty\end{aligned}$$

$$\begin{aligned}&||\Lambda^2 u_1(t) - \Lambda^2 u_2(t)|| \\ &\leq \; M^2\left(\int_0^t L(s)(\int_0^s L(\tau)d\tau)ds\right)||u_1-u_2||_\infty \\ &\quad +M^2\int_0^t\left(\int_0^s K(s,\tau)\left(\int_0^\tau K(\tau,\xi)d\xi\right)d\tau\right)ds||u_1-u_2||_\infty \\ &\leq \; \frac{M^2}{2}(\int_0^t L(\tau)d\tau)^2||u_1-u_2||_\infty + \frac{M^2}{2}(\int_0^t\int_0^s K(s,\tau)d\tau ds)^2||u_1-u_2||_\infty \\ &\leq \; M^2\frac{(||L||_1+||K||_1)^2}{2}||u_1-u_2||_\infty\end{aligned}$$

By Mathematical induction, we get the following estimate

$$||\Lambda^n u_1(t) - \Lambda^n u_2(t)|| \leq M^n\frac{(||L||_1+||K||_1)^n}{n!}||u_1-u_2||_\infty.$$

Since $M^n\frac{(||L||_1+||K||_1)^n}{n!} < 1$, for n sufficiently large, by the fixed point iteration method $\Lambda$ has a unique fixed point $u \in AAP(X)$.

**Theorem 4.4.** Let R(t) be an $(\alpha, \beta, \gamma)$-regularized family on X with generator A that satisfies assumption (A). Let $f \in AAP(\mathcal{R}_+ \times X, X)$ and $g \in AAP(\mathcal{R}_+ \times \mathcal{R}_+ \times X, X)$ and suppose that f and g satisfies Lipschitz condition (5)and(6), L and K a bounded continuous function. Let $\beta(t) = \int_0^t e^{-\omega(t-s)} L(s)ds$ and $\gamma(t) = \int_0^t e^{-\omega(t-s)} \left( \int_0^s g(s, \tau, u(\tau))d\tau \right) ds$ if there are constants $k_1, k_2 < 1$, such that $M\beta(t) \leq k_1 < 1, M\gamma(t) \leq k_2 < 1$ where $M > 0$ is the constant given in assumption ED, then equation (4) has a unique mild solution $u \in AAP(X)$

**Proof:** Define the operator $\Lambda : AAP(X) \to AAP(X)$ by expression (7). We prove that $\Lambda$ is a k-contraction. In fact, given $u, v \in AAP(X)$ we have that

$$\begin{aligned}
&||\Lambda u(t) - \Lambda v(t)|| \\
&\quad \leq M \int_0^t e^{-\omega(t-s)} L(s)||u(s) - v(s)||ds + M \int_0^t e^{-\omega(t-s)} \int_0^s g(s, \tau, u(\tau)d\tau)||u(s) - v(s)||ds \\
&\quad \leq M\beta(t)||u - v||_\infty + M\gamma(t)||u - v||_\infty \\
&\quad \leq k_1||u - v||_\infty + k_2||u - v||_\infty \\
&\quad \leq (k_1 + k_2)||u - v||_\infty \\
&\quad \leq k||u - v||_\infty
\end{aligned}$$

Hence, by Banach's fixed point theorem, $\Lambda$ has a unique fixed point $u \in AAP(X)$.

**Corollary 4.5.** Ler R(t) be an $(\alpha, \beta, \gamma)$-regularized family on X with generator A that satisfies assumption (A). Let $f \in AAP(\mathcal{R}_+ \times X, X)$ and $g \in AAP(\mathcal{R}_+ \times \mathcal{R}_+ \times X, X)$ and suppose that f and g satisfies Lipschitz condition

$$\begin{aligned}
||f(t, x) - f(t, y)|| &\leq k_1||x - y||, \\
||g(t, s, x) - g(t, s, y)|| &\leq k_2||x - y||, \quad \forall x, y \in X, \quad s, t > 0.
\end{aligned}$$

If $\frac{Mk_1}{-\omega} + \frac{Mk_2}{\omega^2} < 1$ where M and $\omega$ are constant given in the assumption (A) then equation(4) has a unique mild solution $u \in AAP(X)$.

Next we prove the existence of asymptotically almost periodic mild solution of the problem (4) when the function f, g are not Lipschitz continuous. To establish our result, we consider the functions $f : \mathcal{R}_+ \times X \to X, g : \mathcal{R}_+ \times \mathcal{R}_+ \times X \to X$ satisfying boundedness condition
(B) There exists a continuous non-decreasing function
$W_1 : \mathcal{R}_+ \to \mathcal{R}_+, cW_2 : \mathcal{R}_+ \times \mathcal{R}_+ \to X$ such that

$$\begin{aligned}
||f(t, x)|| &\leq W_1(||x||) \\
||\int_0^t g(t, s, x)ds|| &\leq W_2(||x||)
\end{aligned}$$

**Theorem 4.6.** Let $f \in AAP(\mathcal{R}_+ \times X; X), g \in AAP(\mathcal{R}_+ \times \mathcal{R}_+ \times X; X)$ be a function that satisfies assumption (B) and the following conditions:
(a) For each $\nu \geq 0$,

$$\begin{aligned}
\lim_{t \to \infty} \frac{1}{h(t)} \int_0^t e^{-\omega(t-s)} W_1(\nu(s))ds &= 0 \\
\lim_{t \to \infty} \frac{1}{h(t)} \int_0^t e^{-\omega(t-s)} W_2(\nu(s))ds &= 0
\end{aligned}$$

(b) For each $\epsilon > 0$ there is $\delta > 0$ such that for every $u, v \in C_h(X)$, $||v-u||_h \leq \delta$ implies that

$$M\int_0^t e^{-\omega(t-s)}||f(s,v(s)) - f(s,u(s))||ds \leq \frac{\epsilon}{2}$$

$$M\int_0^t e^{-\omega(t-s)}||\int_0^t (g(t,s,v(s)) - g(t,s,u(s)))||ds \leq \frac{\epsilon}{2}$$

for all $t \in \mathcal{R}_+$

(c) For all $a, b \in \mathcal{R}_+, a \leq b$ and $r > 0$, the set

$$\{f(s,h(s)x) : a \leq s \leq b, x \in X, ||x|| \leq r\}$$

$$\{g(t,s,h(s)x) : a \leq (s,t) \leq b, x \in X, ||x|| \leq r\}$$

are relatively compact in X.

(d) $\lim\ inf_{\xi\to\infty}\dfrac{\xi}{\beta(\xi)} > 1$, where

$$\begin{aligned}\beta(\nu) &= \Big\|||\alpha R'(.)y|| + ||R(.)y|| + ||\alpha R(.)z|| + M\int_0^t e^{-\omega(t-s)}W_1(\nu h(s))ds \\ &\quad + M\int_0^t e^{-\omega(t-s)}W_2(\nu h(s))ds\Big\|_h\end{aligned}$$

then equation (4) has a asymptotically almost periodic mild solution.

**Proof:** We define a operator $\Lambda$ on $C_h(X)$ by

$$\begin{aligned}\Lambda u(t) &= \alpha R'(t)y + R(t)y + \alpha R(t)z + \int_0^t R(t-s)f(s,u(s))ds \\ &\quad + \int_0^t R(t-s)\int_0^s g((s,\tau,u(\tau))d\tau)ds.\end{aligned}$$

We show that $\Lambda$ has a fixed point in AAP(X).

(i) For $u \in C_h(X)$, we have that

$$\begin{aligned}||\Lambda u(t)|| &\leq (\alpha+1)M||y|| + \alpha M||z|| + M\int_0^t e^{-\omega(t-s)}W_1(||u||_h h(s))ds \\ &\quad + M\int_0^t e^{-\omega(t-s)}W_2(||u||_h h(s))ds\end{aligned}$$

It follows from condition (a) that $\Lambda : C_h(X) \to C_h(X)$.

(ii) The map $\Lambda$ is continuous. In fact, for $\epsilon > 0$ we take $\delta$ involved in condition (b). If $u, v \in C_h(X)$ and $||u - v||_h \leq \delta$ then

$$\begin{aligned}||\Lambda u(t) - \Lambda v(t)|| &\leq M\int_0^t e^{-\omega(t-s)}||f(s,u(s)) - f(s,v(s))||ds \\ &\quad + M\int_0^t e^{-\omega(t-s)}\int_0^t ||(g(t,s,u(s)) - g(t,s,v(s))||ds \\ &\leq \epsilon\end{aligned}$$

(iii) We next show that $\Lambda$ is completely continuous. We set $B_r(Z)$ for the closed ball with center at 0 and radius r in a space Z. Let $V = \Lambda(B_r(C_h(X)))$ and $v = \Lambda(u)$ for $u \in B_r(C_h(X))$. Initially, we prove that $V_b(t)$ is relatively compact subset of X for each $t \in [0, b]$. We get

$$\begin{aligned} v(t) &= \alpha R'(t)y + R(t)y + \alpha R(t)z + \int_0^t R(s)f(t-s, u(t-s))ds \\ &\quad + \int_0^t R(s)\int_0^s g(s-\tau, \tau-\theta, u(\tau-\theta))d\theta ds \\ &\in \alpha R'(t)y + R(t)y + \alpha R(t)z + tc(\bar{K}_1) + tc(\bar{K}_2), \end{aligned}$$

where $c(K_1)$ denotes the convex hull of $K_1$ and $K_1 = \{R(s)f(\xi, h(\xi)x) : 0 \leq s \leq t, 0 \leq \xi \leq t, ||x|| \leq r\}$ and $c(K_2)$ denotes the convex hull of $K_2$ and $K_2 = \{R(s)g(s, \xi, h(\xi)x) : 0 \leq s \leq t, 0 \leq \xi \leq t, ||x|| \leq r\}$. Using the fact that R(.) is strongly continuous and the property (c), we infer that $K_1, K_2$ are relatively compact set, and

$$V_b(t) \subseteq \alpha R'(t)y + R(t)y + \alpha R(t)z + tc(\bar{K}_1) + tc(\bar{K}_2),$$

which establishes our assertion.

We next show that $V_b$ is equicontinuous and in fact we can decompose

$$\begin{aligned} &v(t+s) - v(t) \\ &= \alpha(R'(t+s) - R'(t))y + (R(t+s) - R(t))y + \alpha(R(t+s) - R(t))z \\ &\quad + \int_t^{t+s} R(t+s-\xi)f(\xi, u(\xi))d\xi + \int_0^t (R(\xi+s) - R(\xi))f(t-\xi, u(t-\xi))d\xi \\ &\quad + \int_t^{t+s} (R(t+s-\xi)\int_0^s g(\xi, \theta, u(\theta))d\theta \\ &\quad + \int_0^t R(\xi+s) - R(\xi))\int_0^s g(s-\xi, \xi-\theta, u(\xi-\theta))d\theta ds \end{aligned}$$

For each $\epsilon > 0$ we can choose $\delta_1 > 0, \delta_2 > 0$ such that

$$||\int_t^{t+s} R(t+s-\xi)f(\xi, u(\xi))d\xi|| \leq M\int_t^{t+s} e^{-\omega(t+s-\xi)}W_1(rh(\xi))d\xi \leq \epsilon/7,$$
$$||\int_t^{t+s} R(t+s-\xi)\int_0^s g(\xi, \theta, u(\theta))d\theta|| \leq M\int_t^{t+s} e^{-\omega(t+s-\xi)}W_2(rh(\xi))d\xi \leq \epsilon/7$$

for $s \leq \delta_1, \delta_2$. Moreover $\{f(t-\xi, u(t-\xi)) : 0 \leq \xi \leq t,\ u \in B_r(C_h(X))\}$ and $\{g(s-\xi, \xi-\theta, u(\xi-\theta) : 0 \leq \xi, \theta \leq t, u \in B_r(C_h(X))\}$ are relatively compact set, R(.)and $R'(.)$ are strongly continuous, we can choose $\delta_i > 0, i = 3, 4, ...7$ such that

$$\begin{aligned} ||\alpha(R'(t+s) - R'(t))y|| &\leq \frac{\epsilon}{7}, \quad s \leq \delta_3 \\ ||(R(t+s) - R(t))y|| &\leq \frac{\epsilon}{7}, \quad s \leq \delta_4 \\ ||\alpha(R(t+s) - R(t))z|| &\leq \frac{\epsilon}{7}, \quad s \leq \delta_5 \\ ||(R(\xi+s) - R(\xi))f(t-\xi, u(t-\xi))|| &\leq \frac{\epsilon}{7t}, \quad s \leq \delta_6 \text{ and} \\ ||R(\xi+s) - R(\xi)g(s-\xi, \xi-\theta, u(\xi-\theta)|| &\leq \frac{\epsilon}{7t}, \quad s \leq \delta_7. \end{aligned}$$

Combining these estimate, we get $||v(t+s)-v(t)|| \leq \epsilon$ for s small enough and independent of $u \in B_r(C_h(X))$.
Finally, applying condition (a) we can show that

$$\frac{||v(t)||}{h(t)} \leq \frac{(\alpha+1)M||y||}{h(t)} + \frac{\alpha M||z||}{h(t)} + \frac{M}{h(t)}\int_0^t e^{-\omega(t-s)}W_1(||u||_h h(s))ds + \frac{M}{h(t)}\int_0^t e^{-\omega(t-s)}W_2(||u||_h h(s))ds \to 0,\ t\to\infty$$

and this convergence is independent of $u \in B_r(C_h(X))$. Hence V satisfies conditions (c-1)and (c-2)which completes the proof that V is a relatively compact set in $C_h(X)$.

(iv) If $u^\lambda(.)$ is a solution of the equation $u^\lambda = \lambda\Lambda(u^\lambda)$ for some $0<\lambda<1$, we have the estimate $\frac{||u^\lambda||_h}{\beta(||u^\lambda||_h)} \leq 1$ and, combining with condition (d) we conclude that the set $\tilde{K} = \{u^\lambda : u^\lambda = \lambda\Lambda(u^\lambda), \lambda \in (0,1)\}$ is bounded.

(v) It follows from Lemma 2.7 and Lemma 4.1, that $\Lambda(AAP(X)) \subseteq AAP(X)$ and, consequently we consider $\Lambda : \overline{AAP(X)} \to \overline{AAP(X)}$. Using the properties (i)-(iii), we have that this map is completely continuous. Taking into account that $\tilde{K}$ is bounded and using Leray-Schauder alternative theorem ([10, theorem 6.5.4]), we infer that $\Lambda$ has a fixed point $u \in \overline{AAP(X)}$. Let $(u_n)_n$ be a sequence in $AAP(X)$ that converges to $u$. We see that $(\Lambda u_n)_n$ converges to $\Lambda u = u$ uniformly in $\mathcal{R}_+$. This implies that $u \in AAP(X)$ and this completes the proof.

## 5 CONCLUSION

In this work we have established some sufficient condition for the existence of asymptotically almost periodic solution of integrodifferential equations by using fixed point theory.

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