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Development of N-period Dynamic Programming Model for Determining Economic Order Quantity (EOQ) under Stochastic Demand

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ABSTRACT

This paper demonstrates an approach to determine the Economic Order Quantity (EOQ) of an item under a periodic review inventory system with stochastic demand. Using an N-period dynamic programming planning horizon, we analyze the ordering policy during each period considering when demand is favorable or unfavorable to determine the EOQ and the associated profit at the end of each planning horizon. The objective is to determine in each period of the planning horizon, an optimal EOQ so that the long run profits are maximized for a given state of demands. We also generate a formula to determine the number of matrix transitions for each planning horizon until final stage and then prove the formula by the principle of mathematical induction.

Keywords: Dynamic programming, EOQ, Inventory Management, Planning horizon, Stochastic demand.

1 INTRODUCTION

Inventory is one of the most crucial aspects of any business organization as it pervades businesses world over. Maintaining inventories is necessary for any company dealing with physical products, including manufacturers, wholesalers, and retailers. The manufacturers need inventories of the materials required to manufacture their products and also inventories of the finished products awaiting shipment. Similarly, both wholesalers and retailers need to maintain inventories of goods to be available for purchase by customers. With an effective inventory management system, one can easily track the stock in the warehouse by avoiding overstocking, stocking of obsolete items, under-stocking and then focus on making the business one of the key players in the market space. The importance of inventory management cannot be over emphasized especially for e-commerce and online retail business. Inventory management in businesses must grow as the company expands. With strategic plan in place to optimize the process of overseeing and managing inventory, companies can achieve accurate order fulfilment, better inventory planning and ordering and increased customer satisfaction.

The EOQ is a model that is used to calculate the optimal quantity that can be purchased to minimize the cost of both the carrying inventory and the processing of purchase orders. By using the EOQ model, companies can minimize the costs associated with ordering and holding inventory. EOQ can be a valuable tool for small business owners who need to make decisions about how much inventory to keep on hand, how many items to order each time, and how often to reorder. The EOQ model is widely used by practitioners as a decision-making tool for the control of inventory. In general, the objective of inventory management deals with maximization of profit or minimization of the inventory carrying cost. Therefore, it is important to determine the optimal stock and optimal time of replenishment of inventory to meet the future demand.

[1] Considered stochastic location-allocation strategic decisions and inventory transportation tactical decisions in supply chain network. The model considers the trade-off between the risk pooling effect and transportation cost in two-echelon inventory and logistic system.

[2] Used a stochastic dynamic programming and simulation model to solve a multi-period inventory problem with raw material procurement carried out through a reverse auction. The work was an extension of the earlier work on procurement auctions in single period inventory models.

[3] Proposed a stochastic nonlinear mixed binary integer programming model for a multi-period inventory problem with quantity discounts based on previous orders. The model examined quantity discount contracts between a manufacturer and a retailer in a stochastic, two-period inventory situation in which quantity discounts are provided on the basis of previous order size. [4] Examined the problem of optimally dynamic joint decisions, replenishment and quantity, retail and wholesale price and revenue-sharing allocation over multi-period planning horizon, subject to deteriorating goods and multi-variant demand function. They used a calculus-based formulation combined with dynamic programming techniques to solve the channel coordination decision problem.

[4] Developed an infinite-horizon inventory control model under both yield uncertainty and disruptions. They demonstrated the importance of analyzing a sufficiently long-time horizon when modeling inventory systems subject to supply disruptions and pointed out that using a single-period approximation could lead to a wrong strategy for mitigating supply risks.

[5] Considered the inventory of a stochastic system whose aim was to optimize the order quantity and profits associated with ordering and holding cost of a single item. The decision whether to order additional units of the item or not to order was modelled based on two demand states, which is the favorable state and unfavorable state using dynamic programming.

[6] Considered a joint location inventory replenishment problem involving a chain of supermarkets at designated locations under stochastic demand. They formulated a finite state Markov decision process model where states of a Markov chain represent possible states of demand for milk powder product. The replenishment decisions were made using dynamic programming over a finite period planning horizon.

[7] Proposed a mathematical model that optimizes campaign strategies of a political candidate. Considering uncertainty in voter support and cost implications in holding political rallies, they formulated a finite state Markov decision process model where states of the Markov chain were represented by possible states of support among voters. The decision of whether or not to campaign and hold a political rally at a given location was made using discrete time Markov chains and dynamic programming over a finite period planning horizon.

[8] Considered a multi-period, single-item production-inventory lot sizing problem where demand is stochastic and non-stationary. They formulated a finite state Markov decision model where states of the Markov chain represent possible states of demand for the product using dynamic programming.

[9] Developed a stochastic optimization model that would help decision makers about uncertainty in the supply chain. They proposed a method to transform a robust model into a linear program that requires only the addition of the number of scenarios and total constraints respectively.

[10] Proposed a formulation and solution procedure for inventory planning with the Markov decision process (MDP) models by identifying the chain's state space and the transition probabilities. They considered the cost structure, evaluated its individual components and they obtained the optimal policy using the policy improvement algorithm.

[11] Developed a simultaneous model considering a stochastic demand with risk pooling phenomenon. The simultaneous approach incorporates inventory control decisions such as EOQ and safety stock decisions into typical facility location models which were used to solve the distribution network design problem.

[12] Chao *et al*. [13] discussed joint replenishment and pricing decisions inventory system with stochastically dependent supply capacity. They analyzed a single period, periodic review system with price dependent stochastic demand.

[13] Developed a replenishment inventory model to understand product, sales and supply characteristics of perishables in supermarkets. They defined relevant concepts in controlling the inventories of perishables and investigated how the intelligence in automated store ordering systems in supermarkets can be further improved in terms of random demand.

[14] Presented a probabilistic inventory model for items that deteriorate at a constant rate and the demand is a random variable. They constructed discrete cycle and continuous cycle time models to determine the global optimal ordering policies for the models.

[15] Considered an optimal policy for a stochastic inventory model for deteriorating items with time dependent selling price. The rate of deterioration of the items was constant over time and the selling price decreased monotonically at a constant rate with deterioration of items.

2 MATERIALS AND METHOD

2.1 Model parameters

Table 1: Model parameters

u	Unfavorable state
f	Favorable state
h	Holding cost per unit
s	Selling price per unit
o	Ordering cost per unit

p	Cost price per unit
p^z	Profit matrix for z policy
I^z	Inventory matrix for z policy
i_n, j_n	States of demand for nth stage
z	Ordering policy
n, N	Stages going from n=1 to N
s	Shortage cost per unit
$O^z_{i_n}$	Economic order quantity at nth stage for z policy
p_{i_n, j_n}	Profit in state i_n, j_n
a^z_{N, i_n}	Accumulated total profits at the end of N period
$Q^z_{i_n, j_n}$	Demand transition probability in state i_n, j_n for z policy
$I^z_{i_n, j_n}$	Quantity in inventory in state i_n, j_n for z policy
a^z_{n, i_n}	Expected accumulated total profits at the end of nth period for z policy
$g_n(i_n, p)$	Optimal expected total inventory cost at stage n.

2.2 Dynamic programming formulation

We consider an inventory system of a single item with two states of demand. That is, the favorable state (f) and the unfavorable state (u) based on the situation of the market. The problem can be expressed as a finite-period dynamic programming model to obtain an optimal EOQ. We denoted $g_n(i_n)$ as the expected total profit accumulated during n periods is $n \le N$, given that the state of the system at the beginning of period n is $i_n \in \{f, u\}$.

The recursive equation relating g_n and g_{n+1} is as follows

$$g_n(i_n) = \max_z \{Q^z_{i_n f} (P^z_{i_n f} + g_{n+1}(f), Q^z_{i_n u} (P^z_{i_n u} + g_{n+1}(u))\} \tag{1}$$

$z \in \{0,1\}$, $i_n \in \{f, u\}$

$n = 1, 2 \ldots N$

For, $g_N(f) = g_N(u) = 0$ (2)

The cumulative total profit is given by

$$P^z_{i_n, j_n} + g_{n+1}(j_n) \tag{3}$$

As a result of getting to state $j_n \in \{f, u\}$ at the start of period n+1 from state $i_n \in \{f, u\}$ at the beginning of period n, which occurs with the transition probability $Q^z_{i_n, j_n}$.

This shows that $a^z_{n,i_n} = Q^z (P^z)^T$ (4)

Then the dynamic recursive equation becomes

$$g_n(i_n) = \max_z \{a^z_{n,i_n} + Q^z_{i_n f} g_{n+1}(f) + Q^z_{i_n u} g_{n+1}(u)\} \quad (5)$$

or $g_N(i_n) = \max_z \left[a^z_{n,i_n} \right]$

2.3 Computing the Economic Order Quantity for the planning horizons

We consider an N-period planning horizon in this paper and so the planning starts from $n = 1, 2, 3 \ldots N-1, N$, as follows:

2.3.1 Optimization during period 1 (First planning horizon)

The ordering policy during period 1 when the demand is favorable

$$z = \begin{cases} 1, & if\ a^1_{1,f} > a^0_{1,f} \\ 0, & if\ a^1_{1,f} \le a^0_{1,f} \end{cases} \quad (6)$$

The associated total profit and EOQ are

$$g_1(f) = \begin{cases} a^1_{1,f}, & if\ z = 1 \\ a^0_{1,f}, & if\ z = 0 \end{cases} ; O^z_f = \begin{cases} (D^1_{ff} - I^1_{ff}) + (D^1_{fu} - I^1_{fu}); & if\ z = 1 \\ 0; & if\ z = 0 \end{cases} \quad (7)$$

for $D^z_i > I^z_i$

The ordering policy during period 1 when the demand is unfavorable

$$z = \begin{cases} 1, & if\ a^1_{1,u} > a^0_{1,u} \\ 0, & if\ a^1_{1,u} \le a^0_{1,u} \end{cases} \quad (8)$$

The associated total profit and EOQ are

$$g_1(u) = \begin{cases} a^1_{1,u}, & if\ z = 1 \\ a^0_{1,u}, & if\ z = 0 \end{cases} ;\ O^z_u = \begin{cases} (D^1_{uf} - I^1_{uf}) + (D^1_{uu} - I^1_{uu}); & if\ z = 1 \\ 0; & if\ z = 0 \end{cases} \quad (9)$$

for $D_i^z > I_i^z$

2.3.2 Optimization during period 2 (Second planning horizon)

a_{2,i_n}^z Represents the accumulated profits at the end of the first period as a result of decisions made using recursive equation (5), it follows that:

$$a_{2,i_n}^z = a_{1,i_n}^z + Q_{if}^z g_1(f) + Q_{iu}^z g_1(u) \tag{10}$$

Therefore, the ordering policy when the demand is favorable is

$$z = \begin{cases} 1, & if\ a_{2,f}^1 > a_{2,f}^0 \\ 0, & if\ a_{2,f}^1 \le a_{2,f}^0 \end{cases} \tag{11}$$

The associated total profits and EOQ are

$$g_2(f) = \begin{cases} a_{2,f}^1, & if\ z=1 \\ a_{2,f}^0, & if\ z=0 \end{cases} ;\ O_f^z = \begin{cases} (D_{ff}^1 - I_{ff}^1) + (D_{fu}^1 - I_{fu}^1); & if\ z=1 \\ 0; & if\ z=0 \end{cases} \tag{12}$$

Also, the ordering policy when the demand is unfavorable is

$$z = \begin{cases} 1, & if\ a_{2,u}^1 > a_{2,u}^0 \\ 0, & if\ a_{2,u}^1 \le a_{2,u}^0 \end{cases} \tag{13}$$

The associated total profits and EOQ are

$$g_2(u) = \begin{cases} a_{2,u}^1, & if\ z=1 \\ a_{2,u}^0, & if\ z=0 \end{cases} ;\ O_u^z = \begin{cases} (D_{uf}^1 - I_{uf}^1) + (D_{uu}^1 - I_{uu}^1); & if\ z=1 \\ 0; & if\ z=0 \end{cases} \tag{14}$$

2.3.3 Optimization during period 3 (Third planning horizon)

The accumulated profits at the end of period 3 as a result of decisions made using recursive equation (5), is as follows:

$$a_{3,i_n}^z = a_{2,i_n}^z + Q_{if}^z g_2(f) + Q_{iu}^z g_2(u) \tag{15}$$

Therefore, the ordering policy when the demand is favorable is

$$z = \begin{cases} 1, & if\ a_{3,f}^1 > a_{3,f}^0 \\ 0, & if\ a_{3,f}^1 \le a_{3,f}^0 \end{cases} \tag{16}$$

The associated total profits and EOQ are

$$g_3(f)=\begin{cases} a_{3,f}^1, \; if \; z=1 \\ a_{3,f}^0, \; if \; z=0 \end{cases} ; \quad O_f^z=\begin{cases} (D_{ff}^1-I_{ff}^1)+(D_{fu}^1-I_{fu}^1); \; if \; z=1 \\ 0; \; if \; z=0 \end{cases} \tag{17}$$

Also, the ordering policy when the demand is unfavorable is

$$z=\begin{cases} 1, \; if \; a_{3,u}^1 > a_{3,u}^0 \\ 0, \; if \; a_{3,u}^1 \leq a_{3,u}^0 \end{cases} \tag{18}$$

The associated total profits and EOQ are

$$g_3(u)=\begin{cases} a_{2,u}^{\;1}, \; if \; z=1 \\ a_{2,u}^{\;0}, \; if \; z=0 \end{cases} ; \; O_u^z=\begin{cases} (D_{uf}^1-I_{uf}^1)+(D_{uu}^1-I_{uu}^1); \; if \; z=1 \\ 0; \; if \; z=0 \end{cases} \tag{19}$$

Continuing in this way up to period N-1, as follows:

2.3.4 Optimization during period (N-1) ((N-1)th planning horizon)

The accumulated profits at the end of (N-1)-period as a result of decisions made using recursive equation (5), is as follows:

$$a_{N-1,i_n}^z = a_{N-2,i_n}^z + Q_{if}^z g_{N-2}(f) + Q_{iu}^z g_{N-2}(u) \tag{20}$$

Therefore, the ordering policy when the demand is favorable is

$$z=\begin{cases} 1, \; if \; a_{N-1,f}^1 > a_{N-1,f}^0 \\ 0, \; if \; a_{N-1,f}^1 \leq a_{N-1,f}^0 \end{cases} \tag{21}$$

The associated total profits and EOQ are

$$g_{N-1}(f)=\begin{cases} a_{N-1,f}^1, \; if \; z=1 \\ a_{N-1,f}^0, \; if \; z=0 \end{cases} ; \; O_f^z=\begin{cases} (D_{ff}^1-I_{ff}^1)+(D_{fu}^1-I_{fu}^1); \; if \; z=1 \\ 0; \; if \; z=0 \end{cases} \tag{22}$$

Also, the ordering policy when demand is unfavorable is

$$z=\begin{cases} 1, \; if \; a_{N-1,u}^1 > a_{N-1,u}^0 \\ 0, \; if \; a_{N-1,u}^1 \leq a_{N-1,u}^0 \end{cases} \tag{23}$$

The associated total profits and EOQ are

$$g_{N-1}(u)=\begin{cases}a^1_{N-1,u},\ if\ z=1\\ a^0_{N-1,u},\ if\ z=0\end{cases};\quad O^z_u=\begin{cases}(D^1_{uf}-I^1_{uf})+(D^1_{uu}-I^1_{uu});\ \ if\ z=1\\ 0;\ if\ z=0\end{cases}\tag{24}$$

2.3.5 Optimization during period N (N^th^ planning horizon)

The accumulated profits at the end of period N as a result of decisions made using recursive equation (5), is as follows:

$$a^z_{N,i_n}=a^z_{N-1,i_n}+Q^z_{if}g_{N-1}(f)+Q^z_{iu}g_{N-1}(u)\tag{25}$$

Therefore, the ordering policy when the demand is favorable is

$$z=\begin{cases}1,\ \ if\ a^1_{N,f}>a^0_{N,f}\\ 0,\ \ if\ \ a^1_{N,f}\le a^0_{N,f}\end{cases}\tag{26}$$

The associated total profits and EOQ are

$$g_N(f)=\begin{cases}a^1_{N,f},\ if\ z=1\\ a^0_{N,f},\ if\ z=0\end{cases};\ O^z_f=\begin{cases}(D^1_{ff}-I^1_{ff})+(D^1_{fu}-I^1_{fu});\ \ if\ z=1\\ 0;\ if\ z=0\end{cases}\tag{27}$$

Also, the ordering policy when the demand is unfavorable is

$$z=\begin{cases}1,\ if\ a^1_{N,u}>a^0_{N,u}\\ 0,\ if\ a^1_{N,u}\le a^0_{N,u}\end{cases}\tag{28}$$

The associated total profits and EOQ are

$$g_N(u)=\begin{cases}a^1_{N,u},\ if\ z=1\\ a^0_{N,u},\ if\ z=0\end{cases};\ O^z_u=\begin{cases}(D^1_{uf}-I^1_{uf})+(D^1_{uu}-I^1_{uu});\ \ if\ z=1\\ 0;\ if\ z=0\end{cases}\tag{29}$$

3 MODEL AND DATA

This model considers an inventory system of a single item over a finite planning horizon. The demand for the item can be classified into favorable state and unfavorable state. The transition probabilities over the planning horizon can be expressed as Markov decision process. The decision to order additional units (z=1) or not to order additional units (z=0) is dependent on the state of demand where z is the decision variable. This decision is to be made using dynamic programming techniques over N-period planning horizon. The decision tree for the states is shown in the diagram below:

Key: F= favorable state**,** U= unfavorable state

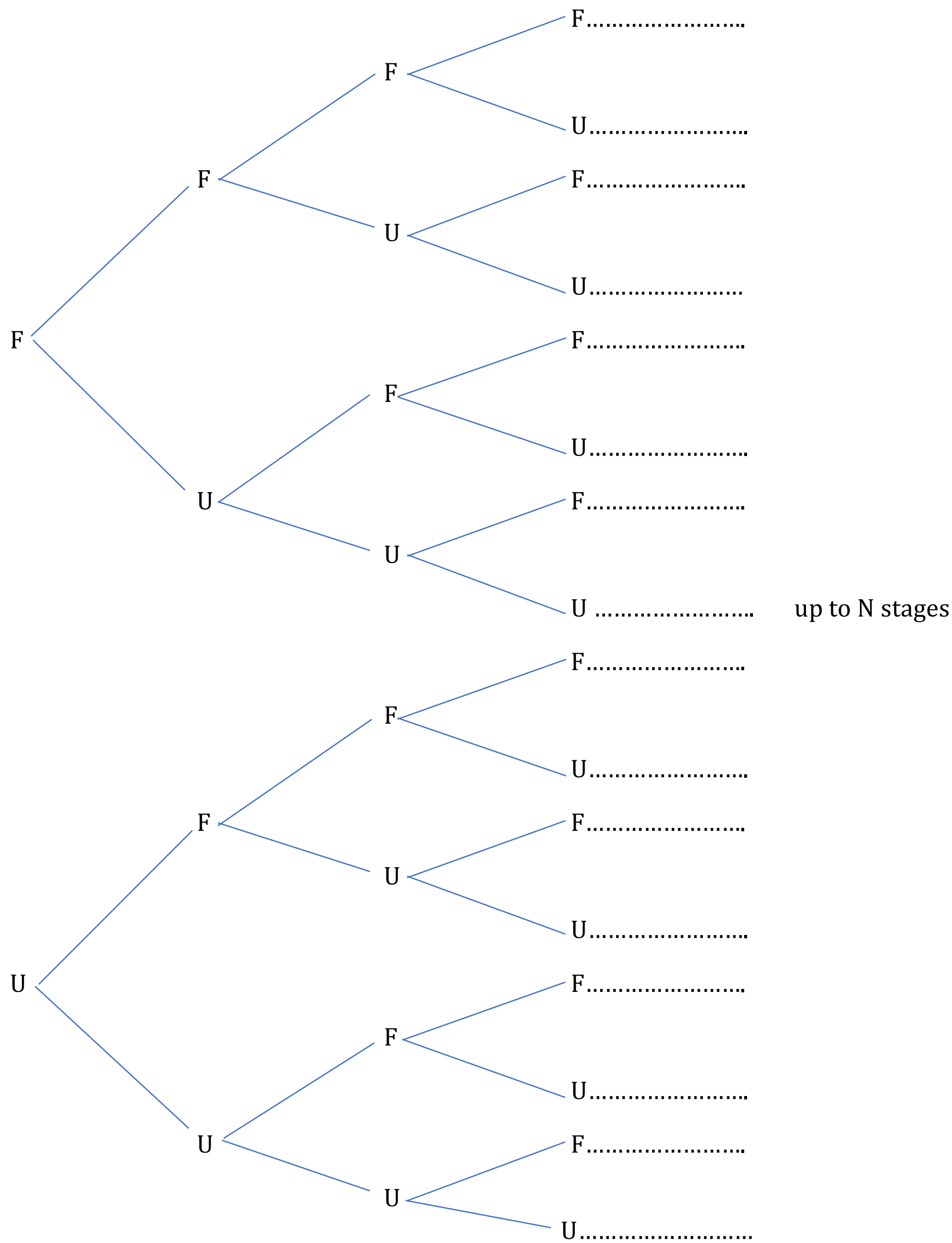

Figure 1: The decision three diagram N-planning horizon

The change in demand has the following possibilities.

$$Q^z_{i_n,j_n} \Leftrightarrow \begin{bmatrix} Q^z{}_{fff\ldots f} & Q^z{}_{fff\ldots u} & Q^z{}_{ffu\ldots f} & \cdots & \cdots & \cdots & Q^z{}_{fuu\ldots f} & Q^z{}_{fuu\ldots u} \\ Q^z{}_{uff\ldots f} & Q^z{}_{uff\ldots u} & Q^z{}_{ufu\ldots f} & \cdots & \cdots & \cdots & Q^z{}_{uuu\ldots f} & Q^z{}_{uuu\ldots u} \end{bmatrix}$$

$$D^z_{i_n,j_n} \Leftrightarrow \begin{bmatrix} D^z{}_{fff\ldots f} & D^z{}_{fff\ldots u} & D^z{}_{ffu\ldots f} & \cdots & \cdots & \cdots & D^z{}_{fuu\ldots f} & D^z{}_{fuu\ldots u} \\ D^z{}_{uff\ldots f} & D^z{}_{uff\ldots u} & D^z{}_{ufu\ldots f} & \cdots & \cdots & \cdots & D^z{}_{uuu\ldots f} & D^z{}_{uuu\ldots u} \end{bmatrix}$$

$$I^z_{i_n,j_n} \Leftrightarrow \begin{bmatrix} I^z{}_{fff\ldots f} & I^z{}_{fff\ldots u} & I^z{}_{ffu\ldots f} & \cdots & \cdots & \cdots & I^z{}_{fuu\ldots f} & I^z{}_{fuu\ldots u} \\ I^z{}_{uff\ldots f} & I^z{}_{uff\ldots u} & I^z{}_{ufu\ldots f} & \cdots & \cdots & \cdots & I^z{}_{uuu\ldots f} & I^z{}_{uuu\ldots u} \end{bmatrix}$$

$$P^z_{i_n,j_n} \Leftrightarrow \begin{bmatrix} P^z{}_{fff\ldots f} & P^z{}_{fff\ldots u} & P^z{}_{ffu\ldots f} & \cdots & \cdots & \cdots & P^z{}_{fuu\ldots f} & P^z{}_{fuu\ldots u} \\ P^z{}_{uff\ldots f} & P^z{}_{uff\ldots u} & P^z{}_{ufu\ldots f} & \cdots & \cdots & \cdots & P^z{}_{uuu\ldots f} & P^z{}_{uuu\ldots u} \end{bmatrix}$$

The possibilities can be converted to multiple transition matrices as follows:

$$Q^z_{i_n,j_n} = \begin{bmatrix} Q^z_{ff} & Q^z_{fu} \\ Q^z_{uf} & Q^z_{uu} \end{bmatrix}\begin{bmatrix} Q^z_{ff} & Q^z_{fu} \\ Q^z_{uf} & Q^z_{uu} \end{bmatrix}\begin{bmatrix} Q^z_{ff} & Q^z_{fu} \\ Q^z_{uf} & Q^z_{uu} \end{bmatrix}\cdots\begin{bmatrix} Q^z_{ff} & Q^z_{fu} \\ Q^z_{uf} & Q^z_{uu} \end{bmatrix} = \begin{bmatrix} Q^z_{FF} & Q^z_{FU} \\ Q^z_{UF} & Q^z_{UU} \end{bmatrix} \text{say}$$

$$D^z_{i_n,j_n} = \begin{bmatrix} D^z_{ff} & D^z_{fu} \\ D^z_{uf} & D^z_{uu} \end{bmatrix}\begin{bmatrix} D^z_{ff} & D^z_{fu} \\ D^z_{uf} & D^z_{uu} \end{bmatrix}\begin{bmatrix} D^z_{ff} & D^z_{fu} \\ D^z_{uf} & D^z_{uu} \end{bmatrix}\cdots\begin{bmatrix} D^z_{ff} & D^z_{fu} \\ D^z_{uf} & D^z_{uu} \end{bmatrix} = \begin{bmatrix} D^z_{FF} & D^z_{FU} \\ D^z_{UF} & D^z_{UU} \end{bmatrix} \text{say}$$

$$I^z_{i_n,j_n} = \begin{bmatrix} I^z_{ff} & I^z_{fu} \\ I^z_{uf} & I^z_{uu} \end{bmatrix}\begin{bmatrix} I^z_{ff} & I^z_{fu} \\ I^z_{uf} & I^z_{uu} \end{bmatrix}\begin{bmatrix} I^z_{ff} & I^z_{fu} \\ I^z_{uf} & I^z_{uu} \end{bmatrix}\cdots\begin{bmatrix} I^z_{ff} & I^z_{fu} \\ I^z_{uf} & I^z_{uu} \end{bmatrix} = \begin{bmatrix} I^z_{FF} & I^z_{FU} \\ I^z_{UF} & I^z_{UU} \end{bmatrix} \text{say}$$

$$P^z_{i_n,j_n} = \begin{bmatrix} P^z_{ff} & D^z_{fu} \\ P^z_{uf} & D^z_{uu} \end{bmatrix}\begin{bmatrix} P^z_{ff} & P^z_{fu} \\ P^z_{uf} & P^z_{uu} \end{bmatrix}\begin{bmatrix} P^z_{ff} & P^z_{fu} \\ P^z_{uf} & P^z_{uu} \end{bmatrix}\cdots\begin{bmatrix} P^z_{ff} & P^z_{fu} \\ P^z_{uf} & P^z_{uu} \end{bmatrix} = \begin{bmatrix} P^z_{FF} & P^z_{FU} \\ P^z_{UF} & P^z_{UU} \end{bmatrix} \text{say}$$

We note that, the number of matrices generated from the transition matrices above followed a trend that coincides with the formulation below:

$$I_N = \sum_{n=0}^{N-2} 2^n = 2^{N-1} - 1 \tag{30}$$

Thus, since N is the number of periods which is the same as the planning horizon, the transition has to start from 2. Then we have:

$$I_2 = \sum_{n=0}^{0} 2^n = 2^{2-1} - 1 = 2 - 1 = 1 \tag{31}$$

$$I_3 = \sum_{n=0}^{1} 2^n = 2^{3-1} - 1 = 2^2 - 1 = 3 \quad (32)$$

$$I_4 = \sum_{n=0}^{2} 2^n = 2^{4-1} - 1 = 2^3 - 1 = 7 \quad (33)$$

and so on.

3.1 Proof of the formulation

The formulation can be proved by the principle of mathematical induction as follows:

For $N = 2,\ I_2 = 2^{2-1} - 1 = 1$. Therefore, the result is true for the first N which is equal 2.

Next, we assume that the result is true for $N = k$ i.e. $I_k = 2^{k-1} - 1$. That is,

$$I_k = 2^0 + 2^1 + 2^2 + \ldots + 2^{k-2} = 2^{k-1} - 1. \quad (34)$$

Then for $N = k + 1$ we have

$$I_{k+1} = 2^0 + 2^1 + 2^2 + \ldots + 2^{k-2} + 2^{k-1}$$

$$= \sum_{n=0}^{k-2} 2^n + 2^{k-1}$$

$$= 2^{k-1} - 1 + 2^{k-1} = 2(2^{k-1}) - 1$$

$$= 2^k - 1 \quad (35)$$

Hence, the result is true for $N = k + 1$ and so the result is true for any N such that $N \geq 2$. Therefore, the result is proved.

3.2 Computation of demand transition matrix, profit matrix and EOQ

For any ordering policy $z \in \{0,1\}$, the demand transition probability from state i_n to j_n could be taken as the quantity demanded when the demand is at initial state i_n and then changing to state j_n divided by the total quantity demanded over all states. As represented in the following equation.

$$Q^z_{i_n, j_n} = \frac{D^z_{i_n, j_n}}{[D^z_{i_n f} + D^z_{i_n u}]} \ ;\ i_n, j_n \in \{f, u\},\ z \in \{0,1\} \quad (36)$$

When demand is greater than the on-hand inventory, the profit matrix P^z can be computed by the following equations:

Profit = selling price – cost price

$$P^z = s - p \tag{37}$$

Therefore, $P^z = P_{i_n,j_n}(D^z) - (o+h)I^z - (o+s)[D^z - I^z]$ (38)

On the other hand, if demand is less than or equal to the inventory at hand, then

$$P^z = P_{i_n,j_n} D^z_{i_n,j_n} - (o+h) D^z_{i_n,j_n} \tag{39}$$

Equations (39) and (40) are combined in equation (41) below:

$$\Rightarrow P^z = \begin{Bmatrix} P_{i_n,j_n} D^z_{i_n,j_n} - (o+h) I^z_{i_n,j_n} - (o+s)[D^z_{i_n,j_n} - I^z_{i_n,j_n}] & if \;\; D^z_{i_n,j_n} > I^z_{i_n,j_n} \\ P_{i_n,j_n} D^z_{i_n,j_n} - (o+h) D^z_{i_n,j_n} & if \;\; D^z_{i_n,j_n} \le I^z_{i_n,j_n} \end{Bmatrix}$$

$$\forall \;\; i_n, j_n \in \{f, u\}; z \in \{0,1\} \tag{40}$$

The justification for equation (41) is that $D^z_{i_n,j_n} - I^z_{i_n,j_n}$ must be ordered to satisfy the excess demand. Thus, the additional units ordered must attract shortage cost and ordering cost since the demand is greater than inventory. Also, when demand is less than or equal to on-hand inventory, no order will be placed, as such only holding cost and ordering cost will be charged on the goods in inventory. The EOQ when demand is in state $i_n \in \{f, u\}$ with ordering policy $z \in \{0,1\}$ can be obtained from the following equation.

$O^z_{i_n} = (D^z_{i_n f} - I^z_{i_n f}) + (D^z_{i_n u} - I^z_{i_n u})$, Provided $D^z_{i_n,j_n} > I^z_{i_n,j_n}$ (41)

$i_n, j_n \in \{f, u\}$, $z \in \{0,1\}$

Otherwise, $O^z_{i_n} = 0$

4 RESULTS AND DISCUSSION

4.1 Numerical example

The demand patterns and inventory levels over state transitions for sample of customers were collected in past weeks in respect to favorable and unfavorable demand of a particular item. For ordering policy z=0, the data is in Table 2 and for ordering policy z=1, the data is in Table 3 below:

Table 2: Z=0

State transition, (i_n, j_n)	Demand, $D^0_{i_n j_n}$	Inventory, $I^0_{i_n j_n}$
ff	04	03
fu	03	02
uf	02	01
uu	03	02

Table 3: Z=1

State transition, (i_n, j_n)	Demand, $D^1_{i_n j_n}$	Inventory, $I^1_{i_n j_n}$
ff	05	03
fu	04	02
uf	03	02
uu	02	01

We can break the above tables into the following matrices:

When additional units are ordered, z=1

$$D^1 = \begin{pmatrix} 05 & 04 \\ 03 & 02 \end{pmatrix}; \quad I^1 = \begin{pmatrix} 03 & 02 \\ 02 & 01 \end{pmatrix}$$

When additional units are not ordered, z=0

$$D^0 = \begin{pmatrix} 04 & 03 \\ 02 & 03 \end{pmatrix}; \quad I^0 = \begin{pmatrix} 03 & 02 \\ 01 & 02 \end{pmatrix}$$

We are considering a 5-period planning horizon in this example, as such the matrices above can be expressed as 15-multiple transition matrices because a 5-period planning horizon generates 15 transitions. The transition matrices for the 5-period can therefore be expressed below as obtained with the help of MATLAB computer package.

When additional units are ordered, z=1

$$D^1_F = \begin{bmatrix} 05 & 04 \\ 03 & 02 \end{bmatrix}^{15} = \begin{bmatrix} 5{,}911{,}518{,}558{,}221 & 4{,}482{,}738{,}450{,}156 \\ 3{,}362{,}053{,}837{,}617 & 2{,}549{,}464{,}720{,}604 \end{bmatrix} \simeq \begin{bmatrix} 5.9115 & 4.4827 \\ 3.3621 & 2.5495 \end{bmatrix} \times 10^{12}$$

$$I^1_F = \begin{bmatrix} 03 & 02 \\ 02 & 01 \end{bmatrix}^{15} = \begin{bmatrix} 1{,}836{,}311{,}903 & 1{,}134{,}903{,}170 \\ 1{,}134{,}903{,}170 & 701{,}408{,}733 \end{bmatrix} \simeq \begin{bmatrix} 1.8363 & 1.1349 \\ 1.1349 & 0.7014 \end{bmatrix} \times 10^{9}$$

When additional units are not ordered, z=0

$$D_F^0 = \begin{bmatrix} 04 & 03 \\ 02 & 03 \end{bmatrix}^{15} \simeq \begin{bmatrix} 2.8211 & 2.8211 \\ 1.8807 & 1.8807 \end{bmatrix} \times 10^{11}$$

$$I_F^0 = \begin{bmatrix} 03 & 02 \\ 01 & 02 \end{bmatrix}^{15} \simeq \begin{bmatrix} 715827883 & 715827882 \\ 357913941 & 357913942 \end{bmatrix}$$

Suppose in each of the cases, the unit selling price (P_s) is ₦1000, the ordering cost (o) is ₦200, the holding cost (h) is ₦100, the shortage cost (s) is ₦50 and the cost price (c) is ₦500.

4.2 Computation of Model Parameters

$$P_{i_n j_n} = 1000 - 500 = 500$$

The matrices for the demand transition and profit are computed using equations (38) and (39).

Substituting for z=0 and z=1, we have the following:

$$Q^1 = \begin{bmatrix} \frac{D_{FF}^1}{D_{FF}^1 + D_{FU}^1} & \frac{D_{FU}^1}{D_{FF}^1 + D_{FU}^1} \\ \frac{D_{UF}^1}{D_{UF}^1 + D_{UU}^1} & \frac{D_{UU}^1}{D_{UF}^1 + D_{UU}^1} \end{bmatrix} = \begin{bmatrix} \frac{5,911,518,558,221}{10,394,257,008,377} & \frac{4,482,738,450,156}{10,394,257,008,377} \\ \frac{3,362,053,837,617}{5,911,518,558,221} & \frac{2,549,464,720,604}{5,911,518,558,221} \end{bmatrix}$$

$$\Rightarrow Q^1 = \begin{bmatrix} 0.569 & 0.431 \\ 0.569 & 0.431 \end{bmatrix}$$

From equation (38), when z=1, we have

$$\Rightarrow P^1 = 500\begin{bmatrix} 05 & 04 \\ 03 & 02 \end{bmatrix}^{15} - 300\begin{bmatrix} 03 & 02 \\ 02 & 01 \end{bmatrix}^{15} - (250)\left(\begin{bmatrix} 05 & 04 \\ 03 & 02 \end{bmatrix}^{15} - \begin{bmatrix} 03 & 02 \\ 02 & 01 \end{bmatrix}^{15}\right)$$

$$\Rightarrow P^1 = \begin{bmatrix} 1.4778 & 1.1206 \\ 0.8405 & 0.6373 \end{bmatrix} \times 10^{15}$$

Also, from (36), for z=0:

$$Q^0 = \begin{bmatrix} \frac{D_{FF}^0}{D_{FF}^0 + D_{FU}^0} & \frac{D_{FU}^0}{D_{FF}^0 + D_{FU}^0} \\ \frac{D_{UF}^0}{D_{UF}^0 + D_{UU}^0} & \frac{D_{UU}^0}{D_{UF}^0 + D_{UU}^0} \end{bmatrix} = \begin{bmatrix} \frac{282,110,990,746}{564,221,981,492} & \frac{282,110,990,746}{564,221,981,492} \\ \frac{188,073,993,830}{376,147,987,660} & \frac{188,073,993,830}{376,147,987,660} \end{bmatrix}$$

$$\Rightarrow Q^0 = \begin{bmatrix} 0.500 & 0.500 \\ 0.500 & 0.500 \end{bmatrix}$$

From equation (39), for z=0, we have:

$$\Rightarrow P^0 = 500\begin{bmatrix} 04 & 03 \\ 02 & 03 \end{bmatrix}^{15} - 300\begin{bmatrix} 04 & 03 \\ 02 & 03 \end{bmatrix}^{15}$$

$$\Rightarrow P^0 = \begin{bmatrix} 1.4106 & 1.4106 \\ 0.9404 & 0.9404 \end{bmatrix}\times 10^{14} - \begin{bmatrix} 8.4633 & 8.4633 \\ 5.6422 & 5.6422 \end{bmatrix}\times 10^{13} \Rightarrow P^0 = \begin{bmatrix} 5.6422 & 5.6422 \\ 3.7615 & 3.7615 \end{bmatrix}\times 10^{13}$$

4.3 Computation of expected total profit for first planning horizon

For z=1, the matrices Q^1 and P^1 yield profit as follows:

$$a_1^1 = Q^1 P^1 = \begin{bmatrix} 0.569 & 0.431 \\ 0.569 & 0.431 \end{bmatrix}\begin{bmatrix} 14,777,878,239,601 & 11,206,278,673,805 \\ 8,404,567,142,457 & 6,373,311,097,143 \end{bmatrix}$$

So that $a_{1,f}^1 = 14,777,878,239,601(0.569) + 11,206,278,673,805(0.431) = 13,238,518,826,742$

$$\Rightarrow a_{1,u}^1 = 8,404,567,142,457(0.569) + 6,373,311,097,143(0.431) = 7,529,095,786,927$$

Also, for z=0, the matrices Q^0 and P^0 yield profit as follows:

$$a_1^0 = Q^0 P^0 = \begin{bmatrix} 0.500 & 0.500 \\ 0.500 & 0.500 \end{bmatrix}\begin{bmatrix} 5,642,219,815 & 5,642,219,815 \\ 3,761,479,877 & 3,761,479,877 \end{bmatrix}$$

$$a_{1,f}^0 = 5,642,219,815(0.500) + 5,642,219,815(0.500) = 5,642,219,816$$

$$a_{1,u}^0 = 3,761,479,877(0.500) + 3,761,479,877(0.500) = 3,761,479,877$$

This shows that z=1 is the optimal ordering policy for favorable state since ₦13,238,518,826,742 is greater than ₦5,642,219,816 with associated total profit of ₦13,238,518,826,742 and EOQ of 4units. Also, it shows that z=1 is the optimal ordering policy for unfavorable state since ₦7,529,095,786,927 is greater than ₦3,761,479,877 with associated profit of ₦7,529,095,786,927 and EOQ of 2units for the first planning horizon. The table below shows the computation till the fifth planning horizon.

Table 4: Table for the planning horizons

FIRST PLANNING	Z=1	Z=0
$a_{1,f}^z$	13,238,518,826,742	5,642,219,816
$a_{1,u}^z$	7,529,095,786,927	3,761,479,877
o_f^z	4	0
o_u^z	2	0
SECOND PLANING	Z=1	Z=0
$a_{2,f}^z$	24,016,276,323,324	10,389,449,526,651
$a_{2,u}^z$	18,306,853,283,509	10,387,568,786,712

o_f^z	4	0
o_u^z	2	0
THIRD PLANNING	Z=1	Z=0
$a_{3,f}^z$	45,571,791,316,488	31,551,014,330,068
$a_{3,u}^z$	39,862,368,276,673	31,549,133,590,129
o_f^z	4	0
o_u^z	2	0
FOURTH PLANNING	Z=1	Z=0
$a_{4,f}^z$	88,682,821,302,816	74,268,094,126,649
$a_{4,u}^z$	82,973,398,263,001	74,266,213,386,710
o_f^z	4	0
o_u^z	2	0
FIFTH PLANNING	Z=1	Z=0
$a_{5,f}^z$	174,904,881,275,472	160,096,203,909,558
$a_{5,u}^z$	169,195,458,235,657	160,094,323,169,619
o_f^z	4	0
o_u^z	2	0

5 C0NCLUSION

In this study, we develop an EOQ model using N-period dynamic programming technique under a periodic review inventory system with stochastic demand. With the aid of dynamic programming, we analyzed the N-period planning horizons for a single item inventory system with favorable and unfavorable states of demand where N is finite. As a result of the generalization carried out in this study, we noticed a trend of the transitions from one state to another which generated a formulation for obtaining the number of matrix transitions from one state to another up to nth state which we proved by the principle of mathematical induction.

Comparing our study with the work of Mubiru (2015), each study involves a dynamic programming model with 2-states but the work of Mubiru (2015) involves 2-planning horizon while our study extends till the Nth planning horizon. With the generalization made in our study, a decision maker in an organization will monitor the trends of events over the planning horizons for either both short- and long-term period in order to make an optimal decision as to know when to order additional items or not order to avoid overstocking and aids profit maximization.

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