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# Singular fuzzy linear systems

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**Abstract:** *The linear system of equations $A\tilde{x} = \tilde{b}$ where $A$ is a $n \times n$ singular crisp matrix and the right-hand side is a fuzzy number vector is called a singular fuzzy linear system of equations. Drazin inverse is one of the generalized inverses. In this paper the effect of Drazin inverse in solving such systems using LU factorization is investigated. Here the computing Drazin inverse and solving singular fuzzy linear systems using MATLAB software are illustrated.*

**Keywords:** *Fuzzy systems; Drazin inverse; Singular systems; Asady's method.*

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## 1 Introduction

The concept of fuzzy numbers and fuzzy arithmetic operations were first introduced by Zadeh [28, 29]. System of simulations linear equations play major mathematics, physics, statistics. engineering and social sciences. One of the major applications using fuzzy number arithmetic is treating linear systems their parameters are all or partially represented by fuzzy numbers. A $n \times m$ linear system whose coefficient matrix is crisp and the right hand side column is an arbitrary fuzzy number vector, is called a fuzzy linear system. Friedman et al. [8] introduced a general model for solving fuzzy linear system. In [3] the original fuzzy linear system

$$A\tilde{x} = \tilde{b},$$

with the nonsingular matrix $A$ is replaced by two $n \times n$ crisp linear systems. A fuzzy linear system where any entry of the coefficient matrix $A$ is a fuzzy number is called a fully fuzzy linear system and is denoted by $\tilde{A}\tilde{x} = \tilde{b}$ [25]. Solving fuzzy linear systems and fully fuzzy linear systems is a current issue in recent years [1, 2, 24, 26, 27].

Unlike the case of the nonsingular matrix, which has a single unique inverse for all purposes, there are different generalized inverses for different purposes. For some purposes, as in the examples of solutions of linear systems, there is not a unique inverse, but any matrix

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of a certain class will do [4]. Drazin inverse is one of the generalized inverses. Drazin inverse has important spectral properties that make it extremely useful in many applications [4, 6]. A special case of the Drazin inverse, is called the group inverse. The Drazin inverse has various applications in the theory of finite Markov chains, the study of singular differentail and difference equations, the investigation of Cesaro-Neumann iterations, cryptograph, iterative methods in numerical analysis, multibody system dynamics and others [4]. In the important paper [5], Cline and Greville extended the Drazin inverse of a square matrix to a rectangular matrix and introduced the notion of W-weighted Drazin inverse. Many properties and applications of the W-weighted Drazin inverse have been discussed later in [15, 16, 17, 18, 19]. Drazin inverse in solving consistent or inconsistent singular linear system [11], singular fuzzy linear system [12], singular constrained linear system [13], singular linear regression [14] is used. The system

$$A\tilde{x} = \tilde{b}$$

with given crisp matrix $A$ and an arbitrary fuzzy number vector $\tilde{b}$, is called a singular fuzzy linear system. Such system is converted to a crisp linear system. In this paper, the effect of Drazin inverse is extended and in solving singular consistent or inconsistent fuzzy linear systems is used.

In section 2, we recall some preliminaries. New results on the singular matrices are given in section 3. Solving consistent or inconsistent singular fuzzy linear systems are investigated in section 4. In section 5 two numerical examples are given.

## 2 Preliminaries

e first present some preliminaries and basic definitions which are needed in this paper. For more details, we refer the reader to ([4, 7, 20, 21, 22, 23]

**Definition 1.** *Let $A \in C^{n\times n}$ . The index of matrix $A$ is equivalent to the dimension of largest Jordan block corresponding to the zero eigenvalue of $A$ and is denoted by $ind(A)$ .*

**Definition 2.** *Let $A \in C^{n\times n}$ , with $ind(A) = k$ . The matrix $X$ of order $n$ is the Drazin inverse of $A$ ,denoted by $A^D$ , if $X$ satisfies the following conditions*

$$AX = XA, \quad XAX = X, \quad A^kXA = A^k,$$

**Theorem 1.** ([6]) *Let $A \in C^{n\times n}$ , with $ind(A) = k$ , $rank(A^k) = r$ . We may assume that the Jordan normal form of $A$ has the form as follows*

$$A = P\begin{pmatrix} D & 0 \\ 0 & N \end{pmatrix} P^{-1},$$

*where $P$ is a nonsingular matrix, $D$ is a nonsingular matrix of order $r$ , and $N$ is a nilpotent matrix that $N^k = \bar{o}$ . Then we can write the Drazin inverse of $A$ in the form*

$$A^D = P\begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix} P^{-1}.$$

*When $ind(A) = 1$ , it is obvious that $N = \bar{o}$ .*

**Example 1.** *Consider the following symmetric matrix*

$$C = \begin{pmatrix} -1 & -1 & -1 \\ -1 & \frac{1}{3} & -1 \\ -1 & -1 & -1 \end{pmatrix}.$$

*The matrix $C$ has the eigenvalues $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = -\frac{8}{3}$. The index of matrix $C$ is equal to one, because $rank(C) = rank(C^2)$ . So Jordan normal form of matrix $C$ has the following form*

$$J = PCP^{-1} = \begin{pmatrix} [1] & 0 & 0 \\ 0 & [-\frac{8}{3}] & 0 \\ 0 & 0 & [0] \end{pmatrix}, P = \begin{pmatrix} \frac{1}{11} & -\frac{3}{11} & \frac{1}{11} \\ -\frac{9}{22} & -\frac{3}{11} & -\frac{9}{22} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}. \tag{1}$$

*$P$ is a nonsingular matrix. The dimension of largest Jordan block corresponding to the zero eigenvalue of (1) is equal to one. By Theorem (1) we have*

$$C_g = P^{-1}JP = \frac{1}{16}\begin{pmatrix} -1 & -6 & -1 \\ -6 & 12 & -6 \\ -1 & -6 & -1 \end{pmatrix}.$$

**Theorem 2.** *([6]) $A^D b$ is a solution of*

$$Ax = b, \quad k = ind(A), \tag{2}$$

*if and only if $b \in R(A^k)$ , and $A^D b$ is an unique solution of (2) provided that $x \in R(A^k)$.*

**Definition 3.** *A fuzzy number $\widetilde{u}$ in parametric form is a pair $(\bar{u}(r), \underline{u}(r))$ of functions $\bar{u}(r), \underline{u}(r), 0 \le r \le 1$, which satisfy the following requirements*

1. *$\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0, 1]$.*
2. *$\bar{u}(r)$ is a bounded left continuous non-increasing function over $[0, 1]$.*
3. *$\underline{u}(r) \le \bar{u}(r), 0 \le r \le 1$.*

**Definition 4.** *Let $A$ be an $n \times n$ matrix be factored as*

$$A = LU,$$

*where $L$ is a lower triangular matrix and $U$ is a upper triangular matrix. This factorization is known as $LU$ factorization.*

If $A$ is factored into $L$ and $U$, a system of equations $Ax = b$ is reduced to the form $LUx = b$. Thus it can be solved by solving two triangular systems: first $Ly = b$ for the unknown $y$, then $Ux = y$ for the unknown $x$.

**Definition 5.** *For arbitrary fuzzy numbers $\widetilde{x} = (\underline{x}(r), \bar{x}(r))$ , $\widetilde{y} = (\underline{y}(r), \bar{y}(r))$ and $k \in R$ , we may define the addition and the scalar multiplication of fuzzy numbers as*

1. $\widetilde{x} + \widetilde{y} = (\underline{x}(r) + \underline{y}(r), \bar{x}(r) + \bar{y}(r))$,
2. $k\widetilde{x} = \begin{cases} (k\underline{x}, k\bar{x}) & k \ge 0 \\ (k\bar{x}, k\underline{x}) & k < 0 \end{cases}$

**Definition 6.** *The system*

$$A\widetilde{x} = \widetilde{b}, \tag{3}$$

*with given crisp matrix $A \in C^{n\times n}$ , $b$ is an arbitrary fuzzy number vector is called a singular fuzzy system. It is shown that*

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_n \end{pmatrix} = \begin{pmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_n \end{pmatrix},$$

*wherein $A = (\alpha_{ij})$ can be extended into a crisp linear system as follows*

$$SX = Y, \tag{4}$$

where $S = (s_{ij})$ are determined as follows :

$$\alpha_{ij} \geq 0 \Rightarrow s_{ij} = \alpha_{ij}, \quad s_{i+n,j+n} = \alpha_{ij},$$
$$\alpha_{ij} < 0 \Rightarrow s_{i,j+n} = -\alpha_{ij}, \quad s_{i+n,j} = -\alpha_{ij},$$

while all the remaining $(s_{ij})$ are taken zero, and

$$X = \begin{pmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ -\bar{x}_1 \\ \vdots \\ -\bar{x}_n \end{pmatrix}, \quad Y = \begin{pmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_n \\ -\bar{y}_1 \\ \vdots \\ -\bar{y}_n \end{pmatrix},$$

and the structure of $S$ implies that $S = (s_{ij}) \geq 0, 1 \leq i \leq 2n$ , $1 \leq j \leq 2n$ and that

$$S = \begin{pmatrix} B & C \\ C & B \end{pmatrix},$$

where $B$ contains the positive entries of $A$ and $C$ contains the absolute value of the negative entries of $A$ , i.e., $A = B - C$ .

**Definition 7.** *The fuzzy linear system (3) is called a singular fuzzy linear system while (4) is a crisp singular linear system of equations.*

**Corollary 1.** *([22]) The fuzzy linear system (3) is consistent if and only if*

$$rank[S] = rank[S|Y].$$

**Definition 8.** *([30]) Let* $X = \{\underline{x}_i(r), -\bar{x}_i(r)), 1 \leq i \leq n\}$ *denote a solution of (4). The fuzzy number vector* $U = \{\underline{u}_i(r), -\bar{u}_i(r)), 1 \leq i \leq n\}$ *defined by*

$$\underline{u}_i(r) = min\{\underline{x}_i(r), \bar{x}_i(r), \underline{x}_i(1), \bar{x}_i(1)\},$$
$$\bar{u}_i(r) = max\{\underline{x}_i(r), \bar{x}_i(r), \underline{x}_i(1), \bar{x}_i(1)\},$$

*is called a fuzzy solution of* $SX = Y$*. If* $(\underline{x}_i(r), \bar{x}_i(r)), 1 \leq i \leq n)$ *, are all fuzzy numbers and* $\underline{x}_i(r) = \underline{u}_i(r), \bar{x}_i(r) = \bar{u}_i(r), 1 \leq i \leq n$ *, then* $U$ *is called a strong fuzzy solution. Otherwise,* $U$ *is a weak fuzzy solution.*

## 3 New results on the singular matrices

In this section, some new results on the Drazin inverse are given.

**Theorem 3.** *Let* $A$ *be a singular matrix, show that*

$$\begin{pmatrix} B & C \\ C & B \end{pmatrix},$$

*is a singular matrix, wherein* $A = B - C$ *.*

**Proof.** The same as the proof of Theorem 1 in [6].

**Corollary 2.** *Let (3) be a fuzzy singular fuzzy linear system. Therefore (4) is a singular crisp linear system.*

**Theorem 4.** *The Drazin inverse of matrix*

$$S = \begin{pmatrix} B & C \\ C & B \end{pmatrix},$$

*is*

$$S^D = \begin{pmatrix} D & E \\ E & D \end{pmatrix}, \tag{5}$$

*where*

$$D = \frac{1}{2}[(B+C)^D + (B-C)^D], \quad E = \frac{1}{2}[(B+C)^D - (B-C)^D].$$

**Proof.** Let $S^D$ be the Drazin inverse of $S$ , it is unique. Without loss generality, suppose that

$$S^D = \begin{pmatrix} D & E \\ E & D \end{pmatrix} ,$$

we know $S^D S = SS^D$, hence

$$\begin{pmatrix} D & E \\ E & D \end{pmatrix} \begin{pmatrix} B & C \\ C & B \end{pmatrix} = \begin{pmatrix} B & C \\ C & B \end{pmatrix} \begin{pmatrix} D & E \\ E & D \end{pmatrix},$$

and get

$$BD + CE = DB + EC \quad , CD + BE = EB + DC. \tag{6}$$

By adding and then by subtracting the two parts of (6), we obtain

$$(B+C)(D+E) = (D+E)(B+C) \quad , (B-C)(D-E) = (D-E)(B-C),$$

also, from

$$\begin{pmatrix} D & E \\ E & D \end{pmatrix} \begin{pmatrix} B & C \\ C & B \end{pmatrix} \begin{pmatrix} D & E \\ E & D \end{pmatrix} = \begin{pmatrix} D & E \\ E & D \end{pmatrix},$$

we get

$$(D+E)(B+C)(D+E) = (D+E) \quad , (D-E)(B-C)(D-E) = (D-E),$$

and by

$$\begin{pmatrix} B & C \\ C & B \end{pmatrix}^k \begin{pmatrix} D & E \\ E & D \end{pmatrix} \begin{pmatrix} B & C \\ C & B \end{pmatrix} = \begin{pmatrix} B & C \\ C & B \end{pmatrix}^k,$$

we have

$$(B+C)^k (D+E)(B+C) = (B+C)^k \quad , (B-C)^k (D-E)(B-C) = (B-C)^k.$$

Thus $S^D$ must have the structure given by (5). In order to calculate $E$ and $D$, we have

$$(B+C)^D = (D+E) \quad , (B-C)^D = (D-E),$$

and consequently we get,

$$D = \frac{1}{2}[(B+C)^D + (B-C)^D], \quad E = \frac{1}{2}[(B+C)^D - (B-C)^D].$$

## 4 Singular fuzzy linear systems

In this section on the singular fuzzy linear systems is discussed.

### *4.1 Shortcomings of the existing methods*

In this subsection, the shortcomings of the existing direct methods for solving fuzzy linear systems are pointed out.

1. Friedman et al. [8] considered the fuzzy linear system $A\tilde{x} = \tilde{b}$ where $A$ is a nonsingular matrix. They converted the $n \times n$ fuzzy linear system (3) into the $2n \times 2n$ crisp linear system (4). They used the ordinary inverse for solving the crisp system (4) while the matix $S$ be a singular matrix.
2. Asady et al. [3] replaced the system $SX = Y$ by two $n \times n$ crisp linear systems $E(\bar{x} - \underline{x}) = (\bar{y} - \underline{y})$ and $A(\bar{x} + \underline{x}) = (\bar{y} + \underline{y})$ when the matrices $S$, $A = B - C$ and $E = B + C$ are nonsingular matrices and solves these systems using ordinary inverse.
3. For solving the fuzzy linear system (3) wherein $A \in C^{n\times n}$ is a nonsingular crisp matrix. Ezzati [34] first solve the following system

$$\begin{cases} a_{11}(\underline{x}_1 + \bar{x}_1) + \cdots + a_{1n}(\underline{x}_n + \bar{x}_n) = & (\underline{y}_1 + \bar{y}_1), \\ a_{21}(\underline{x}_1 + \bar{x}_1) + \cdots + a_{2n}(\underline{x}_n + \bar{x}_n) = & (\underline{y}_2 + \bar{y}_2), \\ \quad \vdots & \quad \vdots \\ a_{n1}(\underline{x}_1 + \bar{x}_1) + \cdots + a_{nn}(\underline{x}_n + \bar{x}_n) = & (\underline{y}_n + \bar{y}_n), \end{cases}$$

and suppose the solution of this system is as

$$d = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} \underline{x}_1 + \bar{x}_1 \\ \vdots \\ \underline{x}_n + \bar{x}_n \end{pmatrix}.$$

Let matrices $B$ and $C$ have defined as definition 6 . Now using matrix notation for (3), He get $A\tilde{x} = \tilde{y}$ or $(B - C)\tilde{x} = \tilde{y}$ and in parametric form

$$(B - C)(\underline{x}(r), \bar{x}(r)) = (\underline{y}(r), \bar{y}(r)).$$

Then he write this system as follows:

$$\begin{cases} B\underline{x}(r) - C\bar{x}(r) = \underline{y}(r), \\ B\bar{x}(r) - C\underline{x}(r) = \bar{y}(r), \end{cases}$$

are equivalent. By substituting of $\bar{x}(r) = d - \underline{x}(r)$ and $\underline{x}(r) = d - \bar{x}(r)$ in the first and second equation of above system, respectively. He get

$$(B + C)\underline{x}(r) = \underline{y}(r) + Cd,$$

and

$$(B + C)\bar{x}(r) = \bar{y}(r) + Cd.$$

If the ordinary inverse of matrix $F = B + C$ exist then, He can solve fuzzy linear system (3) by solving following crisp linear systems

$$\begin{cases} \underline{x}(r) = F^{-1}(\underline{y}(r) + Cd), \\ \bar{x}(r) = F^{-1}(\bar{y}(r) + Cd). \end{cases}$$

4. The authoe et.al [12] formed the normal equation for singular fuzzy linear systems. We used of pseudoinverse and Drazin inverse for solving noemal equations.

The existing methods is incapable to use the Drazin inverse to find a solution for the non-singular fuzzy linear systemm

$$\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} (r, 2-r) \\ (0, 1-r) \end{pmatrix},$$

and the singular fuzzy linear system

$$\begin{pmatrix} \frac{1}{2} & -1 & -\frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} (1+r, 3-r) \\ (r, 2-r) \\ (0, 1-r) \end{pmatrix}.$$

### *4.2 Singular fuzzy Linear systems*

The singular fuzzy linear systems are divided into two parts; consistent and inconsistent. In this subsection, indicial equations of inconsistent singular fuzzy linear systems is introduced and the effect of Drazin inverse in solving consistent or inconsistent fuzzy linear systems are investigated.

**Theorem 5.** *The consistent singular crisp linear system (4) has a set of solutions and $X = S^D Y$ is the element of this set.*

**Proof** . The crisp consistent singular linear system (4) has a set of solutions [9]. The minimal solution is the element of this set having the least Euclidean norm. Therefore from [12]

$$X = S^D Y,$$

is is the element of this set.

**Corollary 3.** *If $S$ is factored into $L$ and $U$ the system of equations (4) is reduced to the form*

$$LUX = Y.$$

*Thus it can be solved by solving two triangular systems: first $LZ = Y$ for the unknown $Z$, then $UX = Z$ for the unknown $x$. Using Drazin inverse $Z = L^D Y$ if and only if*

$$Y \in R(L^l), \quad l = ind(L),$$

*and $X = U^D Z$ if and only if*

$$Z \in R(U^u), \quad u = ind(U).$$

According to [31, 10] and properties of the Drazin inverse [32, 6], in order to obtain the Drazin inverse the projection method solves consistent or inconsistent singular linear system (4) through solving the following indicial equations

$$S^k S X = S^k Y, \quad k = ind(S). \tag{7}$$

Therefore from [10] $X = (S^k S)^D S^k Y$ is solution of (7).

Now, the Asady's method is extended and on the solving inconsistent singular fuzzy linear system is performed.

**Generalized Asady's method.** Let

$$S = \begin{pmatrix} B & C \\ C & B \end{pmatrix}, \quad Y = \begin{pmatrix} \underline{y} \\ -\bar{y} \end{pmatrix}.$$

We obtain the following linear system

$$\begin{cases} B(\underline{x}) + C(-\bar{x}) & = \underline{y}, \\ C(\underline{x}) + B(-\bar{x}) & = -\bar{y}, \end{cases} \tag{8}$$

Which is a crisp linear system. If (4) is consistent, by adding and then subtracting the part of Equation (8), we obtain

$$\begin{cases} (B + C)(\bar{x} - \underline{x}) & = \bar{y} - \underline{y}, \\ (B - C)(\bar{x} + \underline{x}) & = \bar{y} + \underline{y}, \end{cases}$$

We get

$$\begin{cases} E\sigma & = \bar{y} - \underline{y}, \\ A\delta & = \bar{y} + \underline{y}, \end{cases} \tag{9}$$

Wherein , $E = B + C,\ A = B - C$ and $\delta = \bar{x} + \underline{x}$ , $\sigma = \bar{x} - \underline{x}$. By adding and subtracting the two solutions of above systems we obtain

$$\bar{x} = \frac{\sigma + \delta}{2}, \qquad \underline{x} = \frac{\delta - \sigma}{2}.$$

**Corollary 4.** *We can solve the systems (9) using theorem 2.*

### *4.3 MATLAB software*

In this subsection, the algorithms for computing the Drazin inverse of matrix $A$ and solving consistent or inconsistent singular fuzzy linear systems using MATLAB software are given. There is no function in MATLAB software for computing Drazin inverse of matrix $A \in C^{n\times n}$. From [33] we have

$$A^D = A^k (A^{(2k+1)})^{\dagger} A^k,$$

wherein $k = ind(A)$.

**Algorithm 3.1** Computing Drazin inverse of matrix $A \in C^{n\times n}$ using MATLAB software

| | |
|---|---|
| **1 -** | Input $A$ is the n-by-n matrix |
| **2 -** | Input $k$ is the index of matrix $A$ |
| **3 -** | G=pinv(mpower(A,(2*k)+1)) |
| **4 -** | H=mtimes(mpower(A,k),G) |
| **5 -** | D=mtimes(H,mpower(A,k)) |

The MATLAB software incorporates built in functions **pinv** and **mpower** for computing the Moore-Penrose inverse and the matrix power respectively.

**Algorithm 3.2** Sloving consistent singular fuzzy linear system using MATLAB software

| | |
|---|---|
| **1 -** | Input $S$ is the 2n-by-2n matrix |
| **2 -** | Input $k$ is the index of matrix $S$ |
| **3 -** | Input $Y$ is the 2n-vector |
| **4 -** | M=mpower(S,k) |
| **5 -** | D=mtimes(mtimes(M,pinv(mpower(S,(2*k)+1))),M) |
| **6 -** | X=mtimes(D,Y) |

**Algorithm 3.3** Sloving inconsistent singular fuzzy linear system using MATLAB software

| | |
|---|---|
| **1 -** | Input $S$ is the2n-by-2n matrix |
| **2 -** | Input $k$ is the index of matrix $S$ |
| **3 -** | Input $Y$ is the 2n-vector |
| **4 -** | M=mtimes(mpower(S,k),S) |
| **5 -** | D=mtimes(mtimes(M,pinv(mpower(M,(2*k)+1))),M) |
| **6 -** | N=mtimes(mpower(S,k),Y) |
| **7 -** | X=mtimes(D,N) |

## 5 Numerical examples

In this section, the effect of Drazin inverse in solving singular fuzzy linear system are illustrated.

**Example 2.** *Consider the following inconsistent singular fuzzy linear system*

$$\begin{pmatrix} -\frac{1}{2} & -1 & -\frac{3}{2} \\ -1 & -1 & -1 \\ -\frac{3}{2} & -1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} (1+r, 3-r) \\ (r, 2-r) \\ (0, 1-r) \end{pmatrix} \tag{10}$$

*with a given subspace The index of the coefficient matrix of the extended crisp linear system $SX = Y$ of (10) is equal to one. Therefore*

$$X = \begin{pmatrix} \frac{1}{3} + \frac{1}{3}r \\ -\frac{2}{3} + \frac{1}{3}r \\ -\frac{5}{3} + \frac{1}{3}r \\ -\frac{7}{18} - \frac{5}{18}r \\ \frac{1}{9} + \frac{2}{9}r \\ \frac{11}{18} + \frac{13}{18}r \end{pmatrix}.$$

*That is satisfies in the indicial equations $SSX = SY$. We can give a weak fuzzy solution for this system by definition (8).*

**Example 3.** *Consider the following consistent singular fuzzy linear system*

$$\begin{pmatrix} \frac{1}{2} & -1 & -\frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} (1+r, 3-r) \\ (r, 2-r) \\ (0, 1-r) \end{pmatrix} \tag{11}$$

*with a given subspace By generalized Asady's method we can get,*

$$\sigma = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} & -\frac{1}{2} \\ -\frac{5}{4} & \frac{1}{4} & \frac{3}{2} \end{pmatrix}^D \begin{pmatrix} 2-2r \\ 2-2r \\ 1-r \end{pmatrix}$$

$$\delta = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix}^D \begin{pmatrix} 4 \\ 2 \\ 1-r \end{pmatrix}$$

*Therefore*

$$X = \begin{pmatrix} \frac{9}{4} + \frac{3}{4}r \\ -\frac{9}{4} + \frac{5}{4}r \\ \frac{9}{4} - \frac{5}{4}r \\ -\frac{15}{4} + \frac{3}{4}r \\ \frac{3}{4} + \frac{1}{4}r \\ -\frac{7}{4} + \frac{3}{4}r \end{pmatrix}$$

*is a solution of the extended crisp linear system of (11). We can give a weak fuzzy solution for this system by definition (8). The MATLAB software incorporates built in functions* **lu** *for the LR factorization. The statement* $[L, U, P] = lu(A)$ *returns an upper triangular matrix in* $U$*, a lower triangular matrix* $L$ *with a unit diagonal, and a permutation matrix* $P$*, such that* $LU = PS$*. Therefore we give*

$$X = \begin{pmatrix} 4 \\ 2r-4 \\ -2r+4 \\ -2 \\ r-1 \\ 0 \end{pmatrix}$$

*is a solution of the extended crisp linear system of (11). We can give a weak fuzzy solution for this system by definition (8). is a solution of the extended crisp linear system of (11). We can give a weak fuzzy solution for this system by definition (8).*

## 6 Conclusions

There is a difference between the normal equations and indicial equations [11, 12]. In this paper, the indicial equations for inconsistent singular fuzzy linear system is introduced and the effect of Drazin inverse in solving consistent or inconsistent singular fuzzy linear stystems are explained.

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