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The Fixed Points of *b*-Bistochastic-Volterra Quadratic Stochastic Operators On $S^1 \times S^1$

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ABSTRACT

The main focus of this paper is to investigate the simplest non-linear Markov operators which is quadratic one. Study of quadratic stochastic operators (QSOs) is not an easy task as linear operators. Thus, researchers introduced classes of QSOs such as Volterra QSOs, strictly non-Volterra QSOs, Orthogonal preserving QSOs, Centered QSOs and etc. However, all the introduced classes were not yet cover the whole set of QSOs. Thus, we introduce a new class of QSOs, namely b-bistochastic-Volterra QSOs or simply bV-QSOs. In this paper, the canonical form of bV-QSO defined on one dimensional simplex is provided. We note that, the main problem in the nonlinear operator theory is to study their dynamics. Thus, the set of all fixed points of bV-QSOs are then obtained and classified into attracting, repelling, saddle and non-hyperbolic by applying Jacobian matrix. This helps understanding the dynamical behaviours of bV-QSOs.

Keywords: Quadratic Stochastic, Markov operators, b-Bistochastic QSOs, Volterra QSOs

1 INTRODUCTION

A *quadratic stochastic operators* (in short QSOs) was originally introduced by [1] which usually arise from the problems of population genetics (see also [2]). QSOs become one of the main sources of analysis in studying dynamical properties and modelling in a system which requires many interactions. For the sake of comprehension, let us consider the following biological ambiance. Assume that each individual in this population belongs precisely to one of the species (trait) which denoted by $I = 1,2,\dots,n$. The probability of an individual is denoted by $P_{ij,k}$, where an individual in i^{th} species and j^{th} species to cross-fertilize and produce an individual from k^{th} species. These coefficients $P_{ij,k}$ are known as *heredity coefficients* which define a QSO V. Provided initial probability distribution of the species, $x^{(0)} = (x_1^{(0)},\dots,x_n^{(0)})$, the probability distribution of the first generation, $x^{(1)} = (x_1^{(1)},\dots,x_n^{(1)})$ can be found by applying the QSO as a total probability that is,

$$x_k^{(1)} = \sum_{i,j=1}^{n} P_{ij,k} x_i^{(0)} x_j^{(0)} =: V(\mathbf{x})_k, \qquad for\ any \quad k \in \{1,\dots,n\}.$$

Generally, the operator, V starts from the initial state of probability distribution $x^{(0)}$ in a population, then it describes the evolution of the probability distribution of the first generation, $x^{(1)} = V(x^{(0)})$, second generation, $x^{(2)} = V(x^{(1)}) = V(V(x^{(0)}) = V^{(2)}(x^{(0)})$ and iterates continuously. These states of population,

$$x^{(0)}, \quad x^{(1)} = V(x^{(0)}), \quad x^{(2)} = V^2(x^{(0)}), \ldots \quad x^{(n)} = V^n(x^{(0)}),$$

define a dynamical system.

Studying QSOs in general is challenging unlike the linear case. Therefore classes of QSOs were introduced by researchers such as QSOs on Banach Lattices, Volterra QSOs, b-bistochastic QSOs, centered QSOs, Orthogonal preserving QSOs, Lebesgue QSOs, QSOs corresponding to permutations and etc (for example see ([3], [4], [5], [6], [7], [8], [9]). However these classes do not yet cover the whole set of QSO. The introduction of this new class of QSO is to contribute the knowledge in the theory of non-linear operator. The book by [10] serves a comprehensive reference in the theory of QSOs. Recent acheivement of QSOs could be further read ([11], [12], [13]) and the references therein.

The concept of majorization was first introduced by Lorenz in [14] and further investigate by Hardy et al. in [15]. Later on, a new order called majorization was then introduced in [16] by referring the majorization that was popularized by [15] as classical majorization. This new order majorization generalize the classical majorization. Besides, it is indeed a partial order on sequence which is an advantage compared to classical majorization. In this paper, we consider majorization as b-order which is denoted as $\leq^b$. A QSO, namely bistochastic QSO, also called as doubly stochastic is defined in terms of classical majorization [17], where $V(\mathbf{x}) \prec \mathbf{x}$, for all $\mathbf{x}$ from $n-1$ dimensional simplex.

Most well-studied class of QSOs is known as Volterra QSO, V. Biological meaning of this operator is that: *the child could inherit the trait from their parents only*. In the study of the Volterra dynamical systems (acting on finite dimensional simplex) for a given biological population, the following question may arise: *what kind of genotypes will preserve and which of them will disappear?* There are many papers published on the investigations of discrete Volterra operators([18], [19], [20]). We note that most of the studies in the theory of QSOs were done by considering V that maps from S^{n-1} into itself.

Rozikov and Zhamilov extend the domain of mapping V from $S^{n-1} \times S^{n-1}$ to itself where they considered V as Volterra QSOs [21]. Motivated from these ideas, a new class of QSOs on $S^{n-1} \times S^{n-1}$ is introduced, namely b-bistochastic-Volterra QSOs, or simply bV-QSO. This paper is organized as follows: In section 2, we introduced required definition and preliminaries results. The canonical form of operator bV-QSO is developed by applying the properties of b-bistochastic QSO and Volterra QSO simultaneously in Section 3. Section 4 is devoted to the description of all fixed point and their stability properties.

2 PRELIMINARIES

This section will briefly explain the required definitions and preliminaries results. Let S^{n-1} be the set of all probability distribution i.e,

$$S^{n-1} = \left\{ \mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n | x_i \geq 0, \sum_{i=1}^{n} x_i = 1 \right\}, \tag{1}$$

where $n \in \mathbb{N}^* = \{1,2, \dots, n\}$. We simply call S^{n-1} as a *simplex.* Next, a mapping V on S^{n-1} is defined by

$$V(\mathbf{x})_k = \sum_{i,j=1}^{n} P_{ij,k} x_i x_j, \qquad k = \overline{1,n}$$

where $P_{ij,k}$ are heredity coefficients. The properties of heredity coefficients, $P_{ij,k}$ include non-negativity, symmetrical and stochasticity i.e.,

$$P_{ij,k} \geq 0, \qquad P_{ij,k} = P_{ji,k}, \qquad and \ \sum_{k=1}^{n} P_{ij,k} = 1, \tag{2}$$

respectively, where $ij, k = 1,2, \dots, n$ and $n \in \mathbb{N}^*$. Such mapping V is called Quadratic Stochastic Operators (QSOs). Recall that, b-order was introduced as follows [5]:

Definition 1. [5] Let us define functional $u_k: \mathbb{R}^n \to \mathbb{R}$,

$$u_k(x_1, \dots, x_n) = \sum_{i=1}^{k} x_i,$$

where $k = 1,2, \dots, n-1$. *Let* $\mathbf{x}, \mathbf{y} \in S^{n-1}$, *we said that* $\mathbf{x}$ *is b-ordered by* $\mathbf{y}$ *if,*

$$\mathbf{x} \leq^b \mathbf{y} \Leftrightarrow u_k(\mathbf{x}) \leq u_k(\mathbf{y}), \qquad for\ all\ k = 1,2, \dots, n-1.$$

Remark 1. The relation is partial order which satisfies the following conditions:

i. *For any* $\mathbf{x} \in S^{n-1}, \mathbf{x} \leq^b \mathbf{x}$,
ii. *if* $\mathbf{x} \leq^b \mathbf{y}$ *and* $\mathbf{y} \leq^b \mathbf{x}$, *then* $\mathbf{x} = \mathbf{y}$,
iii. *if* $\mathbf{x} \leq^b \mathbf{y}$ *and* $\mathbf{y} \leq^b \mathbf{z}$, *then* $\mathbf{x} \leq^b \mathbf{z}$.

Moreover, it has the following properties:

i. $\mathbf{x} \leq^b \mathbf{y}$ *if and only if* $\lambda\mathbf{x} \leq^b \lambda\mathbf{y}$, *for any* $\lambda > 0$,
ii. *if* $\mathbf{x} \leq^b \mathbf{y}$ *and* $\lambda \leq \mu$, *then* $\lambda\mathbf{x} \leq^b \mu\mathbf{y}$.

Remark 2. From previous remark, the majorization can be defined as following [5]:

For any $\mathbf{x} = (x_1, \dots, x_n) \in S^{n-1}$,

$\mathbf{x}_{[\downarrow]} = \left(x_{[1]}, x_{[2]}, \dots, x_{[n]}\right)$,

where $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ *is non-increasing rearrangement of* $\mathbf{x}$*. Let* $\mathbf{x}, \mathbf{y} \in S^{n-1}$*, then* $\mathbf{x}$ *is majorized by* $\mathbf{y}$ *(or* $\mathbf{y}$ *majorates* $\mathbf{x}$*) which denote as* $\mathbf{x} \prec \boldsymbol{y}$ *(or* $\mathbf{y} \prec \mathbf{x}$*) if* $\mathbf{x}_{[\downarrow]} \leq^b \mathbf{y}_{[\downarrow]}$*. Note that b-order does not require non-increasing rearrangement of* $\mathbf{x}$ *or in other words,* $\mathbf{x}$ *preserves the order.*

Hence, b-order is a generalization of the concept majorization. Moreover, not all elements in the simplex are somparable in terms of b-order, for instance, take $x = \left(\frac{1}{4}, \frac{3}{4}, 0\right)$ and $y = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, we get $\frac{1}{4} \leq \frac{1}{3}$ but $\frac{1}{4} + \frac{3}{4} \geq \frac{1}{3} + \frac{1}{3}$.

Example 1. *Let* $x = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$ *and* $y = (0,0,1)$*. From the terms of majorization mentioned, we have*

$$x_{[\downarrow]} = \left(\frac{2}{3}, \frac{1}{3}, 0\right) \prec y_{[\downarrow]} = (1,0,0).$$

Whereas by definition of b-order, we obtain

$$y = (0,0,1) \leq^b x = \left(\frac{1}{3}, \frac{2}{3}, 0\right).$$

Definition 2. [5] *Let* V *be a QSO defined on* S^{n-1}*, then* V *is called a b-bistochastic if*

$$V(\mathbf{x}) \leq^b \mathbf{x}, \quad for\ all \quad \mathbf{x} \in S^{n-1},$$

where S is simplex and $n \in \mathbb{N}^*$.

The properties and dynamics of b-bistochastic QSOs were intensively studied in ([5], [22], [23], [24], [25]). The most common and well known class of QSO is Volterra QSO. A QSO, $V: S^{n-1} \to S^{n-1}$ is called Volterra QSO if

$$P_{ij,k} = 0 \quad \text{for} \quad k \notin \{i, j\}, \tag{3}$$

where $P_{ij,k}$ is heredity coefficient, and for all $i, j, k \in \mathbb{N}^*$. Let $V(\mathbf{x})_k$ and $V: S^{n-1} \to S^{n-1}$ be a Volterra QSO, and $V(\mathbf{x})_k = \mathbf{x}'_k$, taking into account equation (3), we can write Volterra in the following form:

$$x'_k = x_k \left(1 + \sum_{i=1}^{n} a_{ki} x_i\right), \quad k \in I,$$

where $a_{ki} = 2P_{ik,k} - 1$, for $i \neq k$, $a_{ki} = -a_{ik}$, and $|a_{ki}| \leq 1$.

Now, define an operator V that maps from $S^{n-1} \times S^{v-1}$ where $P_{ij,k}^{(f)}$ and $P_{ij,l}^{(m)}$ are its coefficients of inheritance. In biological point of view, these coefficients respresent as the probability of a female offspring being type k and, respectively a male offspring being type l, when the parental pair is i, j $(i, k = 1,2, \dots, n$; and $j, l = 1,2, \dots, v)$. We have

$$P_{ij,k}^{(f)} \geq 0, \quad \sum_{k=1}^{n} P_{ij,k}^{(f)} = 1, \quad P_{ij,k}^{(m)} \geq 0, \quad \sum_{k=1}^{n} P_{ij,k}^{(m)} = 1. \tag{4}$$

Using these coefficients, we can define the operator V as follows: Let $\mathbf{x} = (x_1, \dots, x_n) \in S^{n-1}$ and $\mathbf{y} = (y_1, \dots, y_n) \in S^{v-1}$

$$V(\mathbf{x}, \mathbf{y}) = \begin{cases} V_{\mathbf{x}} = \left(\sum_{i,j=1}^{n,v} P_{ij,k}^{(f)} x_i y_j \right)_{k=1}^{n} \\ \\ V_{\mathbf{y}} = \left(\sum_{i,j=1}^{n,v} P_{ij,l}^{(m)} x_i y_j \right)_{l=1}^{v} \end{cases}. \tag{5}$$

One may check that $V_{\mathbf{x}}$ and $V_{\mathbf{y}}$ is stochastic, i.e., V maps from $S^{n-1} \times S^{v-1}$ into itself. In this paper, we limited ourselves to $n = 2, v = 2$. Hence, in addition to (4), without the loss of generality, we may assume symmetrical property of heredity coefficients. If not then, we may define

$$\tilde{P}_{ij,k} = \frac{P_{ij,k} + P_{ji,k}}{2},$$

in which symmetriness of $\tilde{P}$ is satisfied.

Definition 3 *A QSO V is called bV-QSO if*

$$V_{\mathbf{x}} \leq^{b} \mathbf{x},$$

and the heredity coefficients for $V_{\mathbf{y}}$ satisfy $P_{ij,k}^{(m)} = 0$ for any $k \notin \{i, j\}$.

3 DESCRIPTION OF BV-QSO ON 1D SIMPLEX

This section aims to give a full description bV-QSOs on $S^1 \times S^1$. One can see that if $n = 2$, then the simplex is reduced to:

$$S^1 = \{x = (x_1, x_2) \in \mathbb{R}^2 | x_1, x_2 \geq 0, x_1 + x_2 = 1\}. \tag{6}$$

Theorem 1 *Let V be a QSO defined on $S^1 \times S^1$. The operator V is a bV-QSO if and only if*

$$V(\mathbf{x},\mathbf{y}) = \begin{cases} x' = axy \\ y' = xy + b(x - 2xy + y), \end{cases} \tag{7}$$

where $a = \overset{(f)}{P}_{11,1}$, $b = \overset{(m)}{P}_{12,1} = \overset{(m)}{P}_{21,1}$, and $\overset{(m)}{P}_{11,1} = 1$.

Proof. Using (5), then one gets

$$V(\mathbf{x},\mathbf{y}) = \begin{cases} x'_1 = \overset{(f)}{P}_{11,1}x_1y_1 + \overset{(f)}{P}_{12,1}x_1y_2 + \overset{(f)}{P}_{21,1}x_2y_1 + \overset{(f)}{P}_{22,1}x_2y_2 \\ x'_2 = \overset{(f)}{P}_{11,2}x_1y_1 + \overset{(f)}{P}_{12,2}x_1y_2 + \overset{(f)}{P}_{21,2}x_2y_1 + \overset{(f)}{P}_{22,2}x_2y_2 \\ y'_1 = \overset{(m)}{P}_{11,1}x_1y_1 + \overset{(m)}{P}_{12,1}x_1y_2 + \overset{(m)}{P}_{21,1}x_2y_1 + \overset{(m)}{P}_{22,1}x_2y_2 \\ y'_2 = \overset{(m)}{P}_{11,2}x_1y_1 + \overset{(m)}{P}_{12,2}x_1y_2 + \overset{(m)}{P}_{21,2}x_2y_1 + \overset{(m)}{P}_{22,2}x_2y_2. \end{cases} \tag{8}$$

First, we assume that V is a bV-QSO. From (6) and the properties of symmetrical we have $x_1 + x_2 = 1$ and $P_{12,1} = P_{21,1}$. Therefore, (8) can be simplified as below,

$$V(\mathbf{x},\mathbf{y}) = \begin{cases} x' = \overset{(f)}{P}_{11,1}xy + \overset{(f)}{P}_{12,1}(x(1-y) + (1-x)y) + \overset{(f)}{P}_{22,1}((1-x)(1-y)) \\ y' = \overset{(m)}{P}_{11,1}xy + \overset{(m)}{P}_{12,1}(x(1-y) + (1-x)y) + \overset{(m)}{P}_{22,1}((1-x)(1-y)). \end{cases} \tag{9}$$

Consider the equation of x' which described by the properties of b-bistochastic QSO. From Definition 3, we get

$$\begin{aligned} \overset{(f)}{P}_{11,1}xy + \overset{(f)}{P}_{12,1}(x(1-y) + (1-x)y) + \overset{(f)}{P}_{22,1}((1-x)(1-y)) &\leq x \\ xy(\overset{(f)}{P}_{11,1} - 2\overset{(f)}{P}_{12,1} + \overset{(f)}{P}_{22,1}) + x(\overset{(f)}{P}_{12,1} - \overset{(f)}{P}_{22,1} - 1) + y(\overset{(f)}{P}_{12,1} - \overset{(f)}{P}_{22,1}) + \overset{(f)}{P}_{22,1} &\leq 0. \end{aligned} \tag{10}$$

To satisfy the above equation, we know that $\overset{(f)}{P}_{ij,k}, x, y \in [0,1]$ for $i,j,k = 1,2$. Now, let $x = 0$ and $y = 0$, we then have from (10), $\overset{(f)}{P}_{22,1} \leq 0$, which implies $\overset{(f)}{P}_{22,1} = 0$. Next, let $x = 0$ and $y = 1$, then (10) becomes $\overset{(f)}{P}_{12,1} + \overset{(f)}{P}_{22,1} - \overset{(f)}{P}_{22,1} \leq 0$, which implies $\overset{(f)}{P}_{12,1} = 0$. Then, let $x = 1$ and $y = 1$, $\overset{(f)}{P}_{11,1} - 1 \leq 0$, implies for any $\overset{(f)}{P}_{11,1} \in [0,1]$ satisfy the equation.

Therefore, we can conclude that for b-bistochastic part, $x' = axy$, where $a = P_{11,1}^{(f)}$.

Next, we consider the equation of y'. From (3), we have, $P_{11,2}^{(m)} = P_{22,1}^{(m)} = 0$. Then, by applying the properties of stochasticity and symmetrical in (2) we obtain $P_{11,1}^{(m)} = 1$, *and* $P_{12,1}^{(m)} = P_{21,1}^{(m)}$.

Therefore, we can conclude that for Volterra part, $y' = xy + b(x - 2xy + y)$.

Corollary 1 *Let V be a bV-QSO defined on $S^1 \times S^1$. Then, the following properties hold:*

i. $P_{12,1}^{(f)} = P_{21,1}^{(f)} = P_{22,1}^{(f)} = 0$.
ii. $P_{11,2}^{(m)} = P_{22,1}^{(m)} = 0$.
iii. $P_{11,1}^{(m)} = 1$.

4 FIXED POINT

Theorem 2 *Let V be bV-QSO defined on $S^1 \times S^1$, then one has the following statements:*

i. (0,0) is always the fixed point.
ii. If $a < 1$ and $b = 1$, $(0, y)$ is the fixed point for any $y \in (0,1]$.
iii. If $b < 1$, $a = 1$, then (1,1) is the fixed points.
iv. If $a = 1$ and $b = 1$, then $(0, y)$ and $(x, 1)$ are the fixed points.

Proof. We first prove i., substitute $x = 0$ and $y = 0$ into (7), we obtain

$$\begin{aligned} x' &= 0, \\ y' &= 0. \end{aligned}$$

Next to prove ii., we equate

$$axy = x, \tag{11}$$

$$xy + b(x - 2xy + y) = y. \tag{12}$$

From (11), we have $axy - x = 0$, which implies

$$x(ay - 1) = 0. \tag{13}$$

This equation can be divided into two cases which are:

Case 1: $x = 0$. Let $x = 0$, taking into account Equation (12), we obtain

$$\begin{aligned} &by = y, \\ &y(b - 1) = 0. \end{aligned} \tag{14}$$

If $b = 1$, then $(0, y)$ is fixed point for any $y \in (0,1]$ hence prove ii.. If $b \neq 1$ then $y = 0$ which implies $(0,0)$ is the only fixed point.

Case 2: $ay = 1$. Since $a, y \in [0,1]$, this implies $a = 1$, $y = 1$. Substitute them in to Equation (12), we obtain

$$\begin{aligned} x + b(x - 2x + 1) &= 1 \\ x - bx + b &= 1 \\ x(1 - b) &= 1 - b \\ x &= 1. \end{aligned}$$

Therefore, the fixed point is $(1,1)$, hence prove iii..

Next, to prove iv., let $a = 1$ and $b = 1$, then we have below equations

$$x(y - 1) = 0. \tag{15}$$

$$x(1 - y) = 0. \tag{16}$$

From (15), there are two cases which are:

Case 1: For $x = 0$, taking into account (16), then $(0, y)$ is the fixed point.

Case 2: For $y = 1$, substitute in (16) then $(x, 1)$ is the fixed point.

The following corollary is obtained as follows:

Corollary 2 *Let V be bV-QSO. If $a < 1$ and $b < 1$, then $(0,0)$ is a unique fixed point.*

Proof. Let $a < 1$ and $b < 1$. From Equation (13), we have only one case which is

$x = 0.$

This is because $ay = 1$ implies $y = \frac{1}{a}$, since $a < 1$ from assumption, then $y > 1$ which is a contradiction.

Next, substitute $x = 0$ into Equation (12), then we obtain from equation (14) which implies

$y = 0.$

Another case $b - 1 = 0$ is not true since from assumption $b < 1$. Thus, $(0,0)$ is the only fixed point.

Next, we want to study the stability of the fixed points. Consider the Jacobian matrix of the operator (7) at a fixed point (x, y):

$$J_V(x, y) = \begin{bmatrix} \mathrm{ay} & ax \\ y + b - 2by & x - 2bx + b \end{bmatrix}.$$

Then find the modulus $|J - \lambda I| = 0$,

$$\begin{aligned}|J - \lambda I| &= \left|\begin{bmatrix} ay - \lambda & ax \\ y + b - 2by & (x - 2bx + b) - \lambda \end{bmatrix}\right| \\ &= |\lambda^2 + \lambda(2bx - ay - x - b) + aby - abx|.\end{aligned}$$

The following are the lists of the eigenvalues associated to the Jacobian matrix at the fixed points stated in Theorem 2. Then, we classify the fixed points into attracting, repelling or saddle.

i. For the fixed point (0,0), we have $\lambda_1(\lambda_2 - b) = 0$. Then, we have two cases.

a. For $b < 1$ we have $\lambda_1 < 1, \quad \lambda_2 < 1$ which implies attracting fixed point.
b. For $b = 1$, we have $\lambda_1 < 1, \quad \lambda_2 = 1$ which implies repelling.

ii. For the fixed point $(0, y)$ for $y \in [0,1]$. Since $b = 1$, we have $\lambda_{1,2} = \frac{(ay+1)\pm,(ay-1)}{2}$. There are also two cases.

a. For $ay < 1$ we have $\lambda_1 < 1, \quad \lambda_2 = 1$ which implies attracting fixed point.
b. For $ay = 1$, we have $\lambda_1 = 1, \quad \lambda_2 = 1$ which implies non-hyperbolic.

iii. For the fixed point (1,1). Since $a = 1$, we have $\lambda_1(\lambda_2 + b - 2) = 0$. There are also two cases.

a. For $b < 1$ we have $\lambda_1 < 1, \quad \lambda_2 > 1$ which implies saddle fixed point.
b. For $b = 1$, we have $\lambda_1 < 1, \quad \lambda_2 = 1$ which implies repelling.

iv. For the fixed point $(x, 1)$. Since $a = 1$ and $b = 1$ we have $\lambda_{1,2} = \frac{(2-x)\pm(x)}{2}$, where we obtain $\lambda_1 = 1, \quad \lambda_2 < 1$ which implies non-hyperbolic fixed point.

5 CONCLUSION

The study of non-linear Markov operator specifically quadratic stochastic operators (QSOs) are tricky in general setting. Therefore, many classes of QSOs were introduced. This paper introduces a new class of QSO namely b-bistochastic-Volterra QSO (bV-QSO) defined on one-dimensional simplex. We give full description of the considered class of QSO. By using the canonical form, we are able to list all fixed points. Then, we study the stability of all fixed points. We note that, a QSO could have various dynamical behavior such as non-ergodic, periodic and regular. A quick example, one can see that Lotka-Volterra system has periodic behavior. Thus, our result here is useful to study the dynamical behavior in which will be done in another work.

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REFERENCES

[1] S. N. Bernstein, "Solution of a mathematical problem related to the theory of inheritance," *Uch. Zap. n.-i. kaf. Ukrainy,* vol. 1, pp. 83-115, 1924.

[2] Y. I. Lyubich, E. Akin, A. E. Karpov, and D. Vulis, *Mathematical structures in population genetics*, vol. 22. Berlin: Springer, 1992.

[3] M. Badocha and W. Bartoszek, "Quadratic stochastic operators on Banach lattices," *Positivity,* vol. 22, no. 2, pp. 477-492, 2018.

[4] R. Ganikhodzhaev, F. Mukhamedov, and U. Rozikov, "Quadratic stochastic operators and processes: results and open problems," *Infinite Dimensional Analysis, Quantum Probability and Related Topics,* vol. 14, no. 2, pp. 279-335, 2011.

[5] F. Mukhamedov and A. F. Embong, "On b-bistochastic quadratic stochastic operators," *Journal of Inequalities and Applications,* no. 1, pp. 1-16, 2015.

[6] K. Bartoszek, J. Domsta, and M. Pułka, "Weak stability of centred quadratic stochastic operators," *Bulletin of the Malaysian Mathematical Sciences Society,* vol. 42, no. 4, pp. 1813-1830, 2019.

[7] F. Mukhamedov and M. H. M. Taha, "On orthogonality preserving quadratic stochastic operators," in *AIP Conference Proceedings*, vol. 660, p. 050057, 2015.

[8] S. N. Karim, N. Z. A. Hamzah, and N. Ganikhodjaev, "New Class of Lebesgue Quadratic Stochastic Operators On Continual State Space," *International Journal of Allied Health Sciences,* vol. 5, no. 1, 2021.

[9] U. U. Jamilov, K. O. Khudoyberdiev, and M. Ladra, "Quadratic operators corresponding to permutations," *Stochastic Analysis and Applications,* vol. 38, no. 5, pp. 929-938, 2020.

[10] F. Mukhamedov and N. Ganikhodjaev, "Quantum Quadratic Stochastic Operators," in *Quantum Quadratic Operators and Processes*, Springer, 2015, pp. 85-101.

[11] M. Saburov and K. Saburov, "Ganikhodjaev's Conjecture on Mean Ergodicity of Quadratic Stochastic Operators," *Lobachevskii Journal of Mathematics,* vol. 41, no. 6, pp. 1014-1020, 2020.

[12] R. Abdulghafor, S. Almotairi, H. Almohamedh, B. Almutairi, A. Bajahzar, and S. S. Almutairi, "A nonlinear convergence consensus: Extreme doubly stochastic quadratic operators for multi-agent systems," *Symmetry,* vol. 12, no. 4, p. 540, 2020.

[13] S. B. Abdurakhimova and U. A. Rozikov, "Dynamical system of a quadratic stochastic operator with two discontinuity points," *arXiv preprint arXiv:2103.14834,* 2021.

[14] M. O. Lorenz, "Methods of measuring the concentration of wealth," *Publications of the American statistical association,* vol. 9, no. 70, pp. 209-219, 1905.

[15] G. H. Hardy, J. E. Littlewood, and G. Pólya, "Inequalities Cambridge University Press," *Cambridge, England,* p. 89, 1952.

[16] D. S. Parker and P. Ram, *Greed and majorization*. Los Angeles, CA: Computer Science Dept., University of California, 1996.

[17] R. N. Ganikhodzhaev, "On the definition of bistochastic quadratic operators," *Russian Mathematical Surveys,* vol. 48, no. 4, p. 244, 1993.

[18] F. Mukhamedov, O. Khakimov, and A. F. Embong, "On omega limiting sets of infinite dimensional Volterra operators," *Nonlinearity,* vol. 33, no. 11, p. 5875, 2020.

[19] U. A. Rozikov and S. K. Shoyimardonov, "Ocean ecosystem discrete time dynamics generated by ℓ-Volterra operators," *International Journal of Biomathematics,* vol. 12, no. 2, p. 1950015, 2019.

[20] R. N. Ganikhodzhaev, "Quadratic stochastic operators, Lyapunov functions, and tournaments," *Sbornik: Mathematics,* vol. 76, no. 2, p. 489, 1993.

[21] U. A. Rozikov and U. U. Zhamilov, "Volterra quadratic stochastic operators of a two-sex population," *Ukrainian Mathematical Journal,* vol. 63, no. 7, pp. 1136-1153, 2011.

[22] F. Mukhamedov and A. F. Embong, "On stable b-bistochastic quadratic stochastic operators and associated non-homogenous Markov chains," *Linear and Multilinear Algebra,* vol. 66, no. 1, pp. 1-21, 2018.

[23] F. Mukhamedov and A. F. Embong, "On mixing of Markov measures associated with b-bistochastic QSOs," in *AIP Conference Proceedings*, vol. 1739, p. 020090 2016.

[24] F. Mukhamedov and A. F. Embong, "Extremity of\(b\)-bistochastic Quadratic Stochastic Operators on 2D Simplex," *Malaysian Journal of Mathematical Sciences,* vol. 11, no. 2, pp. 119-139, 2017.

[25] F. Mukhamedov and A. F. Embong, "b- bistochastic quadratic stochastic operators and their properties," *Journal of Physics: Conference Series*, vol. 697, p. 012010, 2016.