Applied Mathematics and Computational Intelligence
Volume 10, No.1, Dec 2021 [18-31]

Behaviour Of Solutions Of The Volterra Integro-Differential Equations Using Numerical Methods

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Received: 12 November 2020; Accepted: 9 February 2021; Available online: 12 April 2021

ABSTRACT

This paper is interested in solving Volterra integro-differential equation with associated initial condition. The Implicit Euler method was used to approximate the derivatives and the Explicit Euler method to approximate the integral as the numerical schemes. The result shows that the long term qualitative behaviour of the solutions obtained from the schemes with different choices of λ and step-sizes of h values are independent of the parameters. Finally, the comparison of the exact solution of the original integro-differential equation (1) with the numerical schemes shows that the qualitative behaviour of the solutions are all the same.

Keywords: Volterra, Integro-differential equation, explicit method, implicit method and numerical method.

1 INTRODUCTION

The work here is to study the use of different numerical methods in solving Volterra integro-differential equations. Considering the volterra integro-differential equation and its associated initial condition of the form

$$y'(t) = -\int_0^t e^{-\lambda(t-s)} y(s)\,ds, y(0) \tag{1}$$

And λ is a constant.

It also study the different behaviour of solutions. The equation (1) above are use in many applications when the behaviour of the system does not depend only on the present state but completely on its entire history. Most of this studies are related to environmental modelling, such as models of evolution, population, pollution as well as the physical sciences and model equation from engineering [1]. Volterra studied the hereditary influences when he was examining a Population growth models. Hence, Volterra integro-differential equation could be seen as an equation having both differential and integral operators [12,15,10].

The origins of theory and application of integro-differential equations which could be traced back to the work of Volterra (in his first paper on the subject [4,13] he also introduced the name for these functional equations). While his first discoveries dealt with certain partial integro-differential equations arising in the theory of elasticity and hysteresis [5,14]. The solution of systems of integral equations occurring in Physics, biology and engineering are based on numerical methods such as the Euler-Chebyshev and Runge-Kutta methods. In recent years, the systems of integral and integro-differential equations have been solved using the homotopy perturbation and efficient algorithm methods [6,9], the Modified homotopy perturbation method and the differential transformation methods [7,8], the Tau and the variational iteration methods [8,3].

The aim of this paper shall be on assessing the long term qualitative behaviour of the solutions for different choices of λ and step size of h to demonstrate different types of qualitative behaviour in solutions obtained from the methods. These shall be compared with the theoretical solution of (1). The equation (1) will be transformed into an equivalent Volterra integral equation of second kind. Some linear equations can best be understood using both analytical and numerical solutions and these provide the key to understanding the nonlinear problems [2]. Therefore, it will also be seen how the general Θ-method for the integral equation in [1], could be derived and shown how its equivalent to an iterative process but that its not dependent upon θ [1].

2 DIFFERENCE EQUATIONS FOR SOLUTIONS TO A VOLTERRA INTEGRO-DIFFERENTIAL EQUATION

Rewriting equation (1) in the form.

$$\frac{dy}{dx} = f(t, y(t), z(t)), z(t) = \int_0^t K(t, s, y(s))ds, K(t, s, y(s)) = -e^{-\lambda(t-s)}y(s)$$

and then applying the general θ- method to both the derivative and the integral gives

$$\frac{dy}{dx} = f(t, y(t), z(t)) \quad \Rightarrow \frac{y_{n+1} - y_n}{h} = \theta f((nh, y_n, z_n) + (n+1)h, y_{n+1}, z_{n+1}), z(t) = \int_0^t k(t, s, y(s))ds$$

$$\Rightarrow z_n = h\left[\vartheta k(nh, 0, y_0) + \sum_{j=1}^{n-1} k(nh, jh, y_j) + (1-\vartheta)k(nh, jh, y_n) \right] \tag{2}$$

$\forall t_j = jh, y_j = y(t_j), z_j = z(t_j)$. Applying (2) to (1), we have,

$$\frac{y_{n+1} - y_n}{h} = \theta f((nh, y_n, z_n) + (1-\theta)(n+1)h, y_{n+1}, z_{n+1}),$$

$$z_n = -h\left[\vartheta e^{-\lambda nh} + \sum_{j=1}^{n-1} e^{-\lambda(n-j)h} y_j (1-\vartheta) y_n \right] \tag{3}$$

$$z_{n+1} = -h\left[\vartheta e^{-\lambda(n+1)h} + \sum_{j=1}^{n} e^{-\lambda(n+1-j)h} y_j (1-\vartheta) y_{n+1}\right]$$

2.1 Explicit Euler Method

The derivative and the integral equations can both be approximated using Explicit Euler method. When we use $\theta = 1$ and $\vartheta = 1$, the explicit Euler method could be apply to approximate the derivative and the integral as follows

$$\frac{y_{n+1} - y_n}{h} = f(nh, y_n, z_n) = z_n = -h\left(e^{-\lambda nh} + \sum_{j=1}^{n-1} e^{-\lambda(n-j)h} y_j\right)$$

$$\Rightarrow y_{n+1} = y_n - h^2\left(e^{-\lambda nh} + \sum_{j=1}^{n-1} e^{-\lambda(n-j)h} y_j\right)$$

We have the following difference equation, which employs the explicit Euler method to approximate both the derivative and the integral.

$$y_{n+1} = y_n - h^2 e^{-\lambda nh} - \sum_{j=1}^{n-1} h^2 e^{-\lambda(n-j)h} y_j.$$

If we use this iteration to find $y_{n+2} - y_{n+1}e^{-\lambda h}$, we obtain the following result,

$$\begin{aligned} y_{n+2} - e^{-\lambda h} y_{n+1} &= y_{n+1} - h^2 e^{-\lambda h(n+1)} - \sum_{j=1}^{n} h^2 e^{-\lambda(n+1-j)h} y_j - e^{-\lambda h} y_n + h^2 e^{-\lambda h(n+1)} \\ &+ \sum_{j=1}^{n-1} h^2 e^{-\lambda(n+1-j)} y_j \\ &= y_{n+1} - h^2 e^{-\lambda h(n+1)} - \sum_{j=1}^{n-1} h^2 e^{-\lambda(n+1-j)} y_j - h^2 e^{-\lambda h} y_n - e^{-\lambda h} y_n + h^2 e^{-\lambda h(n+1)} \\ &+ \sum_{j=1}^{n-1} h^2 e^{-\lambda(n+1-j)} y_j \\ &= y_{n+1} - e^{-\lambda h} y_n - h^2 e^{-\lambda h} y_n \end{aligned}$$

$$y_{n+2} - y_{n+1}\left(e^{-\lambda h} + 1\right) + y_n e^{-\lambda h}(1 + h^2) = 0 \tag{4}$$

2.2 Implicit Euler Method

The use of Implicit Euler method to approximate derivative and explicit Euler method employed to approximate the integral.

When we use $\theta = 0$ and $\vartheta = 1$, the explicit Euler method could be apply to approximate the derivative and the integral as follows

$$\frac{y_{n+1}-y_n}{h} = f\left((n+1)h, y_{n+1}, z_{n+1}\right), z_{n+1} = -h\left(e^{-\lambda(n+1)h} + \sum_{j=1}^{n} e^{-\lambda(n+1-j)h} y_j\right)$$

$$\Rightarrow y_{n+1} = y_n - h^2\left(e^{-\lambda(n+1)h} + \sum_{j=1}^{n} e^{-\lambda(n+1-j)h} y_j\right)$$

We then have the following difference equation that uses the implicit Euler method to approximate the derivative and also explicit Euler method to approximate the integral.

$$y_{n+1} = y_n - h^2 e^{-\lambda(n+1)h} + \sum_{j=1}^{n} \quad h^2 e^{-\lambda(n+1-j)h} y_j$$

If we use this iteration to find $y_{n+2} - e^{-\lambda h} y_{n+1}$ the following result is obtain:

$$y_{n+2} - e^{-\lambda h} y_{n+1} = y_{n+1} - h^2 e^{-\lambda(n+2)h} - \sum_{j=1}^{n+1} h^2 e^{-\lambda(n+2-j)h} y_j - e^{-\lambda h} y_n + h^2 e^{-\lambda(n+2)h} + \sum_{j=1}^{n} h^2 e^{-\lambda(n+2-j)h} y_j$$

$$= y_{n+1} - h^2 e^{-\lambda(n+2)h} - \sum_{j=1}^{n} h^2 e^{-\lambda(n+2-j)h} y_j - h^2 e^{-\lambda h} y_{n+1} - e^{-\lambda h} y_n + h^2 e^{-\lambda(n+2)h} + \sum_{j=1}^{n} h^2 e^{-\lambda(n+2-j)h} y_j$$

$$= y_{n+1} - h^2 e^{-\lambda h} y_{n+1} - e^{-\lambda h} y_n$$

$$y_{n+2} - y_{n+1}\left(e^{-\lambda h}\left(1-h^2\right)+1\right) + e^{-\lambda h} y_n = 0 \qquad (5)$$

2.3 The Implicit Euler method to approximate the derivative and the Trapezium rule to approximate the integral.

When θ = 0 and, $\vartheta = \frac{1}{2}$ explicit Euler method can be use to approximate the derivative and the integral as seen below:

$$\frac{y_{n+1}-y_n}{h} = f\left((n+1)h, y_{n+1}, z_{n+1}\right), z_{n+1} = -\frac{h}{2}\left(e^{-\lambda(n+1)h} + 2\sum_{j=1}^{n} e^{-\lambda(n+1-j)h} y_j + y_{n+1}\right)$$

$$y_{n+1} = y_n - \frac{h^2}{2}\left(e^{-\lambda(n+1)h} + 2\sum_{j=1}^{n} e^{-\lambda(n+1-j)h} y_j + y_{n+1}\right)$$

Making y_{n+1} the subject formular it becomes

$$y_{n+1} = y_n - \frac{h^2}{2}\left(e^{-\lambda(n+1)h} + 2\sum_{j=1}^{n} e^{-\lambda(n+1-j)h} y_j + y_{n+1}\right)$$

$$\Rightarrow \left(1 + \frac{h^2}{2}\right) y_{n+1} = y_n - \frac{h^2}{2} e^{-\lambda(n+1)h} - \sum_{j=1}^{n} h^2 e^{-\lambda(n+1-j)h} y_j$$

$$\left(2 + \frac{h^2}{2}\right) y_{n+1} = y_n - \frac{h^2}{2} e^{-\lambda(n+1)h} - \sum_{j=1}^{n} h^2 e^{-\lambda(n+1-j)h} y_j$$

$$\Rightarrow y_{n+1} = \frac{2}{2+h^2}\left(y_n - \frac{h^2}{2} e^{-\lambda(n+1)h} - \sum_{j=1}^{n+1} h^2 e^{-\lambda(n+2-j)h} y_j\right)$$

If we use this iteration to find $y_{n+2-} y_{n+1e}{-\lambda h}$ we obtain the following result:

$$y_{n+2} - e^{-\lambda} y_{n+1} = \frac{2}{2+h^2} y_{n+1} - \frac{h^2}{2+h^2} e^{-\lambda(n+2)h} - 2\sum_{j=1}^{n} \frac{h^2}{2+h^2} e^{-\lambda(n+2-j)h} y_j$$

$$- e^{-\lambda h} \frac{2}{2+h^2} y_n + \frac{h^2}{2+h^2} e^{-\lambda(n+2)h} - 2\sum_{j=1}^{n} \frac{h^2}{2+h^2} e^{-\lambda(n+2-j)h} y_j$$

$$= \frac{2}{2+h^2} y_{n+1} + \frac{h^2}{2+h^2} e^{-\lambda(n+2)h} - 2\sum_{j=1}^{n} \frac{h^2}{2+h^2} e^{-\lambda(n+2-j)h} y_j - \frac{4h^2}{2+h^2} e^{-\lambda h} y_{n+1}$$

$$- e^{-\lambda h} \frac{2}{2+h^2} y_n + \frac{h^2}{2+h^2} e^{-\lambda(n+2)h} - 2\sum_{j=1}^{n} \frac{h^2}{2+h^2} e^{-\lambda(n+2-j)h} y_j$$

$$= \frac{2}{2+h^2} y_{n+1} - \frac{4h^2}{2+h^2} e^{-\lambda h} y_{n+1} - e^{-\lambda h} \frac{2}{2+h^2} y_n$$

$$y_{n+2} - y_{n+1}\left(e^{-\lambda h}\left(1 - \frac{4h^2}{2+h^2}\right) + \frac{2}{2+h^2}\right) + \frac{2e^{-\lambda h}}{2+h^2} y_n = 0 \tag{6}$$

2.4 The derivative and the integral.

The derivative and the integral equations can be approximated using Implicit Euler method by changing the value of $\theta = 0$ and $\vartheta = 0$, by applying explicit Euler method to approximate both the derivative and the integral as follows:

$$\frac{y_{n+1} - y_n}{h} = f\left(\left(n+1\right)h, y_{n+1}, z_{n+1}\right), z_{n+1},$$

$$z_{n+1} = -h\left(\sum_{j=1}^{n} e^{-\lambda(n+1-j)h} y_j + y_{n+1}\right)$$

$$\Rightarrow y_{n+1} = y_n - h^2\left(\sum_{j=1}^{n} e^{-\lambda(n+1-j)h} y_j + y_{n+1}\right)$$

Making y_{n+1} the subject of the formular, the result is as follows:

$$y_{n+1} = y_n - h^2\left(\sum_{j=1}^{n} e^{-\lambda(n+1-j)h} y_j + y_{n+1}\right)$$

$$\left(1+h^2\right)y_{n+1} = y_n - \sum_{j=1}^{n} h^2 e^{-\lambda(n+1-j)h} y_j$$

$$\Rightarrow y_{n+1} = \frac{1}{1+h^2}\left(y_n - \sum_{j=1}^{n} h^2 e^{-\lambda(n+1-j)h} y_j\right)$$

Then the following difference equation becomes.

$$y_{n+1} = \frac{1}{1+h^2} y_n - \sum_{j=1}^{n} \frac{h^2}{1+h^2} e^{-\lambda(n+1-j)h} y_j$$

If we use this iteration to find $y_{n+2-e}{-}^{\lambda h} y_{n+1}$ we obtain the following result:

$$y_{n+2} - e^{-\lambda h} y_{n+1} = \frac{1}{1+h^2} y_{n+1} - \sum_{j=1}^{n+1} \frac{h^2}{1+h^2} e^{-\lambda(n+2-j)h} y_j$$

$$-\frac{e^{-\lambda h}}{1+h^2} y_n + \sum_{j=1}^{n} \frac{h^2}{1+h^2} e^{-\lambda(n+2-j)h} y_j$$

$$= \frac{1}{1+h^2} y_{n+1} - \sum_{j=1}^{n} \frac{h^2}{1+h^2} e^{-\lambda(n+2-j)h} y_j - \frac{h^2}{1+h^2} e^{-\lambda h} y_{n+1}$$

$$- \frac{e^{-\lambda h}}{1+h^2} y_n + \sum_{j=1}^{n} \frac{h^2}{1+h^2} e^{-\lambda(n+2-j)h} y_j$$

$$y_{n+2} - y_{n+1}\left(e^{-\lambda h}\left(1 - \frac{h^2}{1+h^2}\right) + \frac{1}{1+h^2}\right) + \frac{e^{-\lambda h}}{1+h^2} y_n = 0 \tag{7}$$

3 IMPLEMENTATION OF MATLAB TO ASSESS THE QUALITATIVE BEHAVIOUR OF THE SOLUTION OF THE NUMERICAL METHODS

According to [1], the exact solution of (1) is

$$y(t) = \frac{1 + \frac{\lambda}{\sqrt{\lambda^2 - 4}}}{2} e^{\frac{-\lambda + \sqrt{\lambda^2 - 4}}{2}} + \frac{1 - \frac{\lambda}{\sqrt{\lambda^2 - 4}}}{2} e^{\frac{-\lambda - \sqrt{\lambda^2 - 4}}{2}} \tag{8}$$

We will implement the numerical methods in (4) and (6), to observe the qualitative behaviour of the numerical solutions as λ and h vary while comparing the results to those of the actual solution described above. Matlab will be used to plot the solutions between $t = 0$ and $t = tmax$ according to the parameters L and h representing λ and h. we shall plot numerical solutions for each of the numerical schemes for (4) and (6) as seen in 2.1 and 2.2 respectively, as earlier stated, the results will be compared with the qualitative behaviours for the exact solutions and that of the numerical. A constant value of $h = 0.001$ will be used for the implementation of the numerical schemes of (4) and (6), but a number of values will be chosen for λ to assess the different qualitative behaviours.

4 RESULTS AND DISCUSSIONS

Figure 1 shows the qualitative behaviour of the solutions used to determine the derivative and the integral equations.

Figure 1(a) uses the values of $\lambda = 5$, $h = 0.001$ and $tmax = 30$ for the numerical solution whose behaviour shows that there is decay with no oscillations, verifying the output predicted in research paper [1] and the actual solution of (8).

Figure 1(b) uses the values of $\lambda = 1$, $h = 0.001$ and $tmax = 15$ for the numerical solution whose behaviour shows that there is decay with infinitely many oscillations of decreasing magnitude, verifying the output predicted in research paper [1] and the actual solution of (8).

Figure 1(c) uses the values of $\lambda = -4$, $h = 0.001$ and $tmax = 5$ for the numerical solution whose behaviour shows that the solution grows without any oscillations, verifying the output predicted in research paper [1] and the actual solution of (8).

Figure 1(d) uses the values of $\lambda = -0.5$, $h = 0.001$ and $tmax = 30$ for the numerical solution whose behaviour shows that the solution grows with infinitely many oscillations of increasing magnitude, verifying the output predicted in research paper [1] and the actual solution of (8).

Figure 1(e) uses the values of $\lambda = 0$, $h = 0.001$ and $tmax = 250$ for the numerical solution whose behaviour shows that for $\lambda = 0$, the actual solution depicts that the behaviour has a constant oscillations between 1 and -1 over a period of 2π, i.e $y(t) = cos(t)$. This is not the case for the numerical solution using (4). Choosing a small $tmax$, the qualitative behaviour of the numerical solution is similar to the behaviour of the actual solution, but as $tmax$ increases, it gives us an insight into the long term behaviour of the numerical solution [3]. We then observed that the oscillations are infinite with increasing magnitude, even though the qualitative behaviour of the numerical solution does not match the behaviour of the actual solution in the long term, which has verified that the behaviour predicted in the research paper [1] is the same as the behaviour observed in figure 1(e).

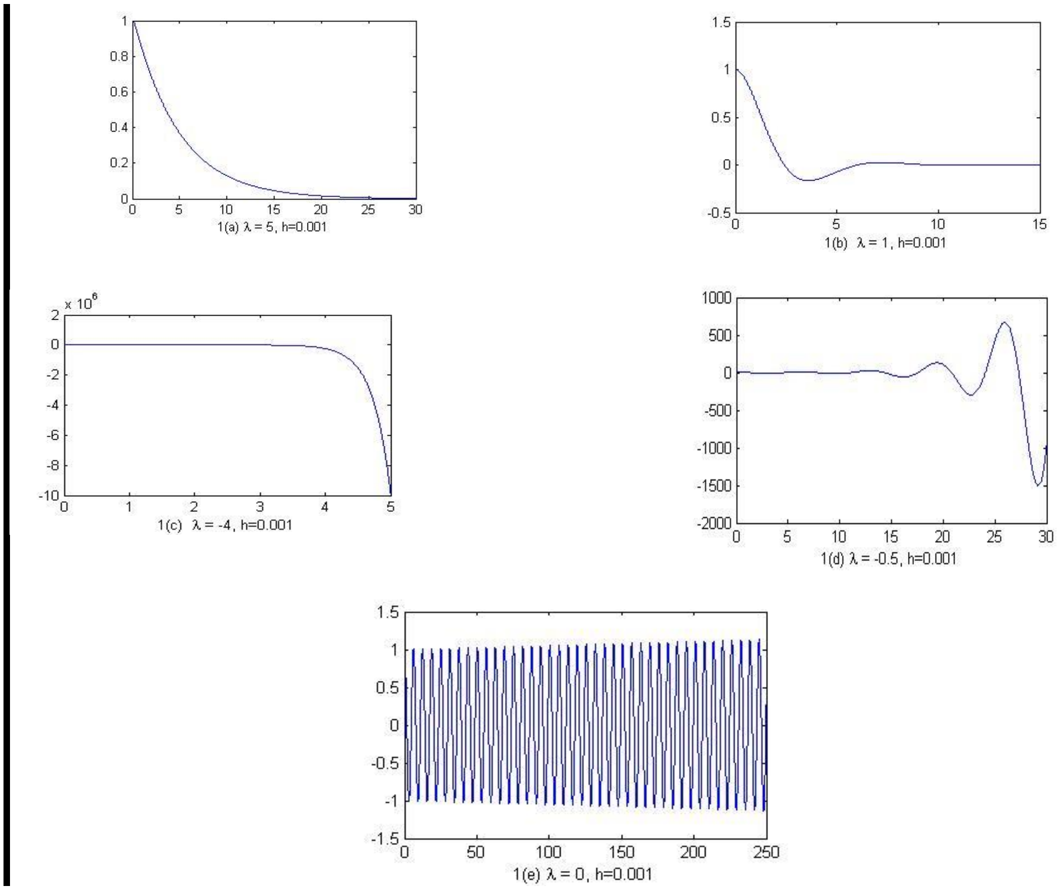


Figure 1: Numerical solutions of equation (4)

Figure 2 shows the qualitative behaviour of the solutions to approximate the derivative and the Trapezium rule to approximate the integral.

Figure $2(a)$ uses the values of $\lambda = 7$, $h = 0.001$ and $tmax = 20$ for the numerical solution whose behaviour shows that there is decay with no oscillations, verifying the output predicted in research paper [1] and the actual solution of (8).

Figure $2(b)$ uses the values of $\lambda = 0.5$, $h = 0.001$ and $tmax = 25$ for the numerical solution whose behaviour shows that there is decay with infinitely many oscillations of decreasing magnitude, verifying the output predicted in research paper [1] and the actual solution of (8).

Figure $2(c)$ uses the values of $\lambda = -5$, $h = 0.001$ and $tmax = 5$ for the numerical solution whose behaviour shows that the solution grows without any oscillations, verifying the output predicted in research paper [1] and the actual solution of (8).

Figure $2(d)$ uses the values of $\lambda = -1$, $h = 0.001$ and $tmax = 20$ for the numerical solution whose behaviour shows that the solution grows with infinitely many oscillations of increasing magnitude, verifying the output predicted in research paper [1] and the actual solution of (8).

Figure 2(e) uses the values of λ = 0, h = 0.001 and $tmax$ = 400 for the numerical solution whose behaviour shows that for λ = 0, the actual solution depicts that the behaviour has a constant oscillations between 1 and −1 over a period of 2π, i.e $y(t) = cos(t)$. This is not the case for the numerical solution using (6). Choosing a small $tmax$, the qualitative behaviour of the numerical solution is similar to the behaviour of the actual solution, but as $tmax$ increases, it gives us an insight into the long term behaviour of the numerical solution [3]. We then observed that the oscillations are infinite with decreasing magnitude, even though the qualitative behaviour of the numerical solution does not match the behaviour of the actual solution in the long term, which has verified that the behaviour predicted in the research paper [1] is the same as the behaviour observed in the figure 2(e).

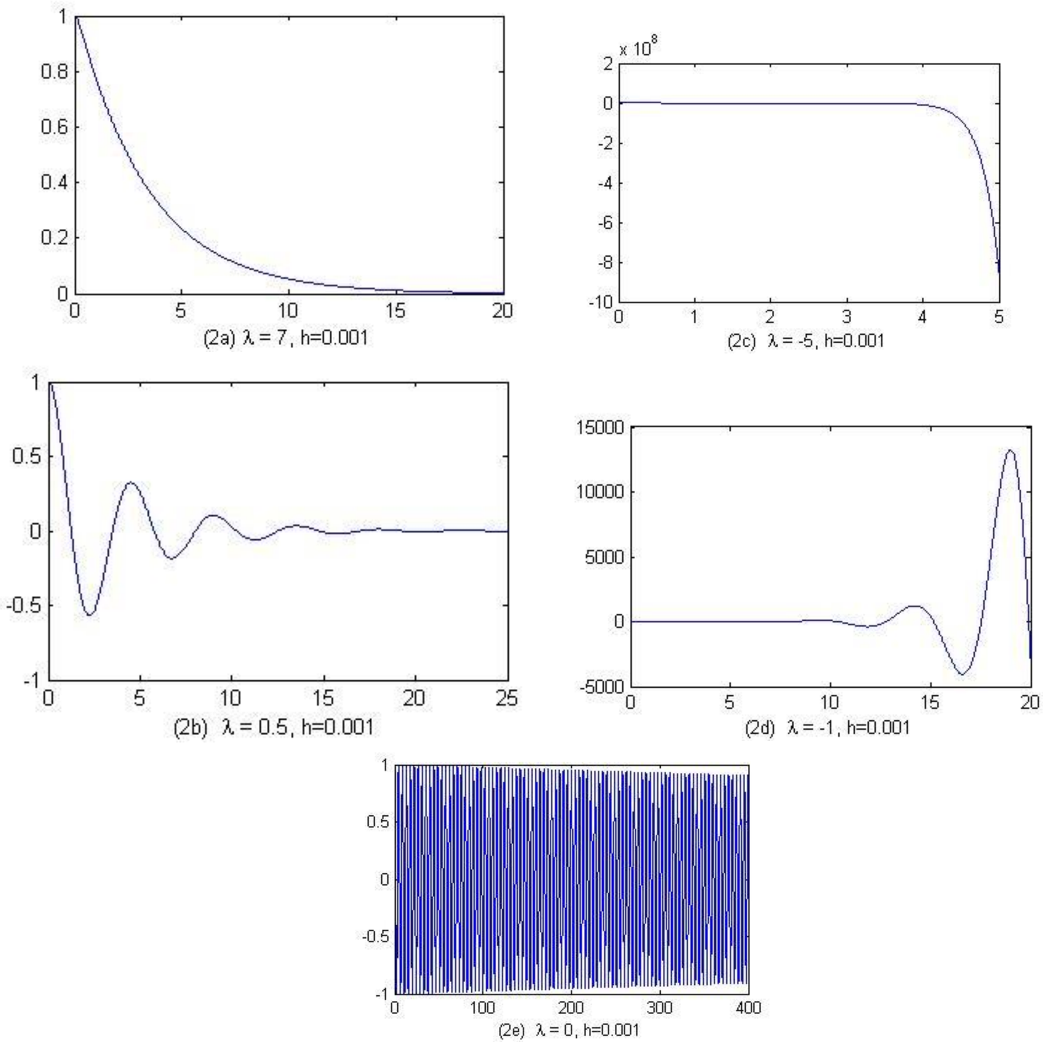
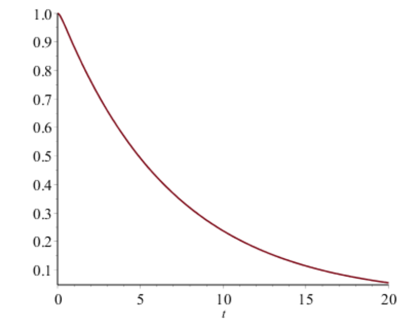
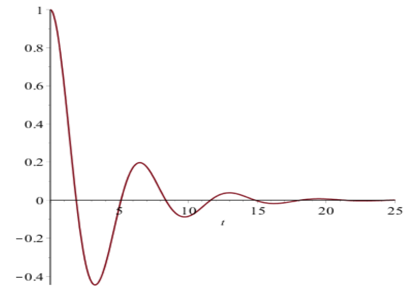


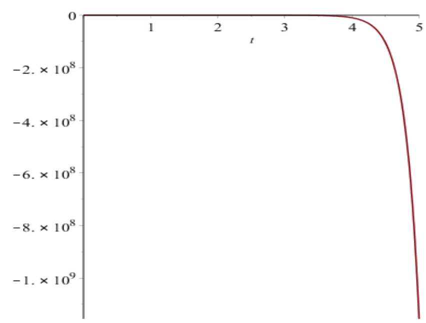
Figure 2. Numerical solutions of equation (6)



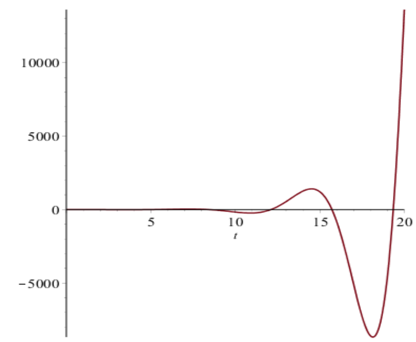
(i) λ = 0.5, h = 0.001



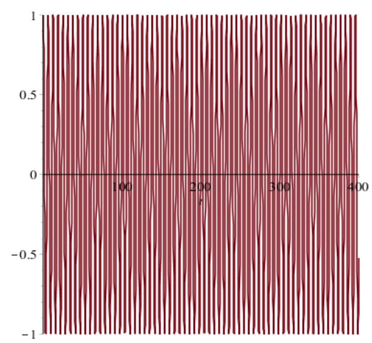
(ii) λ = 1, h = 0.001



(iii) λ = -5, h = 0.001

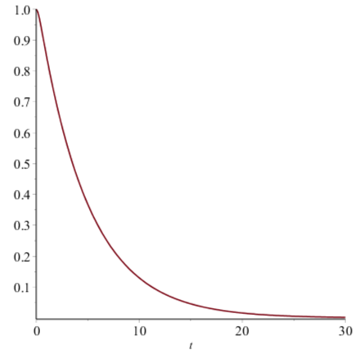


(iv) λ = - 1, h = 0.001

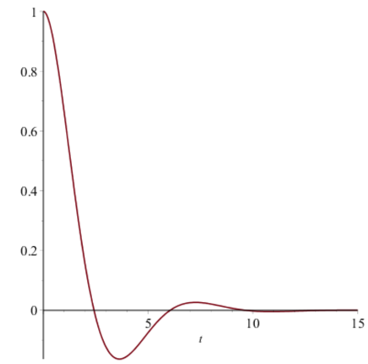


(v) λ = 0, h = 0.001

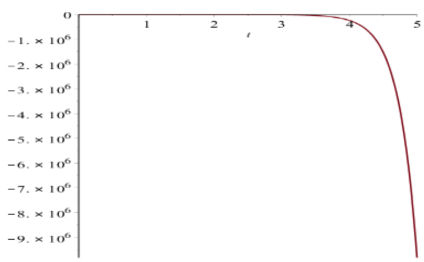
Figure 3a: The exact solution of equation (8)



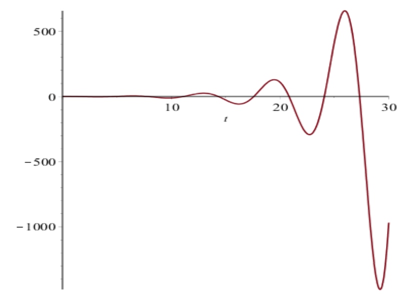
(i) λ = 5, h = 0.001



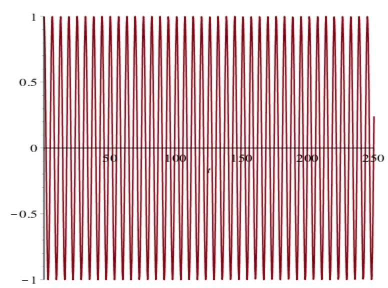
(ii) λ = 1, h = 0.001



(iii) λ = -4, h = 0.001



(iv) λ= -0.5, h = 0.001



(v) λ = 0, h = 0.001

Figure 3b: The exact solution of equation (8)

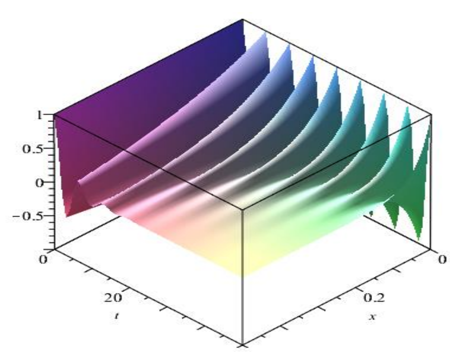


Figure 4.general solution of equation (8)

5 CONCLUSION

This paper has analyzed the long term qualitative behaviour of the Volterra integro-differential equation using numerical schemes. Figures 1,2 and 3, shows that, the qualitative behaviour in the solution obtained from the numerical schemes of (4) and (6) is the same as the behaviour observed in choosing different pairs of λ and step sizes of h values, similarly, the figures 3a and 3b represents the exact solution of equation (8) while figure 4 shows the surface of the general solution of equation (8). One could notice that the behaviour of their solutions are actually the same with figures 1, 2,3a and 3b, but one must understand that in the choice of either λ or step size *h,* the qualitative behaviour is only possible when λ is between -0.5 to -4 and the value of h must always be from 1 to 35 and h must not take the value 0, otherwise there will always be an extinction on the behaviour of the solutions. Finally, a further research can be done using the higher order linear multistep method to determine the qualitative behaviour of the solutions.

ACKNOWLEDGEMENTS

The author is thankful to the reviewers and Dr. Iyakino P. Akpan for all your constructive comments and suggestions to improve this paper and also grateful to my Lecturer Dr. Roberts A. Johnson. who taught me Volterra integro-differential equations at university of Chester, United Kingdom.

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