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Eigenvalue Elasticity Analysis of Mathematical Model of Dynamics of Diabetes and its Complications in a Population

Aye Patrick Olabanji

Dept. of Mathematical Sciences, Adekunle Ajasin University, Akungba Akoko, Ondo State, Nigeria

Corresponding author: patrick.aye@aaua.edu.ng, ayepatricko@gmail.com

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ABSTRACT

In this paper, a mathematical model of dynamics of diabetes and its complications was presented to explore the parameters with the greatest impact on the model. The model allows for the individuals to move from the susceptible class to the treated class. The model exhibit one equilibrum state, namely, the disease prevalent equilibrium state. The local and global asymptotic stability of the disease prevalent equilibrium state was determined using quadratic Lyapunov method for linear system. Eigenvalue elasticity and sensitivity analysis was carried out on the model parameters to determine the parameter that has the highest positive eigenvalue. The analysis revealed that parameter denoted by δ (mortality rate due to complications) has the highest positive eigenvalue elasticity value. Also, using the eigenvalue sensitivity analysis, the parameter denoted by δ has the highest positive value. The overall results showed that parameter δ has the greatest impact on the formulated mathematical model of disease dynamics which must be put into consideration by the health care policy makers in order to reduce the rate of mortality due to the disease.

Keywords: Complications, Diabetes, Eigenvalue Elasticity, Eigenvalue Sensitivity, Quadratic Lyapunov Method.

1 INTRODUCTION

Eigenvalue Elasticity Analysis (EEA) is a method for measuring performance response in a dynamic system. It measures the elastic strength of eigenvalue with respect to different parameters in a dynamic model. The method was first introduced by [1]. Several researchers have discussed the EEA method and applied it in both linear and linearized models [2] - [4]. Eigenvalue elasticity analysis (EEA) is a set of methods to study the effect of structure on behavior in dynamic models. It works by taking into consideration observed model behavior as a mixture of characteristic behavior modes and by examining the relative significance of particular rudiments of system formation in influencing these behavior modes. Elements involved in the formation that have a great impact on model behaviors can offer useful clues to the modeler to identify areas for further testing and study, as well as for policy analysis. The method involves a high level of mathematical effort compared to the previous experimental simulation method applied in the field. The method uses linear systems assumption to divide the observed behavior into its component behavior modes, such as oscillation, growth, and exponential modification, and list how a particular behavior mode and its occurrence in a particular system variable depends upon particular parameters and structural rudiments (links and loops) in the system. In this manner, the method gives a very accurate description of the interaction involving structure and behavior [5].

The EEA method allows large-scale models to be studied analytically in a way that is not feasible or realistic using trial and error simulation. Given the tedious and relative superiority of the method, it may also provide legitimacy to dynamic model analysis in fields that are formerly dominated by analytical mathematics, such as economics and econometrics. The fundamental nature of the elasticity analysis is to look at the relative importance of structural elements, not to calculate approximately the strength of system elements or values of system parameters. Consequently, the method works very greatly from a given model structure and parameter set and then discloses what would happen if you changed the structure and/or parameters. It is useful in the interpretation and policy analysis stages of model construction. It can also prove helpful in the model building and testing stage, to the point that it can help to identify structures that give unnecessary or confusing behavior. Eigenvalue elasticity is dimensionless and assists us to compare elasticities of the eigenvalue with respect to different parameters in a mathematical model [6].

Eigenvalue Sensitivity Analysis (ESA) helps to identify the relative importance of each parameter to disease dynamics. Sensitivity analysis is generally applied to predict the robustness of model performance relative to parameter value, since there are usually errors in supposed parameter value and data collection. Indices of sensitivity analysis allow us to determine the relative difference in a variable when changes occur in parameters [6]. Various researchers have applied the knowledge of mathematical models to provide insight into the study of dynamics of transmission and control of diseases. Boutayeb *et al* [7] formulated a model to analyze the dynamics of diabetes mellitus and its complications in a population. Their model assumption is the constant rate of diabetes person developing complications. Diabetics population were splitted into two groups: Diabetics without complications and Diabetics with complications. The solution of the model was obtained using numerical method. The results show that the incidence of diabetes and occurrence of complications can be controlled with efficient and effective control strategies.

Akinsola and Oluyo [8] developed a model on the dynamics and control of diabetes mellitus and its complications. Their model is an improvement on the work of Boutayeb *et al* [7] and it was based on the size of diabetics without complications and diabetics with complications. The stability analysis of the model was carried out and it was stable. Their study revealed that diabetes persists but its complications can be controlled. They investigated the sensitivity of each parameter to the model and the results obtained established that the size of diabetics with complications can be curtailed with adequate control measures. Adamu *et al* [9] formulated a model for the dynamics of diabetics population. They improved on Boutayeb *et al* [7] by incorporating the impact of treatment and birth rate on the disease dynamics. The model compartments are diabetics with complications, diabetics with controlled sugar and diabetics without complications. The model equations were solved and disease free equilibrum state obtained. The stability of equilibrum state of the model were carried out using Bellman and Coke method. The result obtained showed that diabetic with complications and birth rate determined the stability of the equilibrum solution of the model. The model established that lifestyle and genetic factor determined the dynamics of diabetics population. Enagi *et al* [10] proposed the method of Homotopy Perturbation to solve a system of equations of model of diabetes mellitus disease. An analytical solution was obtained and graphical profile of the solution was shown using Mapple software. The results showed that model parameters have an effect in determining the number of people living with diabetes, Permatasari *et al* [11] proposed method of quadratic Lyapunov function for determining stability of linear system. The method was applied to linear system of dynamics of diabetics population. The model was compartmentalized into healthy class, prediabetic class, diabetics without complications class, diabetics with complications class and disability class. The method was used to examine the global stability of the model. The results obtained established the asymptotic stability of the model globally.

Aye *et al* [12] developed a mathematical model for the dynamics of diabetes mellitus and its complications and carried out analysis of the model. The analytical solution of the model equations was obtained using Homotopy Perturbation Method. Numerical simulation of the model solution was done using Maple 18 Mathematical software. The parameters are varied and their effects on the model dynamics are presented graphically. The results showed that the deaths due to diabetes complications can be reduced drastically if the rate at which complications are treated is high and the rate of developing a complication is slow. The work of Aye [13] improved the existing mathematical models of dynamics of diabetes and its complications by incorporating control measures into the system. The proposed model is compartmentalized into five classes namely, susceptible, healthy, diabetics without complications, diabetics with complications and diabetics with complications undergoing treatment. The equations describing the system were derived and analytic solutions of the system of equations were obtained using Homotopy pertubation method. The numerical simulation of the solution was carried out and the graphical profile of the system responses were presented. The result showed that if the control parameters rate is increased, the number of deaths attributable to diabetes and its complications in a population would be reduced drastically.

In this study, it is intended to carry out the analysis of eigenvalue elasticity and eigenvalue sensitivity of the model parameters in [13] to determine the parameter that has the greatest impact on the formulated mathematical model.

2 MODEL FORMULATION

The model equations are formulated using first-order differential equations. Improving on the work of Enagi *et al* [10] we proposed a mathematical model of diabetes and its complications incorporating a positive lifestyle and effective management of diabetes condition as control. Based on their health status, the model population is classified into five classes. They are healthy class $H(t)$, susceptible class $S(t)$, diabetic without complications class $D(t)$, diabetic with complications class $C(t)$ and diabetic with complications that undergo treatment class $T(t)$. We assume that diabetes disease infections can either be acute or chronic. In this model, we assume that a healthy individual will give birth to a healthy child that will be born into the healthy compartment while parent who is diabetic or has a history of diabetes will give birth to children with genetic factors that will be born into the susceptible compartment. The proportion of children born into the healthy compartment is denoted by θ while proportion of children who are born into the susceptible compartment is $1-\theta$.

To form this model, two control parameters ϕ_1 and ϕ_2 are introduced. ϕ_1 is a measure of a positive lifestyle in the susceptible class, such that $0 \leq \phi_1 \leq 1$. $\phi_1 = 0$ indicate negative lifestyle and $\phi_1 = 1$ indicate positive lifestyle. ϕ_2 is a measure of effective management of diabetes condition in the compartment of diabetics without complications, such that $0 \leq \phi_2 \leq 1$. $\phi_2 = 0$ indicate ineffective management of diabetes condition and $\phi_2 = 1$ indicate effective management of diabetes condition.

Table 1: Description of variables of the model

Variables	**Description**

H(t)	Healthy Class
S(t)	Susceptible Class
D(t)	Diabetics without complications
D(t)	Diabetics with complications
T(t)	Diabetics with complications undergoing treatment class
N(t)	Total population

Table 2: Description of parameters of the model

Parameters	**Description**
α	Probability rate of incidence of diabetes
β	Birth rate
μ	Natural mortality rate
τ	Rate at which healthy individual become susceptible
σ	Rate at which susceptible individual become healthy
λ	Rate at which *D(t)* develop a complications
γ	Rate at which *C(t)* are treated
ω	Rate at which *C(t)* after treatment return to *D(t)*
δ	Mortality rate due to complications
θ	Proportion of children born into the healthy class
ϕ_1	Measure of positive lifestyle in *S(t)* class
ϕ_2	Measure of effective management of diabetes condition in *D(t)* class
$1-\theta$	Proportion of children born into the susceptible class

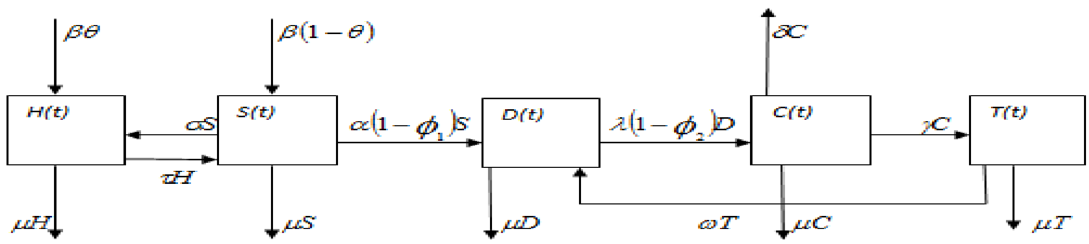

Figure 1: Schematic diagram of the model

2.1 The Model Equations

Based on the model formulation, the model equations are obtained as follows in (1) to (5)

$$\frac{dH(t)}{dt} = \sigma S(t) - \mu H(t) - \tau H(t) + \beta\theta \tag{1}$$

$$\frac{dS(t)}{dt} = \beta(1-\theta) - \mu S(t) + \tau H(t) - \alpha(1-\phi_1)S(t) - \sigma S(t) \tag{2}$$

$$\frac{dD(t)}{dt} = \alpha(1-\phi_1)S(t) + \omega T(t) - \lambda(1-\phi_2)D(t) - \mu D(t) \tag{3}$$

$$\frac{dC(t)}{dt} = \lambda(1-\phi_2)D(t) - \gamma C(t) - \delta C(t) - \mu C(t) \tag{4}$$

$$\frac{dT(t)}{dt} = \gamma C(t) - \omega T(t) - \mu T(t) \tag{5}$$

The initial values conditions are $H(o) = H_o, S(o) = S_o, D(o) = D_o, C(o) = C_o$ and $T(o) = T_o$.

2.2 Disease Prevalence Equilibrium State of the Model

To obtain the Disease Prevalent Equilibrium (DFE) state of the of the model, the system of equation (1) - (5) are rearranged and equated to zero. The new equations are as follows:

$$\beta\theta - \tau H(t) + \sigma S(t) - \mu H(t) = 0 \tag{6}$$

$$\beta(1-\theta) + \tau H(t) - \mu S(t) - \alpha(1-\phi_1)S(t) - \sigma S(t) = 0 \tag{7}$$

$$\alpha(1-\phi_1)S(t) + \omega T(t) - \mu D(t) - \lambda(1-\phi_2)D(t) = 0 \tag{8}$$

$$\lambda(1-\phi_2)D(t) - \mu C(t) - \delta C(t) - \gamma C(t) = 0 \tag{9}$$

$$\gamma C(t) - \omega T(t) - \mu T(t) = 0 \tag{10}$$

The disease prevalence equilibrium state is $E^*(H^*, S^*, D^*, C^*, T^*)$ of the system (1) to (5) is given as follows:

$$H^* = \frac{-\beta[(\mu + \sigma + \alpha(1-\phi_1))\theta + \sigma(1-\theta)]}{[\tau\sigma - (\mu + \sigma + \alpha(1-\phi_1))(\mu + \tau)]} \tag{11}$$

$$S^* = \frac{1}{\sigma}\left[\frac{-(\mu+\tau)[(\mu + \sigma + \alpha(1-\phi_1))\theta + \sigma(1-\theta)]\beta}{[\tau\sigma - (\mu + \sigma + \alpha(1-\phi_1))(\mu + \tau)]} - \beta\theta\right] \tag{12}$$

$$D^* = \frac{\alpha(1-\phi_1)(\mu+\delta+\gamma)(\mu+\omega)}{\omega\lambda\gamma\sigma(1-\phi_2) - \sigma(\mu + \lambda(1-\phi_2))(\mu+\delta+\gamma)(\mu+\omega)} \left[\frac{(\mu+\tau)[(\mu + \sigma + \alpha(1-\phi_1))\theta + \sigma(1-\theta)]\beta}{[\tau\sigma - (\mu + \sigma + \alpha(1-\phi_1))(\mu + \tau)]} + \beta\theta\right] \tag{13}$$

$$C^* = \frac{\alpha\lambda(1-\phi_1)(1-\phi_2)(\mu+\omega)}{\omega\lambda\gamma\sigma(1-\phi_2) - \sigma(\mu + \lambda(1-\phi_2))(\mu+\delta+\gamma)(\mu+\omega)} \left[\frac{(\mu+\tau)[(\mu + \sigma + \alpha(1-\phi_1))\theta + \sigma(1-\theta)]\beta}{[\tau\sigma - (\mu + \sigma + \alpha(1-\phi_1))(\mu + \tau)]} + \beta\theta\right] \tag{14}$$

$$T^* = \frac{\alpha\lambda\gamma(1-\phi_1)(1-\phi_2)}{\omega\lambda\gamma(1-\phi_2) - \sigma(\mu+\lambda(1-\phi_2))(\mu+\delta+\gamma)(\mu+\omega)} \left[\frac{(\mu+\tau)[(\mu+\sigma+\alpha(1-\phi_1))\theta+\sigma(1-\theta)]\beta}{[\tau\sigma-(\mu+\sigma+\alpha(1-\phi_1))(\mu+\tau)]} + \beta\theta\right] \tag{15}$$

3 STABILITY ANALYSIS OF THE DISEASE PREVALENCE EQUILIBRUM STATE OF THE MODEL

For the stability analysis of the mathematical model, the method of quadratic Lyapunov for linear system by Permatasari *et al* [11] was adopted to analyzed the stability of linear system of equations (1) to (5).

3.1 Lyapunov Stability of Linear System

Given that the dynamical system is of linear form:

$$\dot{x} = Ex^* \tag{16}$$

Let $M > 0$ be a symmetric, positive definite matrix, then we define

$$V(x) = x^T Mx \tag{17}$$

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T Mx + x^T M\dot{x} \\ &= E^T x^T Mx + x^T MEx \\ &= x^T (E^T M + ME)x \\ &= E^T M + ME \end{aligned} \tag{18}$$

Let $E^T M + ME = -Q$, Q is definite positive. The existence of definite positive Q guaranteed stability (global asymptotic) of the linear system. $E^T M + ME = -Q$ is called the equation of Lyapunov. Before solving for *M*, we established that *E* is stable, so that given any $Q > 0$, we have $M > 0$, the normal method is to solve for M and set $Q = 1$ [14].

Theorem 3.1: A linear system $\dot{x} = Ex$ is local asymptotically stable if and only if for any symmetric, positive definite *Q*, there exist a corresponding symmetric, positive definite *M* so that

$$E^T M + ME = -Q \tag{19}$$

Theorem 3.2: Let $\overset{*}{x} = E(x), x \in \Re^n$.The system (origin) is globally asymptotically stable if and only if there exists a positive definite matrix $M = M^T > 0$ so that $E^T M + ME$ is negative definite or $E^T M + ME$. Equivalently if, for a given $Q = Q^T > 0$, it is possible to find a $M = M^T > 0$ so that

$$E^T M + ME = -Q$$

then the system is globally asymptotically stable [11].

Theorem 3.3: If $\Re e \lambda_K(E) < 0 \quad \forall k$, then for given every $Q = Q^T > 0$ there exists a unique $M = M^T > 0$ satisfying the Lyapunov equation $E^T M + ME = -Q$ so that the system is globally asymptotically stable [11].

Proof:

$$Jacobian\, matrix\, E = \begin{bmatrix} -(\mu+\tau) & \sigma & 0 & 0 & 0 \\ \tau & -\left(\mu+\sigma+\alpha\left(1-\varphi_1\right)\right) & 0 & 0 & 0 \\ 0 & \alpha\left(1-\varphi_1\right) & -\left(\mu+\lambda\left(1-\varphi_2\right)\right) & 0 & \omega \\ 0 & 0 & \lambda\left(1-\varphi_2\right) & -(\mu+\delta+\gamma) & 0 \\ 0 & 0 & 0 & \gamma & -(\mu+\omega) \end{bmatrix} \tag{20}$$

Let

$$\eta_1 = \mu + \tau \tag{21}$$
$$\eta_2 = \mu + \sigma + \alpha(1-\phi_1) \tag{22}$$
$$\eta_3 = \mu + \lambda(1-\phi_2) \tag{23}$$
$$\eta_4 = \mu + \delta + \gamma \tag{24}$$
$$\eta_5 = \mu + \omega \tag{25}$$
$$\varepsilon = \alpha(1-\phi_1) \tag{26}$$
$$\varphi = \lambda(1-\phi_2) \tag{27}$$

Substituting equations (21) - (27) into equation (20) gives equation (28) below.

$$E = \begin{bmatrix} -\eta_1 & \sigma & 0 & 0 & 0 \\ \tau & -\eta_2 & 0 & 0 & 0 \\ 0 & \varepsilon & -\eta_3 & 0 & \omega \\ 0 & 0 & \varphi & -\eta_4 & 0 \\ 0 & 0 & 0 & \gamma & -\eta_5 \end{bmatrix} \tag{28}$$

The Jacobian determinant of equation (28) is given by

$$\begin{bmatrix} -\eta_1-\rho & \sigma & 0 & 0 & 0 \\ \tau & -\eta_2-\rho & 0 & 0 & 0 \\ 0 & \varepsilon & -\eta_3-\rho & 0 & \omega \\ 0 & 0 & \varphi & -\eta_4-\rho & 0 \\ 0 & 0 & 0 & \gamma & -\eta_5-\rho \end{bmatrix} = 0 \tag{29}$$

To show that the disease prevalent equilibrium state (11) - (15) is stable, we first determine the eigenvalue of the system of equation (1) to (5).

$$(-\eta_1-\rho)\begin{vmatrix} -\eta_2-\rho & 0 & 0 & 0 \\ \varepsilon & -\eta_3-\rho & 0 & \omega \\ 0 & \varphi & -\eta_4-\rho & 0 \\ 0 & 0 & \gamma & -\eta_5-\rho \end{vmatrix} - \sigma\begin{vmatrix} \tau & 0 & 0 & 0 \\ 0 & -\eta_3-\rho & 0 & \omega \\ 0 & \varphi & -\eta_4-\rho & 0 \\ 0 & 0 & \gamma & -\eta_5-\rho \end{vmatrix} = 0$$

$$(-\eta_1-\rho)(-\eta_2-\rho)\begin{vmatrix} -\eta_3-\rho & 0 & \omega \\ \varphi & -\eta_4-\rho & 0 \\ 0 & \gamma & -\eta_5-\rho \end{vmatrix} - \tau\sigma\begin{vmatrix} -\eta_3-\rho & 0 & \omega \\ \varphi & -\eta_4-\rho & 0 \\ 0 & \gamma & -\eta_5-\rho \end{vmatrix} = 0$$

$$((-\eta_1-\rho)(-\eta_2-\rho)-\tau\sigma)\begin{vmatrix} -\eta_3-\rho & 0 & \omega \\ \varphi & -\eta_4-\rho & 0 \\ 0 & \gamma & -\eta_5-\rho \end{vmatrix} = 0$$

$$(\eta_1\eta_2+\eta_1\rho+\eta_2\rho+\rho^2-\tau\sigma)(-\eta_3-\rho)(-\eta_4-\rho)(-\eta_5-\rho)=0 \tag{30}$$

Equating each term of (30) separately to zero, we have

$$\left(\eta_1\eta_2+\eta_1\rho+\eta_2\rho+\rho^2-\tau\sigma\right)=0,\ (-\eta_3-\rho)=0,\ (-\eta_4-\rho)=0,\ (-\eta_5-\rho)=0$$

The eigenvalues are obtained as follows:

$$\rho_{1,2}=\frac{-(\eta_1+\eta_2)\pm\sqrt{(\eta_1+\eta_2)^2-4(\eta_1\eta_2-\tau\sigma)}}{2},\ \rho_3=-\eta_3,\ \rho_4=-\eta_4,\ \rho_5=-\eta_5$$

All the eigenvalues are negative, which implies that the disease prevalent equilibrium states are stable.

$$E^T=\begin{bmatrix} -\eta_1 & \tau & 0 & 0 & 0 \\ \sigma & -\eta_2 & \varepsilon & 0 & 0 \\ 0 & 0 & -\eta_3 & \varphi & 0 \\ 0 & 0 & 0 & -\eta_4 & \gamma \\ 0 & 0 & \omega & 0 & -\eta_5 \end{bmatrix} \tag{31}$$

We choose $Q=Q^T=I$ so from equation $E^TM+ME=-I$, we obtain

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ M_{12} & M_{22} & M_{23} & M_{24} & M_{25} \\ M_{13} & M_{23} & M_{33} & M_{34} & M_{35} \\ M_{14} & M_{24} & M_{34} & M_{44} & M_{45} \\ M_{15} & M_{25} & M_{35} & M_{45} & M_{55} \end{bmatrix} \tag{32}$$

$$E^{T}M = \begin{bmatrix} \tau M_{12}-\eta_1 M_{11} & \tau M_{22}-\eta_1 M_{12} & \tau M_{23}-\eta_1 M_{13} & \tau M_{24}-\eta_1 M_{14} & \tau M_{25}-\eta_1 M_{15} \\ \sigma M_{11}-\eta_2 M_{12}+\varepsilon M_{13} & \sigma M_{12}-\eta_2 M_{22}+\varepsilon M_{23} & \sigma M_{13}-\eta_2 M_{23}+\varepsilon M_{33} & \sigma M_{14}-\eta_2 M_{24}+\varepsilon M_{34} & \sigma M_{15}-\eta_2 M_{25}+\varepsilon M_{35} \\ \phi M_{14}-\eta_3 M_{13} & \phi M_{24}-\eta_3 M_{23} & \phi M_{34}-\eta_3 M_{33} & \phi M_{44}-\eta_3 M_{34} & \phi M_{45}-\eta_3 M_{35} \\ \gamma M_{15}-\eta_4 M_{14} & \gamma M_{25}-\eta_4 M_{24} & \gamma M_{35}-\eta_4 M_{34} & \gamma M_{45}-\eta_4 M_{44} & \gamma M_{55}-\eta_4 M_{45} \\ \omega M_{13}-\eta_5 M_{15} & \omega M_{23}-\eta_5 M_{25} & \omega M_{33}-\eta_5 M_{35} & \omega M_{34}-\eta_5 M_{45} & \omega M_{35}-\eta_5 M_{55} \end{bmatrix} \tag{33}$$

$$ME = \begin{bmatrix} \tau M_{12}-\eta_1 M_{11} & \sigma M_{11}-\eta_2 M_{12}+\varepsilon M_{13} & \phi M_{14}-\eta_3 M_{13} & \gamma M_{15}-\eta_4 M_{14} & \omega M_{13}-\eta_5 M_{15} \\ \tau M_{22}-\eta_1 M_{12} & \sigma M_{12}-\eta_2 M_{22}+\varepsilon M_{23} & \phi M_{24}-\eta_3 M_{23} & \gamma M_{25}-\eta_4 M_{24} & \omega M_{23}-\eta_5 M_{25} \\ \tau M_{23}-\eta_1 M_{13} & \sigma M_{13}-\eta_2 M_{23}+\varepsilon M_{33} & \phi M_{34}-\eta_3 M_{33} & \gamma M_{35}-\eta_4 M_{34} & \omega M_{33}-\eta_5 M_{35} \\ \tau M_{24}-\eta_1 M_{14} & \sigma M_{14}-\eta_2 M_{24}+\varepsilon M_{34} & \phi M_{44}-\eta_3 M_{34} & \gamma M_{45}-\eta_4 M_{44} & \omega M_{34}-\eta_5 M_{45} \\ \tau M_{25}-\eta_1 M_{15} & \sigma M_{15}-\eta_2 M_{25}+\varepsilon M_{35} & \phi M_{45}-\eta_3 M_{35} & \gamma M_{55}-\eta_4 M_{45} & \omega M_{35}-\eta_5 M_{55} \end{bmatrix} \tag{34}$$

$$-Q = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{35}$$

Evaluating equation (19), the following system of linear equations in (36) are obtained.

$$\left.\begin{array}{l}
2\tau M_{12}-2\eta_1 M_{11}=-1\\
\tau M_{22}-(\eta_1+\eta_2)M_{12}+\sigma M_{11}+\varepsilon M_{13}=0\\
\tau M_{23}-(\eta_1+\eta_3)M_{13}+\varphi M_{14}=0\\
\tau M_{24}-(\eta_1+\eta_4)M_{14}+\gamma M_{15}=0\\
\tau M_{25}-(\eta_1+\eta_5)M_{15}+\omega M_{13}=0\\
\sigma M_{11}-(\eta_1+\eta_2)M_{12}+\varepsilon M_{13}+\tau M_{22}=0\\
2\sigma M_{12}-2\eta_2 M_{22}+2\varepsilon M_{23}=-1\\
\sigma M_{13}-(\eta_2+\eta_3)M_{23}+\varepsilon M_{33}+\varphi M_{24}=0\\
\sigma M_{14}-(\eta_2+\eta_4)M_{24}+\varepsilon M_{34}+\gamma M_{25}=0\\
\sigma M_{15}-(\eta_2+\eta_5)M_{25}+\varepsilon M_{35}+\omega M_{23}=0\\
\varphi M_{14}-(\eta_1+\eta_3)M_{13}+\tau M_{23}=0\\
\varphi M_{24}-(\eta_2+\eta_3)M_{23}+\sigma M_{13}+\varepsilon M_{33}=0\\
2\varphi M_{34}-2\eta_3 M_{33}=-1\\
\varphi M_{44}-(\eta_3+\eta_4)M_{34}+\gamma M_{35}=0\\
\varphi M_{45}-(\eta_3+\eta_5)M_{35}+\omega M_{33}=0\\
\gamma M_{15}-(\eta_1+\eta_4)M_{14}+\tau M_{24}=0\\
\gamma M_{25}-(\eta_2+\eta_4)M_{24}+\sigma M_{14}+\varepsilon M_{34}=0\\
\gamma M_{35}-(\eta_3+\eta_4)M_{34}+\varphi M_{44}=0\\
2\gamma M_{45}-2\eta_4 M_{44}=-1\\
\gamma M_{55}-(\eta_4+\eta_5)M_{45}+\omega M_{34}=0\\
\omega M_{13}-(\eta_1+\eta_5)M_{15}+\tau M_{25}=0\\
\omega M_{23}-(\eta_2+\eta_5)M_{25}+\sigma M_{15}+\varepsilon M_{35}=0\\
\omega M_{33}-(\eta_3+\eta_5)M_{35}+\varphi M_{45}=0\\
\omega M_{34}-(\eta_4+\eta_5)M_{45}+\gamma M_{55}=0\\
2\omega M_{35}-2\eta_5 M_{55}=-1
\end{array}\right\}\qquad(36)$$

Solving the system of linear equations in (36) we obtain the component of symmetric matrix M.

3.2 Condition for local stability

To analyse the condition for local stability, Theorem (3.2) was adopted. The theorem states that local asymptotic stability of the equilibrum points of linear system (1) to (5) holds, provided the following conditions are met.

$$\Delta_1=|M_{11}|>0$$

$$\Delta_2=\begin{vmatrix} M_{11} & M_{12}\\ M_{21} & M_{22}\end{vmatrix}>0$$

$$\Delta_3=\begin{vmatrix} M_{11} & M_{12} & M_{13}\\ M_{12} & M_{22} & M_{23}\\ M_{13} & M_{23} & M_{33}\end{vmatrix}>0$$

$$\Delta_4=\begin{vmatrix} M_{11} & M_{12} & M_{13} & M_{14}\\ M_{12} & M_{22} & M_{23} & M_{24}\\ M_{13} & M_{23} & M_{33} & M_{34}\\ M_{14} & M_{24} & M_{34} & M_{44}\end{vmatrix}>0$$

$$\Delta_5 = \begin{vmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ M_{12} & M_{22} & M_{23} & M_{24} & M_{25} \\ M_{13} & M_{23} & M_{33} & M_{34} & M_{35} \\ M_{14} & M_{24} & M_{34} & M_{44} & M_{45} \\ M_{15} & M_{25} & M_{35} & M_{45} & M_{55} \end{vmatrix} > 0$$

$\Delta_r > 0, r = 1,\ldots,5$, it means that M is definite positive matrix. The model is locally asymptotically stable.

3.3 Condition for global stability

The analysis for the global stability of the model was done using theorem (3.3).

$$\Delta_M = \begin{vmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ M_{12} & M_{22} & M_{23} & M_{24} & M_{25} \\ M_{13} & M_{23} & M_{33} & M_{34} & M_{35} \\ M_{14} & M_{24} & M_{34} & M_{44} & M_{45} \\ M_{15} & M_{25} & M_{35} & M_{45} & M_{55} \end{vmatrix} = \Delta_{M^T} = \begin{vmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ M_{12} & M_{22} & M_{23} & M_{24} & M_{25} \\ M_{13} & M_{23} & M_{33} & M_{34} & M_{35} \\ M_{14} & M_{24} & M_{34} & M_{44} & M_{45} \\ M_{15} & M_{25} & M_{35} & M_{45} & M_{55} \end{vmatrix} = 0$$

$M = M^T > 0$ satisfying the Lyapunov equation $E^T M + ME = -Q$. The model is globally asymptotically stable.

4 EIGENVALUE ELASTICITY AND EIGENVALUE SENSITIVITY ANALYSIS

Eigenvalue elasticities measures the transient - response sensitivities of the model to parameters [15], [1] and since the values of elasticities are dimensionless, they can be compared with each other. This can aid us identify the parameter which could greatly influence the system. Eigenvalue Elasticity is dimensionless and enables us to compare elasticities of the eigenvalue with respect to different parameters in a mathematical model.

4.1 Eigenvalue Sensitivity with Respect to a Parameter

This is defined as the partial derivative of the eigenvalue with respect to that parameter [16]. The eigenvalue sensitivity $S_i\,(i = 1,\ldots,N)$ and N is the dimension of the state vector with respect to the j^{th} parameter of the system p_j is given in the form;

$$S_i(P_j) = \lim_{\Delta P_j \to 0} \frac{\Delta \lambda_i}{\Delta P_j} = \frac{\partial \lambda_i}{\partial P_i} = I_i^T \frac{\partial J}{\partial P_j} r_i \tag{37}$$

4.2 Eigenvalue Elasticity with Respect to a Parameter

This is defined as the partial derivative of the eigenvalue with respect to that parameter normalized for the size of the parameter and the size of the eigenvalue.

$$E_i(P_j)=\lim_{\Delta P_j\to 0}\frac{\frac{\Delta\lambda_i}{\lambda_i}}{\frac{\Delta p_j}{P_j}}=\frac{\frac{\partial\lambda_i}{\lambda_i}}{\frac{\partial p_j}{P_j}}=\frac{P_j\,\partial\lambda_i}{\lambda_i\,\partial p_j}=I_i^T\frac{\partial J}{\partial p_j}r_i\frac{P_j}{\lambda_i} \tag{38}$$

With these equations, the eigenvalue elasticity and sensitivity with respect to a parameter can be computed using the left eigenvectors (I_i) and the right eigenvectors (r_i) with the partial derivatives of the linearized Jacobian matrix (J) with respect to a parameter (p_i). Where (λ_i) are the eigenvalues, usually we make use of the dominant eigenvalues for the computations.

Table 3: Value of Parameters

Parameters	Values	Source
α	0.05	Adamu *et al*
β	0.01623	Enagi *et al*
μ	0.02	Permatasari *et al*
τ	0.04	Adamu *et al*
σ	0.08	Permatasari *et al*
λ	0.05	Adamu *et al*
γ	0.08	Permatasari *et al*
ω	0.08	Adamu *et al*
δ	0.05	Permatasari *et al*
θ	0.923	Enagi *et al*
ϕ_1	0.5	Assumed
ϕ_2	0.5	Assumed

Table 4: Eigenvalue Sensitivity and Eigenvalue Elasticity Analysis Indices of the Model

Parameters	*Eigenvalue Sensitivityv Values*	*Eigenvalue Elasticity Values*
α	-0.125459728	-0.033533886
β	0	0
γ	-0.026376318	-0.001128011983
λ	0.000354951228	0.0009487422286
μ	-1.000000	-0.010691521
θ	0	0
σ	0.055641208	0.000120373161
ω	0.00177093	0.000757357515
δ	0.16014993	0.0428061629
τ	-0.13595920	-0.029072217
ϕ_1	0.125459729	0.033533886
ϕ_2	-0.035495122	-0.094874221

5 CONCLUSION

This study presented a mathematical model of dynamics of diabetes and its complications in a population. The model equations have no disease free equilibrium state; this is consistent with the dynamics of the disease as it has no cure and hence the disease remains prevalent in the population. The disease prevalence equilibrium state of the model was obtained in (11 - 15). Quadratic Lyapunov method for stability of linear system was constructed in (19) and used to analyze the condition for local and global stability of Disease Prevalence Equilibrium state of the model. The model was found to be locally and globally asymptotically stable. Eigenvalue elasticity and sensitivity analysis was carried out on the model parameters to determine the elasticity and sensitivity of each parameter of the model and to know the parameter that has the highest impact factor on the model. Using the MATLAB software package, the computer program was written for the evaluation of the values of eigenvalue elasticity and sensitivity of the mathematical model given in equations (1 - 5). The results obtained are shown in Table 4. From the results obtained, it was found that the parameter denoted by δ which is mortality rate due to complications has the highest positive eigenvalue elasticity and sensitivity value. This means that the parameter denoted by δ has the greatest impact value on the formulated mathematical model of diabetes mellitus and its complications. This finding established the need for serious attention from government, medical and health practitioners to intensify their effort in curbing the menace of dearth attributed to the complications of diabetes in a population.

The following management options as revealed by the model are:

(i) The need for members of the public to adopt regular checking of blood sugar level to know if they are susceptible (prediabetic) to the disease or affected.
(ii) Introduction of government policy of free blood sugar checking at the Hospitals.
(iii) Aggressive awareness campaigns by the government must be introduced from time to time to sensitize the people on the need to adopt healthy lifestyle.
(iv) Government should put in place a medical support program that will help people that are suffering from the disease to assess medical facilities for treatment.
(v) Public enlightenment campaigns and media sensitization discourage high rate of smoking and alcoholism.
(vi) Introduction of medical drugs and herbs that will helps in attenuating the spread of the disease.
(vii) Promotion of regular physical exercise among the people of old age.

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