App. Math. and Comp. Intel., Vol. 2 (1) (2013) 85–94

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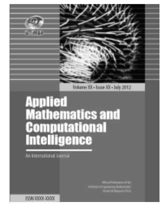

Higher order Homotopy Taylor-Perturbation via start-system

N. H. Abd Rahman[a,*], A. Ibrahim[b], and M. I. Jayes[b]

[a] *Faculty of Computer and Mathematical*
University Teknologi MARA (UiTM)
13500 Permatang Pauh, Pulau Pinang, Malaysia.

[b] *Faculty of Computer and Mathematical Sciences*
University Teknologi MARA Malaysia (UiTM)
40450 Shah Alam, Selangor, Malaysia.

Received: 23 May 2012; Revised: 24 November 2012, Accepted: 12 December 2012

Abstract: *One of the problems in iterations is to determine the best initial value and to ensure the convergence of the iterations. So, the objective is to evaluate the effectiveness of higher order approximated solutions of Homotopy Taylor-perturbation using start-system (ss) to overcome the problems. Successive approximation procedures using start-system technique are applied to the Classical Newton-Raphson, the Newton-perturbation, the Higher Order Taylor-perturbation and the new higher order Homotopy Taylor-perturbation (HHTP). The results are compared and evaluated. Numerical examples are given to illustrate and support the suggested algorithms. Results show that HHTPss offers as an alternative and effective way in solving nonlinear equations.*

Keywords: *Higher order Homotopy, perturbation, start-system, iterations, nonlinear*

PACS: *02.60.-x*

1 Introduction

The classical Newton method is a well-known method for solving non-linear equation due to its high efficient in the convergence speed. However, the need to guess the initial value in the iteration process is a disadvantage. Good initial guess value can solve the equation quickly, or vice versa. Homotopy continuation method can guarantee the answer by a certain path if the suitable auxiliary homotopy function (λ), or a start-system function, $p(x)$ is used [1, 2]. Some useful rules for the choice of the adjustable auxiliary homotopy and start-system functions are discussed and evaluated.Later, the homotopy perturbation method (HPM) was introduced by He [9, 11,12]. The concepts of higher order correctional terms in perturbation techniques are also introduced and applied, and almost all are based on an assumption that a small parameter must exist in the equation

* Corresponding Author: norhanim@ppinang.uitm.edu.my (N. H. Abd Rahman)

[1,2]. Commonly in perturbation methods, the correction terms are calculated once and no iterations are made. Here, iterations are made and numerically evaluated using Maple version 14.

2 Project Design

Definition 1 *A polynomial* $f(x)$ *of degree n is defined as* [3-5],

$$f(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + ... + a_ix^i = \sum_{i=0}^{n} a_ix^i. \tag{1}$$

where, a_i *are any real values for* $i = 0$ *to n.*

Definition 2 *We define the nth Taylor polynomial* $P_n(x)$ *about* $x = x_0$ *as* [3-5],

$$\begin{aligned} P_n(x) = f(x - x_0) &\cong f(x_0) + f'(x_0)(x - x_0) + ... + \frac{(f^n(x_0)(x - x_0)^n)}{n!} \\ &= \sum_{i=0}^{n} \frac{(f^i(x_0)(x - x_0)^i)}{i!}. \end{aligned} \tag{2}$$

Definition 3*Perturbation expansion is defined as*, [1]

$$p(x) = \varepsilon^0 x_0 + \varepsilon^1 x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + ... + \varepsilon^n x_n = \sum_{i=0}^{n} \varepsilon^i x_i \tag{3}$$

where, ε_i *are any real values for* $i = 0$ *to n.*

Definition 4 *The following non-linear algebraic equation is considered,* $f(x) = 0$ *and we construct a convex homotopy for the function* $H(x, \lambda) : \Re \times [0,1] \to \Re$ *as,*[1]

$$H(x, \lambda) = (1 - \lambda)p(x) + \lambda q(x) = 0; \tag{4}$$

where, λ *is an embedded parameter and* $\lambda \in [0,1]$;

$H(x,0) = p(x) \,\&\, H(x,1) = q(x) = f(x)$.

There are several ways to identify a start-system of a linear homotopy as mentioned by Nor Hanimet. al. [1,2] and Palancz et. al. [6]. However, here the start-system of the Taylor-homotopy is defined as,

Collorary 1 *We defined the convex homotopy for the function* $H^*(x, \lambda) : \Re \times [0,1] \to \Re$ *is defined as,*

$$H^*(x, \lambda) = (1 - \lambda)p(x) + \lambda q(x); \tag{5}$$

where, $H^*(x,0) = p(x) = x^n - \mathrm{C}$ *is the start-system function;* $H^*(x,1) = q(x) = f(x)$ *is the target-*

system function; n is preferably the highest power of x *of a nonlinear function* $f(x)$; *C is any real number in* $f(x)$.

Substituting Eq.(3) into Eq.(2), apply it to nonlinear as defined by Eq.(1), and later adding another step, where we convert the nonlinear $f(x)$ into $H(x,\lambda)$ as in Eq. (4) and Eq. (5), thus we are creating a n^{th} order HPM.

Hereafter, our discussion will only proceed with the above method, Eq. (5), for its flexibility to choose the values of *n* and C. Various perturbation methods have been widely introduced and applied, and almost all are based on an assumption that a small parameter must exist in the equation [7]. Commonly in perturbation methods, the correction terms are calculated once and no iterations are made.

Besides the ease of Newton-homotopy, it does not guarantee to converge [10]. Hereafter, our discussion will only proceed with the above method, Eq. (5), for its flexibility to choose the values of *n* and c. Below are some of the *Maple14 algorithms* used to create the Newton-perturbation (NP), Homotopy Taylor-perturbation (HTP) and Higher Order Homotopy Taylor-perturbation (HHTP) using start-system. Most of the algorithms have been simplified to a simpler form. The iterations will follow the following procedures:

i) Identify $q(x) = f(x) = 0$

ii) Identify $p(x)$, such as $p(x) = x^n - c$ where c is a any real number, and n be the highest power of x or, $p(x)$ be a part of $f(x)$ with trivial solution(s),

iii) Find the initial value by setting $p(x) = x^n - c = 0 \rightarrow x_0$.

$$restart; q(x) := x \rightarrow e^{-x} - x; p := x \rightarrow -x; Digits := 15; fsolve(q(x));; fsolve(p(x));$$

iv) Simplify $H(x,\lambda) = (1-\lambda).(x^n - c) + \lambda.q(x)$;

$$H := x \rightarrow (1-\lambda)p(x) + \lambda q(x); simplify(H(x,\lambda));$$

v) Substitute Perturbation Expansions into the Taylor's Series of order n such as n=1, 2, 3, 4 or 5;

$$restart;\ \exp and\left[f(x_0) + \sum_{i=1}^{3} \frac{D^{(i)}(f)(x_0)\left(\sum_{i=1}^{3}(a^i x_i)\right)^i}{i!} \right]$$

vi) Sort the expansion of degree 1 and equate it to zero, in order to calculate Newton-perturbation (Single Correctional Terms);

$$solve(f(x_0) + D(f)(x_0)x_1 a = 0$$

vii) Sort the expansion of degree 2 and equate it to zero, in order to calculate Taylor-perturbation (Double Correctional Terms);

$$solve(D(f)(x_0)x_2a^2 + \frac{1}{2}D^{(2)}(f)(x_0){x_1}^2a^2 = 0$$

viii) Sort the expansion of degree n and equate it to zero, in order to calculate Taylor-Perturbation (Triple Correctional Terms) such as n=3;

$$solve(D(f)(x_0)x_3a^3 + D^{(2)}(f)(x_0)x_1x_2a^3 + \frac{1}{6}D^{(3)}(f)(x_0){x_1}^3a^3 = 0$$

ix) Iterate $H(x,\lambda)=(1-\lambda)(x^n-c)+\lambda q(x)$, where $\lambda\in[0,1]$ e.g. 0.0, 0.2, 0.4, 0.6, 0.8 and 1.0 by using the Higher Order Homotopy Taylor-perturbation; can change the step size , $\frac{1}{n}$:

$$newt := x \to evalf[10]\left(x - \frac{H(x)}{DH(x)}\right) \tag{6}$$

Or,

$$newt := x \to evalf[10]\left(x - \frac{H(x)}{H1(x)} - \frac{H2(x)\cdot(H(x))^2}{2\cdot(H1(x))^3}\right) \tag{7}$$

Or,

$$newt := x \to evalf[10]\left(x - \frac{H(x)}{H1(x)} - \frac{H2(x)\cdot(H(x))^2}{2\cdot(H1(x))^3} + B\right) \tag{8}$$

where,

$$B = \left((H(x))^3 \cdot \frac{H1(x)\cdot H3(x) - 3(H2(x))^2}{6\cdot(H1(x))^5}\right);$$

Or,

$$newt := x \to evalf[10]\left(x - \frac{H(x)}{H1(x)} - \frac{H2(x)\cdot(H(x))^2}{2\cdot(H1(x))^3} + B + C\right)$$

where,

$$B = \left(+(H(x))^3 \cdot \frac{H1(x)\cdot H3(x) - 3(H2(x))^2}{6\cdot(H1(x))^5}\right);$$

$$C = \left(+\frac{H2(x)^2\cdot H(x)^4}{4\cdot(H1(x))^6}\right); \tag{9}$$

Or,

$$newt := x \rightarrow evalf[10]\left(x - \frac{H(x)}{H1(x)} - \frac{H2(x)\cdot(H(x))^2}{2\cdot(H1(x))^3} + B + C + D \right)$$

where,

$$B = \left(+(H(x))^3 \cdot \frac{H1(x)\cdot H3(x) - 3(H2(x))^2}{6\cdot(H1(x))^5} \right);$$

$$C = \left(+ \frac{H2(x)^2 \cdot H(x)^4}{4\cdot(H1(x))^6} \right);$$

$$D = \left(-\frac{1}{12\cdot(H1(x))^8}((H2(x))^2\cdot(H(x))^5.(H3(x) - 3\cdot H1(x))) \right) \quad (10)$$

3 Numerical Examples and Analysis

Table 1 shows the list of nonlinear equations used in applying the higher order Homotopy Taylor-Perturbation. The actual roots and the Homotopy functions are calculated and compared.

Table 1: Nonlinear equations

No.	Nonlinear	$\lambda \rightarrow H^*(x,\lambda)$, where $\lambda \in [0,1]$; Actual roots	Previously used by
a	$x^4 + 6x - 40$	$x^4 - 40 + 6\lambda x$ 2.66728462,-2.66728462,	[1, 2] and [6].
b	$x^2 - 3x + 2 - e^x$	$\lambda x^2 - 3x + 2 - \lambda e^x$ 0.2575302854	[2] and [8].
c	$\cos(x) - x$	$\cos(x) - \lambda x$ 0.7390837332	[2] and [8].
d	$\sin(x) + e^x + x^4 - 2$	$\lambda \sin(x) + \lambda e^x + x^4 - 2$ 0.4342159162	-
e	$(x-2)^2 - \ln(x)$	$(x-2)^2 - \ln(x)$ 3.057103550, 1.412391172	[7]
f	$(x-2)^2.(x^4 + 6x - 40)$	-2.514866859, 2.514866859	-
g	$(x-2)^2.(\sin(x))$	1.259921050	-
h	$(x^2 - e^x - 3x + 2).(\cos(x))$	0.0, 3.000000000	-

Meanwhile, The dashed-green line, $p(x)$, is the start-system line where we freely choose a new equation from $f(x)$ represented by the solid-red line, where $p(x) \subset f(x)$. Figure 1 and Figure 2 illustrate the closeness of the startsystem values suggested to the real roots for Eq. (b) and Eq. (c), respectively. The advantage of this procedure, it eliminates the time needed to decide on the initial value, x_0.

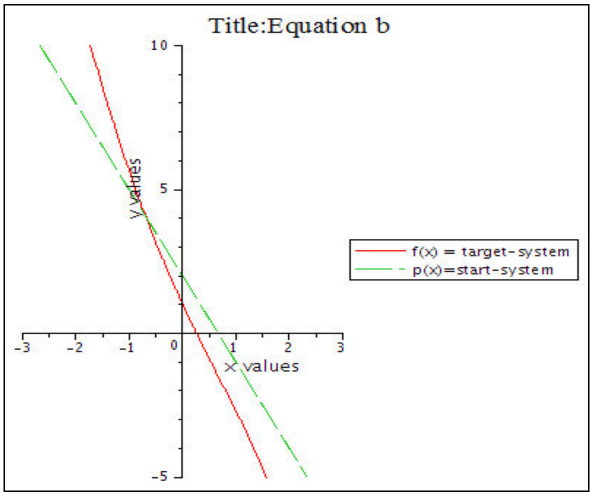

Figure 1: Eq. (b)

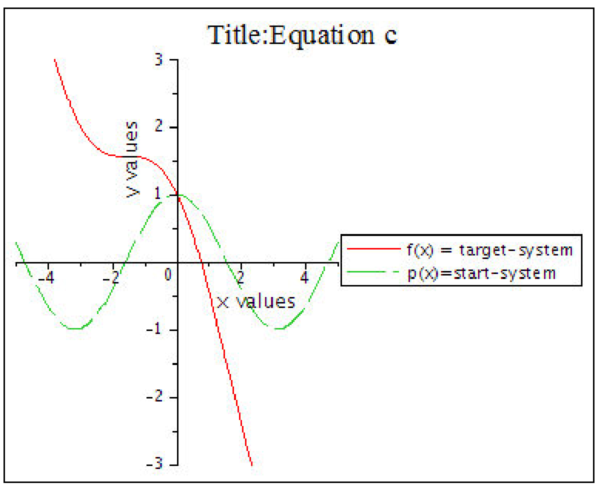

Figure 2: Eq. (c)

The list of the higher order correctional terms Homotopy Taylor-perturbation (HHTP) method by using start-system can be referred at Table 2. Most of the algorithms have been simplified into simpler forms and iterations are done using mathematical software *Maple14*. The choice of a suitable $p(x)$ is not unique and different choices of $p(x)$ work better for different types of equations. Here, the step size $h = \frac{1}{N}$, *where* $N = 5$. So it is set to 0.2 and the stopping-criteria are set to 1×10^{-6}.To determine the initial value x_0, only equate $p(x)$ to zero.

Table 2: The iteration scheme of the Higher Order of Homotopy Taylor-perturbation (HHTP) Method with start-system

Correc-tional Terms	Higher Order Homotopy Taylor-perturbation Method (iterative form, $x = x_{(i+1)}$) ([4; [1,2,3])
1^{st} order	$x_i - \frac{H(x_i,\lambda)}{H'(x_i,\lambda)}$
2^{nd} order	$x_i - \frac{H(x_i,\lambda)}{H'(x_i,\lambda)} - \frac{H''(x_i,\lambda).H^2(x_i,\lambda)}{2H'^3(x_i,\lambda)}$
3^{rd} order	$x_i - \frac{H(x_i,\lambda)}{H'(x_i,\lambda)} - \frac{H''(x_i,\lambda).H^2(x_i,\lambda)}{2.H'^3(x_i,\lambda)} + H^3(x_i,\lambda).\frac{H'(x_i,\lambda).H'''(x_i,\lambda) - 3.H''^2(x_i,\lambda)}{6H'^5(x_i,\lambda)}$
4^{th} order	$x_i - \frac{H(x_i,\lambda)}{H'(x_i,\lambda)} - \frac{H''(x_i,\lambda).H^2(x_i,\lambda)}{2.H'^3(x_i,\lambda)} + H^3(x_i,\lambda).\frac{H'(x_i,\lambda).H'''(x_i,\lambda) - 3.H''^2(x_i,\lambda)}{6H'^5(x_i,\lambda)} + \frac{H''^2(x_i,\lambda).f^4(x_i,\lambda)}{4.H'^6(x_i,\lambda)}$
5^{th} order	$x_i - \frac{H(x_i,\lambda)}{H'(x_i,\lambda)} - \frac{H''(x_i,\lambda).H^2(x_i,\lambda)}{2.H'^3(x_i,\lambda)} + H^3(x_i,\lambda).\frac{H'(x_i,\lambda).H'''(x_i,\lambda) - 3.H''^2(x_i,\lambda)}{6H'^5(x_i,\lambda)} + \frac{H''^2(x_i,\lambda).f^4(x_i,\lambda)}{4.H'^6(x_i,\lambda)} - \frac{H''^2(x_i,\lambda).f5(x_i,\lambda).[f'''(x_i,\lambda) - 3.f'(x_i,\lambda)]}{2.H'^8(x_i,\lambda)}$.

In Table 3, five (5) nonlinear equations were iterated using 1st, 2nd and 3rd order HHTP with and without start-system functions. Results indicate the 2nd order produces less numbers of iterations needed to converge to its root/s.

Table 3: The approximated zeros using Higher Order Homotopy Taylor-perturbation (HHTP) using start-system which involve the 1st, 2nd and 3rd Order Correctional Terms [1]

Functions $q(x) = f(x)$ = target system=0	Start-system = $p(x) = 0$; initial value, x_0	NHP-1st Steps (i)-(ix) ; Eq.(6) (woss)	(ss)	HHTP-2nd Steps (i)-(ix); Eq.(7) (woss)	(ss)	HHTP-3rd Steps (i)-(ix); Eq.(8) (woss)	(ss)
$x^4 + 6x - 40$	$x^4 - 40$;	4*		3*		3*	
	2.514866859 or,	4	(3)	3	(3)	2	(3)
	-2.514866859	4		3*		3*	
$x^2 - 3x + 2 - e^x$	$-3x + 2$;	3*		3*		3*	
	0.6666666667	3*	(4)	3*	(3)	2	(4*)
$\cos(x) - x$	$\cos(x)$;	5		2		3	
	1.570796327	4	(4)	3	(3)	3	(4)
$\sin(x) + e^x + x^4 - 2$	$x^4 - 2$;	3		2		2	
	0.4342159162	4	(4)	3	(3)	3*	(3)

$(x-2)^2-\ln(x)$	$x-2$; 2.000000000	3 3	(6)	3 2	(5)	2 3	(C)

In addition, Table 4 (below) shows that the efficiency of the iterative 4th order and 5th order homotopy perturbation method using start-system which gives equal or better results in terms of convergence rate as compared to the classical Newton-Raphson. The given test functions (a) - (h) are used and the some of the approximated zeros are displayed in Table 3. It seems that the computations converge in less than 5 iterations, even for the multiroots functions (e), (f), (g) and (h). Also refer to the numbers that are bolded, which show that in some cases, higher order can be a better choice compared to the lower orders.

Table 4: The approximated zeros using Higher Taylor-perturbation (HTP) and Higher Order Homotopy Taylor-perturbation (HHTP) using startsystem: 4th and 5th Order Correctional Terms on single functions and multiple functions

Functions $q(x)=f(x)=$ target system=0	Start-system (ss) = $p(x)=0$; initial value, x_0	HTP-4th. Steps (i)-(ix) ; Eq.(9) (woss), (ss)	HTP-5th Steps (i)-(ix) ; Eq.(9) (woss), (ss)	HHTP-4th Steps (i)-(ix) ; Eq.(10) (woss), (ss)	HHTP-5th Steps (i)-(ix); Eq.(10) (woss), (ss)
(a) $x^4+6x-40$	x^4-40 ; 2.514866859 or, -2.514866859	-2.5 (3) 2.5 (3) 2.0 (3), ss (3)	-2.5 (3) 2.5 (3) 2.0 (3), ss (4)	-2.5 (2) 2.5 (3) 2.0 (3), ss (3)	-2.5 (3) 2.5 (3) 2.0 (3), ss (3)
(b) $x^2-3x+2-e^x$	$-3x+2$; 0.6666666667	0.3 (3) 0.5 (3), ss (3)	0.3 (2) 0.5 (3), ss (3)	0.3 (3) 0.5 (3), ss (3)	0.3 (2) 0.5 (2), ss (3*)
(c) $\cos(x)-x$	$\cos(x)$; 1.570796327	0.9 (2) 0.5 (3), ss (3)	0.9 (2) 0.5 (3), ss (3)	0.9 (2) 0.5 (2), ss (3)	0.9 (2) 0.5 (3), ss (3)
(d) $\sin(x)+e^x+x^4-2$	x^4-2 ; -1.189 207115, 1.189207115	0.8 (3) 0.5 (2), ss (3)	0.8 (3) 0.5 (2), ss (3)	0.8 (4) 0.5 (3), ss (2,3)	0.8 (3) 0.5 (2), ss (2,3)
(e) $(x-2)^2-\ln(x)$	$x-2$; 2.000000000	1.5 (2) 2.5 (7) 3.0(div), ss (div)	1.5 (2) 2.5 (7) 3.0(div), ss (div)	1.5 (2) 2.5 (7) 3.0 (2), ss (div)	1.5 (2) 2.5 (7) 3.0 (2), ss (div)
(f) $(x-2)^2.(x^4+6x-40)$	(x^4-40) 2.514866859	-	-	2.5 (3*) -3 (3), ss (3*)	2.5 (3*) -3 (3), ss (3*)

(g) $(x-2)^2.(\sin(x))$	$(x-2)^2.(\sin(x))$ 1.259921050	-	-	1.5 (3) 3.0 (3), ss (3)	1.5 (5) 3.0 (3), ss (3)
(h) $(x^2-e^x-3x+2).(\cos(x)$	(x^2-3x) 3.000000000	-	-	1.5 (2) 0.5 (4), ss (3)	1.5 (2) 0.5 (4), ss (4)

Moreover, the systematic convergence of higher order homotopy Taylor-perturbation via start-system (HHTPss) algorithms, even though it might takes a bit longer in comparison to Newton (1^{st} order) but all 5 orders of HHTPss seem stable and consistently converge to its roots. This hybrid HHTPss method may converge to a root differently, faster or slower, depending on the different selection of the start-system function.

The advantages of HHTPss are (i) it's solvable for a more complex nonlinear single-equations (ii) the existence of the convex homotopy that are bounded from zero to one, guaranteed in most complex cases to converge to its root/s in a faster and more steady and stable iterations. Thus, we believe that HHTPss is a good enough approximation to $f(x)=0$.

4 Conclusions

As a conclusion, it is very important to choose an initial point, a proper startsystem and order of correctional terms in order to ensure convergence is fast and computing time is reduced. The combinations of the startsystem and the higher order homotopy Taylor-perturbation serve as reliable and flexible tools that offer convergence at equal or faster than the existing methods in solving nonlinear equations.

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