App. Math. and Comp. Intel., Vol. 1 (2012) 48–55
http://amci.unimap.edu.my

Numerical solution of integro differential equations based on double exponential transformation in the sinc-collocation method

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Received: 11 January 2012; Revised: 19 April 2012; Accepted: 18 July 2012

Abstract: *In this paper, we consider double exponential transformations* **(DE)** *to solve integro differential equations by Sinc collocation method. Numerical examples illustrate the validity and applicability of the method, in addition the method is easy to use and yields very accurate results.*

Keywords: *Integro differential equations, Sinc collocation method, Double exponential transformation.*

PACS: *02.60.Nm*

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1 Introduction

Integro differential equations occur in various areas of engineering, mechanics, physics, chemistry, astronomy, biology, economies, potential theory, electrostatic, etc [15,16]. Many methods were used to handle these equations such as the successive approximations, Adomian decomposition, Homotopy perturbation, Chebyshev and Taylor collocation, Haar Wavelet, Tau series methods, etc [1,2]. The main purpose of the present research is to consider the double exponential transformation in the Sinc collocation method for integro differential equations. The Double Exponential transformation, abbreviated as **DE** was first proposed by Takahasi and Mori [3,14] in 1974 for one dimensional numerical integration and it has come to be widely used in applications. It is known that the double exponential transformation gives an optimal result for numerical evaluation of definite integral of an analytic function [6,7]. In 1997, Sugihara [11,12] established the "Meta-Optimality " of the DE formulas in a mathematically rigorous manner, and since then it has turned out that DE transformation is also useful for other various kinds of numerical methods. Indeed it has been demonstrated that the use of the Sinc method in cooperate with the DE transformation gives highly efficient numerical methods for approximation of function, indefinite numerical

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integration and solution of differential equations. Recently, Muhammad et al. in [8] established a method of indefinite numerical integration based on DE transformation incorporated into Sinc expansion of the integrand which gives results with high efficiency [9,13]. In the standard setup of the Sinc numerical methods, the error are known to be $O(\exp(-k\sqrt{N}))$ with some $k > 0$, where N is the number of nodes or bases which is used in the methods[10]. However, Sugihara has recently in [13] found that the errors in the Sinc numerical methods are $O(\exp(-cN/logN))$ with some $c > 0$, which is also meaningful practically.

The purpose of this paper is to develop the works proposed in [7] and [4], for the numerical solution of Integro differential equations, which has taken a wide spectrum of applications, by DE transformation based on the Sinc collocation method and reliability and efficiency of the proposed scheme are demonstrated by some numerical experiments. The layout of the paper is as follows: in section 2,we give basic definitions, assumptions and preliminaries of the Sinc approximations and related topics. In section 3, a Sinc collocation method is considered for numerical solution of integro differential equations. Finally, section 4 contains the details of our numerical implementation and some experimental results.

2 Basic definitions and preliminaries

In this paper, we consider the Sinc collocation method for the numerical solution of the following two order linear integro differential equation:

$$\sum_{i=0}^{2} \beta_i(x)u^{(i)}(x) = f(x) + \int_a^x K(x,t)u(t)dt, \tag{1}$$

$$u(a) = a_0, u'(a) = a_1 \tag{2}$$

where a, a_0, a_1 are real constants $\beta_0(x), \beta_1(x)\beta_2(x), f(x), K(x,t)$ are given functions and $u(x)$is to be determined. Let f be a function defined on$\mathbb{R}$ and $h > 0$ is step size then the Whittaker cardinal defined by the series

$$C(f,h)(x) = \sum_{j=-\infty}^{\infty} f(jh)S(j,h)(x) \tag{3}$$

whenever this series convergence, and

$$S(j,h)(x) = \frac{\sin[\pi(x-jh)/h]}{\pi(x-jh)/h}, \quad j = 0, \pm 1, \pm 2, .. \tag{4}$$

where $S(j,h)(t)$ is known as j-th Sinc function evaluated at t.

Throughout of this paper, let $d > 0$, and $D_d = \{z = x+iy \quad |y| < d\}$ in the complex plan $\mathcal{C}$ and ϕ the conformal map of a simply connected domain D in the complex domain onto D_d, such that $\phi(a) = -\infty, \phi(b) = \infty$, where a, b are boundary points of D with $a, b \in \partial D$. Let ψ denotes the inverse map of ϕ, and let the curve Γ, with end points $a, b \ (a, b \in \Gamma)$, given by $\Gamma = \psi(-\infty, \infty)$. For $h > 0$, let the points x_k on Γ given by $x_k = \phi^{-1}(kh), \ \ k \in Z$.

Moreover, let us consider $H^1(D_d)$ be the family of all functions g analytic in D_d, such that

$$N_1(g, D_d) = \lim_{\epsilon\to 0} \int_{\partial D_{d(\epsilon)}} |g(t)||dt| < \infty,$$

$$D_{d(\epsilon)} = \{t \in C, \quad |Ret| < \tfrac{1}{\epsilon}, \quad |Imt| < d(1-\epsilon)\}.$$

We recall the following definitions from [6,7], that will become instrumental in establishing our useful formulas:

Definition 1. *A function g is said to be decay double exponentially, if there exist constants α and C, such that:*

$$|g(t)| \le C\exp(-\alpha \exp|t|), \quad t \in (-\infty, \infty)$$

equivalently, a function g is said to be decay double exponentially with respect to conformal map ϕ, if there exist positive constants α and C such that:

$$|g(\phi(t))\phi'(t)| \le C\exp(-\alpha \exp|t|), \quad t \in (-\infty, \infty).$$

Here, we suppose that $K^{\alpha}_{\phi}(D_d)$ denotes the family of functions g where $g(\phi(t))\phi'(t)$ belongs to $H^1(D_d)$ and decays double exponentially with respect to ϕ. If f belongs to $K^{\alpha}_{\phi}(D_d)$ with respect to ϕ, then we have the following formulas for definite and indefinite integrals based on DE transformation which is given and fully discussed in [3,7]:

$$\int_a^b f(x)dx = h\sum_{j=-N}^{j=N} f(\phi(jh))\phi'(jh) + O(\exp(\frac{-2\pi dN}{log(2\pi dN/\alpha)})),$$

and

$$\begin{aligned}\textstyle\int_a^s f(x)dx &= h\textstyle\sum_{j=-N}^{j=N} f(\phi(jh))\phi'(jh)(\frac{1}{2} + \frac{1}{\pi}si(\frac{\pi\psi(s)}{h} - j\pi)) \\ &\quad + O(\tfrac{logN}{N}\exp(-\tfrac{\pi dN}{log(\pi dN/\alpha)})),\end{aligned}$$

where $Si(t)$ is the Sine integral defined by:

$$Si(t) = \int_0^t \frac{\sin w}{w}dw,$$

and the mesh size h satisfies $h = \frac{1}{N}\log(\pi dN/\alpha)$.

3 Main Idea

To explain Sinc collocation method, we suppose in the right-hand side of (1) that $K(x, .)u(.) \in K^{\alpha}_{\phi}(D_d)$. Then by using indefinite integration formula for second term in right-hand side of Volterra integral equations (1) we obtain:

$$\int_a^x K(x,t)u(t)dt \simeq h\sum_{j=-N}^{N} K(x,t_j)\phi'(jh)(\frac{1}{2} + \frac{1}{\pi}Si(\frac{\pi\phi^{-1}(x)}{h} - j\pi))u_j \tag{5}$$

where u_j denotes an approximate value of $u(t_j), t_j = \phi(jh)$ where [6]

$$\phi(t) = \frac{b-a}{2}\tanh(\frac{\pi}{2}\sinh t) + \frac{a+b}{2}, \tag{6}$$

and

$$\phi'(t) = \frac{b-a}{2}\frac{\pi/2\cosh(t)}{\cosh^2(\pi/2\sinh(t))} \tag{7}$$

We assume that $u(x)$, the solution of (1) is approximated by the finite expansion of Sinc basis functions:

$$u_m(x) = \sum_{j=-N}^{N} u_j S(j,h)o\phi(x), \quad m = 2N+1. \tag{8}$$

The n-th derivative of function $u_m(x)$ at points $x_k = \phi(kh)$ can be approximated by using finite number of terms as:

$$u_m^{(n)}(x) = h^{-n} \sum_{j=-N}^{N} \delta_{jk}^{(n)} u_j, \tag{9}$$

where

$$\delta_{jk}^{(m)} = \frac{d^m}{d\phi^m} S(j,h) o\phi(t)|_{t=kh}, \quad m = 0,1,2 \tag{10}$$

by simple calculations we have in particular:

$$\delta_{jk}^{(0)} = \begin{cases} 1 & \text{if} \quad j = k \\ 0 & \text{if} \quad j \neq k, \end{cases}$$

$$\delta_{jk}^{(1)} = \begin{cases} 1 & \text{if} \quad j = k \\ \frac{(-1)^{(k-j)}}{(k-j)} & \text{if} \quad j \neq k, \end{cases}$$

$$\delta_{jk}^{(2)} = \begin{cases} \frac{-\pi^2}{3} & \text{if} \quad j = k \\ \frac{-2(-1)^{(k-j)}}{(k-j)^2} & \text{if} \quad j \neq k. \end{cases}$$

If we replace $u(x)$ in (1) by (8) and using (5) we have:

$$\sum_{j=-N}^{N} \left\{ \sum_{i=0}^{2} \beta_i(x) \frac{d^i}{dx^i} S(j,h) o\phi(x) - hK(x,t_j)\phi'(jh)\eta_{j,h}(x) \right\} u_j \simeq f(x), \tag{11}$$

where

$$\eta_{j,h}(x) = \left(\frac{1}{2} + \frac{1}{\pi} Si\left(\frac{\pi\phi^{-1}(x)}{h} - j\pi\right)\right)$$

Setting[5]

$$\frac{d^i}{d\phi^i} S(j,h) o\phi(x) = S_j^{(i)}(x), \quad i = 0,1,2 \tag{12}$$

We note that

$$\tfrac{d}{dx} S(j,h) o\phi(x) = S_j^{(1)}(x)\phi'(x),$$

$$\tfrac{d^2}{dx^2} S(j,h) o\phi(x) = S_j^{(2)}(x)[\phi'(x)] + S_j^{(1)}(x)\phi''(x)$$

To find unknown $u_j, j = -N, ..., N$, we can apply the Sinc collocation points $x_k = \phi(kh), k = -N, ..., N$, so we have following system of $(2N+1)(2N+1)$ unknowns u_j

$$\sum_{j=-N}^{N} [\sum_{i=0}^{2} g_i(x_k) \frac{(-1)^i \delta_{kj}^{(i)}}{h^i} - h\delta_{kj}^{(-1)} K(x_k, x_j)] u_j = f_k, \quad k = -N, ..., N \tag{13}$$

where

$$g_0(x_k) = \beta_0(x_k), \quad g_2(x_k) = \beta_2(x_k)(\phi'(x_k))^2,$$

$$g_1(x_k) = \beta_1(x_k)\phi'(x_k) + \beta_2(x_k)\phi''(x_k),$$

$$\delta_{kj}^{(-1)} = \left(\tfrac{1}{2} + \tfrac{1}{\pi} Si(\pi(k-j))\right), \quad k,j = -N..N.$$

We can write the algebraic system (13) in a matrix form as

$$M\mathbf{u} = \mathbf{f} \tag{14}$$

where the component of matrix $M = (M_{kj})$ is:

$$M_{kj} = \sum_{i=0}^{2} g_i(x_k) \frac{(-1)^i \delta_{kj}^{(i)}}{h^i} - h\delta_{kj}^{(-1)} K(x_k, x_j), \quad k, j = -N..N$$

and the vectors **u** and **g** are

$$\mathbf{u} = \begin{bmatrix} u_{-N} \\ \vdots \\ u_N \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} f(x_{-N}) \\ \vdots \\ f(x_N) \end{bmatrix}$$

By solving the above linear system of equations, we obtain u_j which approximate $u(x)$ at Sinc point $u(x_j)$. Furthermore in order to get an approximate value of $u(x)$ at arbitrary point x, we can apply the method in [8].

4 Numerical Experiments

In this section, the theoretical results of the previous sections are illustrated by two numerical examples. In all examples, d taken to be $d = \pi/2$ and $\alpha = 1$, also for the computation of $Si(x)$, we evaluated it directly using the integral representation:

$$Si(x) = \frac{\pi}{2} - f_1(x)\cos(x) - f_2 \sin(x)$$

where

$$f_1(x) = \int_0^\infty \frac{x \exp(-t)}{t^2 + x^2} dt, \quad f_2(x) = \int_0^\infty \frac{t \exp(-t)}{t^2 + x^2} dt$$

We consider the following two test problems.

Example 1.

For the sake of comparison, we consider the problem discussed by Mohsen and El-Gamel in [5]. They used single exponential transformation (SE) and numerical results are listed below, where $\|Es\|$ is maximum absolute error.

$$u''(x) + \tfrac{1}{\sin(\pi x)} u(x) = f(x) + \textstyle\int_0^x K(x,t)u(t)dt$$

$$f(x) = -\pi^2 \sin(\pi x) - 1 + \tfrac{\pi(1+x)\cos(\pi x) - \sin(\pi x) - \pi}{\pi^2(1+x^2)},$$

$$K(x,t) = \tfrac{1+t}{1+x^2}, \quad u(0) = u(1) = 0,$$

with the exact solution: $u(x) = \sin(\pi x)$.

The maximum errors for$N = 10, 20, 40, 60, 80$ are listed in Table 1:

Our calculations are carried out in double precision arithmetic with Maple 13. We use DE transformations to approximate $u(x)$ by applying $\phi(t)$ and Sinc points. The results are shown in Table 2:

Comparing results show that, DE transformation has high efficiency and accuracy because we take $N = 3, 5, 7, 10, 15$ and have very accurate approximation in comparing by the results in [5].

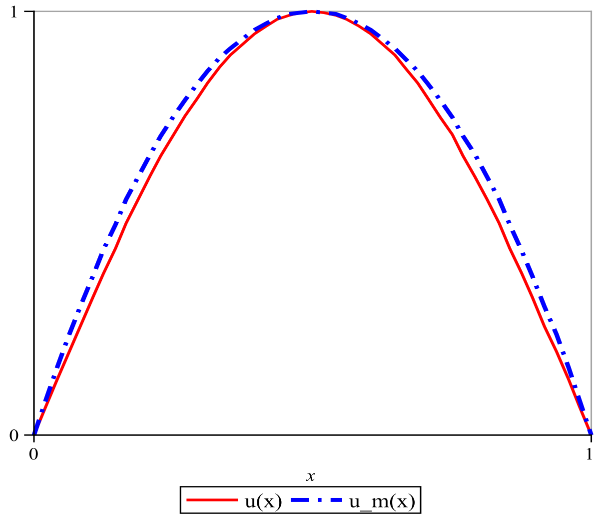

Figure 1: $u(x)$ the exact and $u_m(x)$ approximate solution of Example 1

Table 1: Max-Error for Example 1

N	$\|Es\|$
10	1.490000E-003
20	9.800000E-005
40	1.836250E-006
60	8.255625E-008
80	7.820003E-009

Table 2: Max-Error for Example 1 using DE transformation

N	$\|Es\|$
3	2.300000E-003
5	4.200000E-005
7	4.800250E-007
10	8.255625E-008
15	8.320003E-012

Example 2.

Let the following integro differential equation:

$$u''(x) + xu'(x) + u(x) = f(x) + \int_0^x K(x,t)u(t)dt$$

$$f(x) = 3x^2 - 2x - \exp(2x)(2x-3) - 3\exp(x) + 2$$

$$K(x,t) = \exp(x+t), \quad u(0) = u(1) = 0,$$

with the exact solution: $u(x) = x(x-1)$

The maximum errors for $N = 3, 5, 7, 9, 10$ ares shown in Table 3.

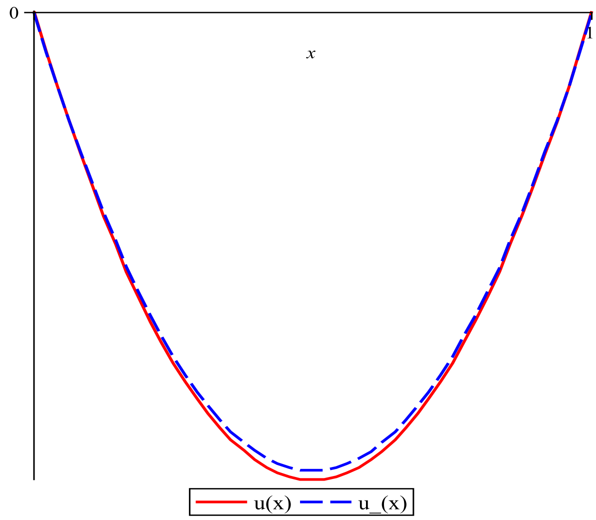

Figure 2: $u(x)$ the exact and $u_m(x)$ approximate solution of Example 2.

Table 3: Max-Error for Example 2.

N	$\|Es\|$
3	1.261000E-001
5	5.885000E-002
7	0.636250E-005
9	8.144705E-008
10	4.232003E-009

By taking small value for N, we also obtain very accurate approximation, which is important in the point of view of numerical methods and computational algorithms.

5 Conclusion

We applied the Sinc collocation method based on double exponential transformation to integro differential equations, we observe that significant improvement have been obtained compared with numerical results reported by others. Also, we improved the accuracy of the solution by selecting the appropriate shape parameters and selecting the large values of N. The results of the examples showed the high accuracy of the proposed method. This method is also able to save the time and decrease the number of calculations. In addition this method is portable to other area of problems and easy to programming.

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