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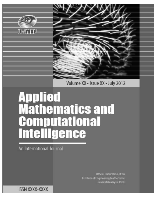

Positive periodic solutions of singular first order difference equations

O. Omar^{a,*}, **M. Mohamed**^{b}, **and S. Muhammad**^{c}

^{a}*Faculty of Computer and Mathematical Sciences*
Universiti Teknologi Mara (Perlis)
02600 Arau, Perlis, Malaysia.

^{b}*Department of Mathematics, Universiti Teknologi Mara (Perlis)*
02600 Arau, Perlis, Malaysia.

^{c}*Department of Civil Engineering, Universiti Teknologi Mara (Perlis)*
02600 Arau, Perlis, Malaysia.

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Abstract: *In this paper, we employ Kranoselskii fixed point theorem and obtain sufficient conditions for the existence and multiplicity of positive periodic solution to the singular first order difference equation*

$$\Delta x(k) = -a(k)x(k) + \lambda b(k)f(x(k)), \quad k \in \mathbb{Z}.$$

Keywords: *Periodic solutions, Singular first order, Kranoselskii fixed point theorem.*

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1 Introduction

Let $\mathbb{R}$ denote the real numbers, $\mathbb{Z}$ the integers and $\mathbb{R}_+ = [0, \infty)$, the positive real numbers. Given $a < b$ in $\mathbb{Z}$, let $[a, b] = \{a, a + 1, \dots, b\}$.

In this paper, we investigate the existence and multiplicity of positive periodic solutions for singular first order difference equation

$$x(k + 1) = (1 - a(k))x(k) + \lambda b(k)f(x(k)), \quad k \in \mathbb{Z} \tag{1}$$

where $\mathbb{Z}$ is the set of integer numbers, $\omega \in \mathbb{N}$ is a fixed integer, $\lambda > 0$ and $b : \mathbb{Z} \to [0, \infty)$, $a(k)$ are $\omega-$periodic and $a(k)$ is continuous with $0 < a(k) \leq 1$ for all $k \in [0, \omega - 1]$ and $f \in C(\mathbb{R}^n_+ \setminus \{0\}, (0, \infty))$.

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[*] Corresponding Author: zincrocker@yahoo.com (O. Omar)

The study of the existence of periodic solutions in difference equations was motivated by the observance of periodic phenomena in mathematical ecological difference models, discrete single-species models and discrete populations models, see for examples, [3, 4, 5, 6, 8, 9, 10, 13, 14, 15]. Although most models are described with differential equations, the discrete models are more appropriate than the continuous ones when the size of the population is rarely small or the population has non-overlapping generations [1].

Recently, Kranoselskii fixed point theorem has become an effective tool in proving the existence of periodic solutions. It seems that the Kranoselskii fixed point theorem on compression and expansion of cones is quite effective in dealing with the problem. In fact, by choosing appropriate cones, the singularity of the problem is essentially removed and the associated operator becomes well-defined for certain ranges of functions even there are negative terms.

Wang [12] employed the Kranoselskii fixed point theorem to establish the existence and multiplicity of positive periodic solutions for first non-autonomous singular systems

$$x_i'(t) = -a_i(t)x_i(t) + \lambda b_i(t) f_i(x_1(t), \ldots, x_n(t),$$

where $i = 1, \ldots, n$. In [1, 2], the authors showed the existence of periodic solutions for singular first order differential equations. On the other hand, [15] Zeng proved the existence of positive periodic solutions for a class of non-autonomous difference equation

$$\Delta x(k) = -a(k)x(k) + f(k, u(k))$$

where the operator Δ is defined as $\Delta x(k) = x(k+1) - x(k)$.

Inspired by the above work, we consider to carry the work of Wang, [12] to the discrete case for scalar difference equations. We shall establish a new result on the existence and multiplicity of positive solutions of equation (1) by utilizing the well-known theory of Kranoselskii fixed point theorem.

2 Preliminaries

In this section we state some preliminaries in the form of lemmas that are essential to proofs our main results.

Let X be the set of all real ω-periodic sequences $x : \mathbb{Z}_+ \to \mathbb{R}_+^n$, endowed with the maximum norm

$$\|x\| = \max_{k \in [0, \omega-1]} |x(k)| .$$

Thus X is a Banach space. Throughout this paper, we denote the product of $x(k)$ from $k = a$ to $k = b$ with the understanding that $\prod_{k=a}^{b} x(k) := 1$ for all $a > b$. Let $\mathbb{R}_+^n = \prod_{i=1}^{n} \mathbb{R}_+$. We make the following assumptions:

(H1) $0 < a(k) \leq 1$ for all $k \in [0, \omega - 1]$.

(H2) $f : \mathbb{R}_+^n \setminus \{0\} \to (0, \infty)$ is continuous.

We now state the Kranoselskii fixed point theorem [7].

Lemma 1. *Let X be a Banach space, and let $K \subset X$ be a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$, and let*

$$T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that either *(i)* $\|Tx\| \leq \|x\|, x \in K \cap \partial\Omega_1$ *and* $\|Tx\| \geq \|x\|, x \in K \cap \partial\Omega_2$; *or*

(ii) $\|Tx\| \geq \|x\|, x \in K \cap \partial\Omega_1$ *and* $\|Tx\| \leq \|x\|, x \in K \cap \partial\Omega_2$;

Then T *has a fixed point in* $T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$.

Lemma 2. *[15] Assume (H1), (H2) hold. If* $x \in X$ *then* x *is a solution of (1) if and only if*

$$x(k) = \sum_{s=k}^{k+\omega-1} G(k,s)\lambda b(s) f(x(s)),$$

where

$$G(k,s) = \frac{\prod_{r=s+1}^{k+\omega-1}(1-a(r))}{1-\prod_{r=0}^{\omega-1}(1-a(r))}, \quad s \in [k, k+\omega-1]. \tag{2}$$

Note that the denominator in $G(k,s)$ is not zero since $0 < a(k) < 1$ for $k \in [0, \omega-1]$.

It is clear that $G(k,s) = G(k+\omega, s+\omega)$ for all $(k,s) \in \mathbb{Z}^2$. A direct calculation shows that

$$m := \frac{\prod_{r=0}^{\omega-1}(1-a(r))}{1-\prod_{r=0}^{\omega-1}(1-a(r))} \leq G(k,s) \leq \frac{1}{1-\prod_{r=0}^{\omega-1}(1-a(r))} =: M.$$

Define $\sigma = \prod_{r=0}^{\omega-1}(1-a(r))$ satisfying

$$\frac{\sigma}{1-\sigma} \leq G(k,s) \leq \frac{1}{1-\sigma}, \quad k \leq s \leq k+\omega.$$

Thus, clearly $\sigma = \frac{m}{M} > 0$,

$$\|x\| = \max_{k \in [0,\omega-1]} |x(k)| \leq M \sum_{k=0}^{\omega-1} \lambda b(k) f(x(k)).$$

Therefore

$$\begin{aligned} x(k) &\geq m\lambda \sum_{k=0}^{\omega-1} b(k) f(x(k)) \\ &\geq \frac{m}{M}\lambda \sum_{k=0}^{\omega-1} b(k) f(x(k)) \\ &\geq \sigma \|x\|. \end{aligned}$$

Now we define a cone

$$K = \left\{ x \in X, k \in [0,\omega], x(k) \geq \frac{m}{M}\|x\| = \sigma\|x\| \right\}.$$

It is clear that K is a cone in X and $\min_{k \in [0,\omega]} |x(k)| \geq \sigma\|x\|$ for $x \in K$. For $r > 0$, define $\Omega_r = \{x \in K : \|x\| < r\}$. Note that $\partial\Omega_r = \{x \in K : \|x\| = r\}$. Define a mapping $T : X \to X$ by

$$Tx(k) = \lambda \sum_{s=k}^{k+\omega-1} G(k,s) b(s) f(x(s)), \tag{3}$$

where $G(k,s)$ is given by (2). By the nonnegativity of $\lambda, f, a, b,$ and G, $Tx(k) \geq 0$ on $[0, \omega-1]$. It is clear that $Tx(k+\omega) = Tx(k)$.

Lemma 3. *$T : K \backslash \{0\} \subset K$ is well-defined.*

Proof. For any $x \in K\backslash \{0\}$, for all $k \in [0, \omega]$ we have

$$\|Tx\| = \max_{k\in[0,\omega-1]} |Tx(k)| \leq M \sum_{s=0}^{\omega-1} \lambda b(s) f(x(s)).$$

Therefore

$$\begin{aligned} Tx(k) &= \lambda \sum_{s=k}^{k+\omega-1} G(k,s) b(s) f(x(s)) \\ &\geq \lambda m \sum_{s=0}^{\omega-1} b(s) f(x(s)) \\ &\geq \frac{m}{M} \|Tx\|. \end{aligned}$$

Hence $Tx(k) \geq \sigma \|Tx\|$. This implies that $T : K\backslash \{0\} \subset K$. □

Lemma 4. *If (H1) and (H2) hold, then the operator $T : K\backslash\{0\} \to K$ is completely continuous.*

Proof. Let $x_m(k), x_0(k) \in K\backslash \{0\}$ with $x_m(k) \to x_0(k)$ as $m \to \infty$. From (3) and since $f(k, \xi)$ is continuous in ξ, as $m \to \infty$, we have

$$|Tx_m(k) - Tx_0(k)| \leq M \sum_{s=0}^{\omega-1} \lambda |b(s)| |f(x_m(s)) - f(x_0(s))| \to 0.$$

Hence $\|Tx_m(k) - Tx_0(k)\| \to 0$, it follows that the operator T is continuous. Further if $x \subset X$ is a bounded set, then $\|x\| \leq C_1 = \mathit{const}$ for all $x \in K\backslash\{0\}$. Set $C_2 = \max f(x(k)), x \in K\backslash\{0\}$ then from (3) we get, for all $x \in K\backslash\{0\}$,

$$\|Tx\| \leq M \sum_{s=k}^{k+\omega-1} \lambda |b(s)| |f(x(k))| \leq M\omega C_2.$$

This shows that $T(K\backslash\{0\})$ is a bounded set in K. Since K is n-dimensional, $T(K\backslash\{0\})$ is relatively compact in K. Therefore T is a completely continuous operator. □

For the next following lemmas, we now introduce some notations. For $r > 0$, let

$$\Gamma = \sigma m \sum_{s=0}^{\omega-1} b(s),\ \chi = M \sum_{s=0}^{\omega-1} b(s),$$

$$C(r) = \max\{f(x) : x \in \mathbb{R}_+, \|x\| \leq r\} > 0.$$

Lemma 5. *Assume that (H1), (H2) holds. For any $\eta > 0$ and $x \in K\backslash \{0\}$, if there exists a f such that $f(x(k)) \geq x(k)\eta$ for $k \in [0, \omega]$, then $\|Tx\| \geq \lambda\Gamma\eta \|x\|$.*

Proof. Since $x \in K \setminus \{0\}$ and $f(x(k)) \geq x(k)\eta$ for $k \in [0,\omega]$, we have

$$\begin{aligned} Tx(k) &= \lambda \sum_{s=k}^{k+\omega-1} G(k,s)b(s)f(x(s)) \\ &\geq \lambda m \sum_{s=0}^{\omega-1} b(s)f(x(s)) \\ &\geq \lambda m \sum_{s=0}^{\omega-1} b(s)x(k)\eta \\ &\geq \lambda m \sum_{s=0}^{\omega-1} b(s)\sigma \|x\| \eta \end{aligned}$$

Thus $\|Tx\| \geq \lambda\Gamma\eta \|x\|$. This completes the proof. ☐

Let $\hat{f} : [1,\infty) \to \mathbb{R}_+$ be the function given by

$$\hat{f}(\theta) = \max\{f(x) : x \in \mathbb{R}_+, \text{and } 1 \leq \|x\| \leq \theta\}.$$

It is easy to see that $\hat{f}(\theta)$ is nondecreasing function on $[1,\infty)$. The following lemma is essentially the same as Lemma 3.6 in [12] and Lemma 2.8 in [11].

Lemma 6. *([12, 11]) Assume (H2) holds. If* $\lim_{x\to\infty} \frac{f(x)}{x}$ *exists (which can be infinty) then* $\lim_{\theta\to\infty} \frac{\hat{f}(\theta)}{\theta}$ *exists and* $\lim_{\theta\to\infty} \frac{\hat{f}(\theta)}{\theta} = \lim_{x\to\infty} \frac{f(x)}{x}$.

Lemma 7. *Assume that (H1) and (H2) holds. Let* $r > \frac{1}{\sigma}$ *and if there exists an* $\varepsilon > 0$ *such that* $\hat{f}(r) \leq \varepsilon r$*, then* $\|Tx\| \leq \lambda\chi\varepsilon \|x\|$ *for* $x \in \partial\Omega_r$.

Proof. From the definition of T for $x \in \partial\Omega_r$, we have

$$\begin{aligned} \|Tx\| &\leq \lambda M \sum_{s=0}^{\omega-1} b(s)f(x(s)) \\ &\leq \lambda M \sum_{s=0}^{\omega-1} b(s)\hat{f}(r)\eta \\ &\leq \lambda M \sum_{s=0}^{\omega-1} b(s)\varepsilon r \\ &\leq \lambda M \sum_{s=0}^{\omega-1} b(s)\varepsilon \|x\|. \end{aligned}$$

This implies that $\|Tx\| \leq \lambda\chi\varepsilon \|x\|$. ☐

In views of definition $C(r)$, it follows that

$$0 < f(x(k)) \leq C(r) \quad \text{for } k \in [0,\omega],$$

if $x \in \partial\Omega_r, r > 0$. Thus it is easy to see the following lemma can be shown in similar manner as in Lemma 7.

Lemma 8. *Assume (H1), (H2) holds. If* $x \in \partial\Omega_r, r > 0$ *then* $\|Tx\| \leq \lambda\chi C(r)$.

Proof. From the definitions of T for $x \in \partial\Omega_r$ we have

$$\begin{aligned}\|Tx\| &\le \lambda M \sum_{s=0}^{\omega-1} b(s) f(x(s)) \\ &\le \lambda M \sum_{s=0}^{\omega-1} b(s) C(r) \\ &\le \lambda\chi C(r).\end{aligned}$$

Thus it implies that $\|Tx\| \le \lambda\chi C(r)$. □

3 Main Result

In this section, we establish conditions for the existence and multiplicity of positive periodic solution of (1).

Theorem 1. *Let (H1), (H2) hold, we assume that* $\lim_{x\to 0} f(x) = \infty$.

- *(a) If* $\lim_{x\to\infty} \frac{f(x)}{x} = 0$*, then for all* $\lambda > 0$ *(1) has a positive solution.*
- *(b) If* $\lim_{x\to\infty} \frac{f(x)}{x} = \infty$*, then for all small* $\lambda > 0$ *(1) has two positive solutions.*
- *(c) If If there exists a* $\lambda_0 > 0$ *such that (1) has a positive periodic solution for* $0 < \lambda < \lambda_0$.

Proof. (a): From the assumptions, $\lim_{x\to 0} f(x) = \infty$ there is an $r_1 > 0$ such that

$$f(x) \ge \eta x$$

for $x \in K\backslash\{0\}$ and $0 < x < r_1$, where $\eta > 0$ is chosen so that

$$\lambda\Gamma\eta > 1.$$

Let $\Omega_{r_1} = \{x \in K : \|x\| < r_1\}$. If $x \in \partial\Omega_{r_1}$, then

$$f(x(k)) \ge x(k)\eta.$$

Lemma 5 implies that

$$\|Tx\| \ge \lambda\Gamma\eta \|x\| > \|x\| \quad \text{for } x \in \partial\Omega_{r_1}. \tag{4}$$

We now determine Ω_{r_2}. Let $\Omega_{r_1} = \{x \in K : \|x\| < r_2\}$. Note that $\lim_{x\to\infty} \frac{f(x)}{x} = 0$, it follows from Lemma 6, $\lim_{\theta\to\infty} \frac{\hat{f}(\theta)}{\theta} = 0$. Therefore there is an $r_2 > \max\left\{2r_1, \frac{1}{\sigma}\right\}$ such that

$$\hat{f}(r_2) \le \varepsilon r_2,$$

where the constant $\varepsilon > 0$ satisfies

$$\lambda\varepsilon\chi < 1.$$

Thus, we have by Lemma 7 that

$$\|Tx\| \le \lambda\varepsilon\chi \|x\| < \|x\| \quad \text{for } x \in \partial\Omega_{r_2}. \tag{5}$$

By Lemma 1 applied to (4) and (5), it follows that T has a fixed point in $\bar{\Omega}_{r_2}\backslash\Omega_{r_1}$, which is the desired positive solution of (1). □

Proof. (b): Fix two numbers $0 < r_3 < r_4$, there exists a λ_0 such that

$$\lambda_0 < \frac{r_3}{\chi C(r_3)}, \quad \lambda_0 < \frac{r_4}{\chi C(r_4)},$$

where $\chi C(r)$ defined in Lemma 8. Thus, in Lemma 8 implies that, for $0 < \lambda < \lambda_0$,

$$\begin{aligned} \|Tx\| &\leq \lambda \chi C(r_j) \\ &\leq \frac{r_j}{\chi C(r_j)} \chi C(r_j) = r_j = \|x\|. \end{aligned}$$

Thus

$$\|Tx\| < \|x\| \quad \text{for } \; x \in \partial\Omega_{r_j}, \quad (j = 3, 4). \tag{6}$$

On the other hand, in view of the assumptions $\lim_{x\to\infty} \frac{f(x)}{x} = \infty$ and $\lim_{x\to 0} f(x) = \infty$, there are positive numbers $0 < r_2 < r_3 < r_4 < \hat{H}$ such that

$$f(x) \geq \eta x$$

for $x \in K\backslash\{0\}$ and $0 < x \leq r_2$ or $x \geq \hat{H}$ where $\eta > 0$ is chosen so that

$$\lambda \Gamma \eta > 1.$$

Thus if $x \in \partial\Omega_{r_2}$, then

$$f(x) \geq \eta x.$$

Let $r_1 = \max\left\{2r_4, \frac{\hat{H}}{\sigma}\right\}$ if $x \in \partial\Omega_{r_1}$, then

$$\min_{k\in[0,\omega]} x(k) \geq \sigma \|x\| = \sigma r_1 \geq \hat{H},$$

which implies that

$$f(x) \geq \eta x.$$

Thus Lemma 5 implies that

$$\|Tx\| \geq \lambda \Gamma \eta \|x\| > \|x\| \qquad \text{for } \; x \in \partial\Omega_{r_1}, \tag{7}$$

and

$$\|Tx\| \geq \lambda \Gamma \eta \|x\| > \|x\| \qquad \text{for } \; x \in \partial\Omega_{r_2}. \tag{8}$$

It follows from Lemma 1 applied to (6), (7) and (8), T has two fixed points x_1 and x_2 such that $x_1 \in \bar{\Omega}_{r_3}\backslash\Omega_{r_2}$ and $x_2 \in \bar{\Omega}_{r_1}\backslash\Omega_{r_4}$, which are the desired distinct positive periodic solutions of (1) for $\lambda < \lambda_0$ satisfying

$$r_2 < \|x_1\| < r_3 < r_4 < \|x_2\| < r_1.$$

□

Proof. (c): Choose a number $r_3 > 0$. By Lemma 8 we infer that there exists a $\lambda_0 = \frac{r_3}{\chi C(r_3)} > 0$ such that

$$\|Tx\| < \|x\| \quad \text{for } \; x \in \partial\Omega_{r_3} \qquad 0 < \lambda < \lambda_0. \tag{9}$$

On the other hand, in view of assumption $\lim_{x\to 0} f(x) = \infty$, there exists a positive number $0 < r_2 < r_3$ such that

$$f(x) \geq \eta x$$

for $x \in K\backslash\{0\}$ and $0 < x < r_2$ where $\eta > 0$ is chosen so that

$$\lambda\Gamma\eta > 1.$$

Thus if $x \in \partial\Omega_{r_2}$, then

$$f(x) \geq \eta x.$$

Lemma 5 implies that

$$\|Tx\| \geq \lambda\Gamma\eta \|x\| > \|x\|, \qquad \text{for } x \in \partial\Omega_{r_2}. \tag{10}$$

It follows from Lemma 1 applied to (9) and (10), that T has a fixed point $x \in \bar{\Omega}_{r_3}\backslash\Omega_{r_2}$. The fixed point $x \in \bar{\Omega}_{r_3}\backslash\Omega_{r_2}$ is the desired positive periodic solution of (1). ☐

4 Conclusion

In this paper, we employed Kranoselskii fixed point theorem to investigate the existence and multiplicity of positive periodic solutions of difference equations (1). It still remains open to generalize it for systems of first order difference equations.

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