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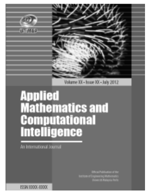

A Study on Reliability of an Efficient Technique

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Abstract: *In this paper, two reliable modifications of Variational Iteration Method (VIM) are tested for two nonlinear mathematical problems like Emden-Fowler Equation and Lane-Emden Equation, which arises in diverse fields of physics. It has been observed that these modifications are very efficient and reliable for the solution of the non-linear problems. Numerical results represent the reliability, effectiveness and efficiency of the proposed modifications.*

Keywords: *Singular differential equation, Emden-Fowler Equation, Lane-Emden Equation, Variational iteration method.*

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1 Introduction

Most of the problems in natural and engineering sciences are modeled by differential equations. These equations arise in various scientific models such as the fluid mechanics, chemical reaction diffusion, propagation of shallow water waves, Schrodinger equation models. To solve such models, a large amount of work has been invested several techniques [1-10] including Homotopy Perturbation, Methods of Characteristic, Multi Grid, Periodic Multi Grid Wave form, Riemann invariants, Finite Difference, Polynomial and Non-polynomial Spline, Variational of Parameter, Sink Galerkin, Parameter Expansion, Energy Balance, Homotopy analysis have been developed for the solution of the natural and engineering problems. Most of the techniques have their limitations and encounter the inbuilt deficiencies like linearization, limited convergence, divergent results, unrealistic assumptions and a lot of computational work.

Recently, Ghorbani et. al. [11] introduced He’s polynomials by splitting the non-linear term. He’s polynomial are calculated from He’s Homotopy Perturbation method [12-14]. More recently,

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Noor and Mohyud-Din [15,16] combined correction functional and He's polynomials of the Variational Iteration Method (VIMHP) and applied this reliable modified form of VIM to a wide class of physical problems. The basic motive of the present study is the implementations of the reliable modifications of Variational Iteration Method to the singular initial value problems.

2 Analysis of Variational Iteration Method (VIM)

To elucidate the basic of the variational Iteration Method (VIM), we consider the differential equation in the general form

$$Lu+Nu=f(x), \tag{1}$$

where L is a linear operator, N is non-linear operator and $f(x)$ is source term respectively. According to variational iteration method, the correction functional of Eq. (1) can be written as,

$$u_{n+1}(x)=u_n(x)+\int_0^x \lambda(\tau)\left(Lu_n(\tau)+N\tilde{u}_n(\tau)-f(\tau)\right)d\tau, \tag{2}$$

where λ is a Lagrange multiplier, which can be identified optimally via variational theory. After determined the Lagrange multiplier, the successive approximation u_{n+1}, $n\geq 0$, of the solution u will be readily obtained by using determined Lagrange multiplier and any selective function u_0. Consequently, the solution is given by

$$u(x)=\lim_{n\to\infty} u_n(x). \tag{3}$$

3 Variational Iteration Method using He's Polynomials (VIMHP)

Variational Iteration Method using He's polynomials is a modified form of Variational Iteration Method. This modification is obtained by coupling of correction functional of Variational Iteration Method with He' Polynomials and is given by

$$\sum_{n=0}^{\infty} p^{(n)} u_n(x)=u_0(x)+p\int_0^x \lambda\left(\sum_{n=0}^{\infty} p^{(n)} L(u_n)+\sum_{n=0}^{\infty} p^{(n)} N(\tilde{u}_n)\right)d\tau-\int_0^x \lambda f(\tau)d\tau. \tag{4}$$

By comparing the like indexes of *p,* give solution of various order.

4 Variational Iteration Method using Adomian's Polynomials (VIMAP)

Variational Iteration Method using Adomian's Polynomials is another modified form of Variational Iteration Method. This Modification is obtained by coupling of correction functional of Variational Iteration Method with Adomian's Polynomials and is given by

$$u_{n+1}(x)=u_n(x)+\int_0^x \lambda(\tau)\left(Lu_n(\tau)+\sum_{n=0}^{\infty} A_n - f(\tau) \right) d\tau, \tag{5}$$

where A_n , are called Adomian's Polynomials which can be generated for all type of non-linearity, and determined by the algorithm defined in [17].

$$A_0 = F(u_0),$$

$$A_1 = u_1 F'(u_0),$$

$$A_2 = u_2 F'(u_0)+\frac{u_1^2}{2!} F''(u_0),$$

$$A_3 = \frac{u_1^3}{3!} F'''(u_0)+u_1 u_2 F''(u_0)+u_3 F'(u_0),$$
$$\vdots$$

4 Analysis of VIM for Singular initial value Problem

Consider the singular initial value problem

$$y''(x)+\frac{k}{x} y'(x)+g(x)y(x)=f(x), \tag{6}$$

Subject to the conditions

$$y(0)=0,\ y'(0)=1.$$

According to variational Iteration Method, the correction functional of Eq. (6) can be written as,

$$y_{n+1}(x)=y_n(x)+\int_0^x \lambda(x,\tau)\left(y_n''(\tau)+\frac{2}{\tau} y_n'(\tau)+\frac{k-2}{\tau} \tilde{y}_n'(x)+g(\tau)\tilde{y}_n(\tau)-f(\tau)\right) d\tau, \tag{7}$$

Where λ is called general Lagrange multiplier, which can be identified optimally via variational theory, and $\tilde{y}_n$, is considered as restricted Variations so, taking δ on both sides, we get

$$\delta y_{n+1}(x)=\delta y_n(x)+\delta\int_0^x \lambda\left(y_n''(\tau)+\frac{2}{\tau} y_n'(\tau)+\frac{k-2}{\tau} \tilde{y}_n'(x)+g(\tau)\tilde{y}_n(\tau)-f(\tau) \right) d\tau, \tag{8}$$

The Lagrange multiplier can be identified via variational theory.

$$\lambda(x,\tau)=\frac{\tau^2}{x}-\tau,$$

Now, Eq. (5) becomes,

$$y_{n+1}(x)=y_n(x)+\int_0^x\left(\frac{\tau}{x}-1\right)\left(\tau\, y_n''(\tau)+k\, y_n'(\tau)+\tau\, g(\tau)\, y_n(\tau)-\tau\, f(\tau)\right)d\tau,\ n\geq 0, \tag{9}$$

Consequently, the solution is given by

$$y(x)=\lim_{n\to\infty} y_n(x).$$

5 Numerical Applications

5.1 Example

Consider the classical Emden-Fowler equation of the second kind

$$\frac{d^2 y}{d x^2}+\frac{2}{x}\frac{d y}{d x}+\alpha x^m y^r=0, \tag{10}$$

subject to the initial conditions,

$$y(0)=1,\ y'(0)=0.$$

where α, m and r are constants.

VIMHP

According to VIM, the correction functional for the Eq. (10) can be written as

$$y_{n+1}(x)=y_n(x)+\int_0^x \lambda(x,\tau)\left(\frac{d^2 y_n(\tau)}{d\tau^2}+\frac{2}{\tau}\frac{d y_n}{d\tau}+\alpha\tau^m\, \tilde{y}_n{}^r(\tau)\right)d\tau, \tag{11}$$

The Lagrange Multiplier can be identified via variational theory.

$$\lambda(x,\tau)=\frac{\tau^2}{x}-\tau,$$

Now Eq. (11) becomes

$$y_{n+1}(x) = y_n(x) + \int_0^x \left(\frac{\tau^2}{x} - \tau\right)\left(\frac{d^2 y_n(\tau)}{d\tau^2} + \frac{2}{\tau}\frac{d y_n}{d\tau} + \alpha\tau^m y_n{}^r(\tau)\right) d\tau. \quad (12)$$

According to VIMHP, Eq. (12) can be written as,

$$\sum_{n=0}^{\infty} p^n y_n(x) = y_0 + p\int_0^x \left(\frac{\tau^2}{x} - \tau\right)\left[\sum_{n=0}^{\infty} p^n \frac{d^2 y_n}{d\tau^2} + \frac{2}{\tau}\sum_{n=0}^{\infty} p^n \frac{d y_n}{d\tau} + \alpha\tau^m \sum_{n=0}^{\infty} p^n y_n{}^r\right] d\tau,$$

$$\sum_{n=0}^{\infty} p^n y_n(x) = 1 + p\int_0^x \left(\frac{\tau}{x} - 1\right)\left[\sum_{n=0}^{\infty} p^n \frac{\tau d^2 y_n}{d\tau^2} + 2\sum_{n=0}^{\infty} p^n \frac{d y_n}{d\tau} + \alpha\tau^{m+1} \sum_{n=0}^{\infty} p^n y_n{}^r\right] d\tau,$$

for $r = 1$,

$$\sum_{n=0}^{\infty} p^n y_n = 1 + p\int_0^x \left(\frac{\tau}{x} - 1\right)\left[\sum_{n=0}^{\infty} p^n \frac{\tau d^2 y_n}{d\tau^2} + 2\sum_{n=0}^{\infty} p^n \frac{d y_n}{d\tau} + \alpha\tau^{m+1} \sum_{n=0}^{\infty} p^n y_n\right] d\tau, \quad (13)$$

Now, comparing the co-efficient of like powers of *p*,

$p^{(0)}$: $\quad y_0 = 1,$

$p^{(1)}$: $\quad y_1 = \int_0^x \left(\frac{\tau}{x} - 1\right)\left(\alpha\tau^{m+1}\right) d\tau,$

$$= -\frac{\alpha x^{m+2}}{(m+3)(m+2)},$$

$p^{(2)}$: $\quad y_2 = \int_0^x \left(\frac{\tau}{x} - 1\right)\left(\tau\frac{d^2 y_1}{d\tau^2} + 2\frac{d y_1}{d\tau} + \alpha\tau^{m+1} y_1\right) d\tau,$

$$= \frac{\alpha^2 x^{2m+4}}{2(2m+5)(m+3)(m+2)^2},$$

⋮

Therefore,

$$y(x) = y_0 + y_1 + y_2 + \cdots,$$

$$= 1 - \frac{\alpha x^{m+2}}{(m+3)(m+2)} + \frac{\alpha^2 x^{2m+4}}{2(2m+5)(m+3)(m+2)^2} - \cdots, \quad (14)$$

This is the same result is as obtained by Chowdhury [18].

VIMAP

According to VIMAP, Eq. (10) can be written in the form,

$$y_{n+1}(x)=y_0(x)+\int_0^x\left(\frac{\tau}{x}-1\right)\left(\frac{\tau d^2 y_n(\tau)}{d\tau^2}+2\frac{d y_n}{d\tau}+\alpha\tau^{m+1}\sum_{n=0}^{\infty}A_n\right)d\tau. \tag{15}$$

The initial approximate solution is reads as

$$y_0=1,$$

Consequently, for $r=1$, we have

$$y_1=\frac{-\alpha x^{m+2}}{(m+3)(m+2)},\; m>-2,$$

$$y_2(x)=\frac{\alpha x^{2m+4}}{2(2m+5)(m+3)(m+2)},\; m>-2,$$

And so on. Finally, we have

$$y(x)=1-\frac{\alpha x^{m+2}}{(m+3)(m+2)}+\frac{\alpha\; x^{2m+4}}{2(2m+5)(m+3)(m+2)^2}\cdots, \tag{16}$$

This is the same result as obtained by Chawdhary [18].

Particularly, we obtained the exact solution as

For $m=0,\; and\;\; r=0,$

$$y_0=1,$$
$$y_1=-\frac{\alpha x^2}{6},$$
$$y_2=0,$$

Therefore, the exact solution will be

$$y(x)=1-\frac{\alpha x^2}{6}, \tag{15}$$

for $m=0$, and $r=1$,

$$y_0=1,\ y_1=-\frac{\alpha x^2}{6},\ y_2=\frac{\alpha^2 x^4}{120},\ y_3=-\frac{\alpha^3 x^6}{5040},\cdots,$$

Therefore,

$$y(x)=1-\frac{\alpha x^2}{6}+\frac{\alpha^2 x^4}{120}-\frac{\alpha^3 x^6}{5040}+\cdots,$$

$$y(x)=\frac{1}{x}\left(x-\frac{\alpha x^3}{6}+\frac{\alpha^2 x^5}{120}-\frac{\alpha^3 x^7}{5040}+\cdots\right),$$

$$=\frac{1}{\sqrt{\alpha}\,x}\left(\sqrt{\alpha}\,x-\frac{\alpha^{\frac{3}{2}} x^3}{6}+\frac{\alpha^{\frac{5}{2}} x^5}{120}-\frac{\alpha^{\frac{7}{2}} x^7}{5040}+\cdots\right),$$

$$y(x)=\frac{Sin\left(\sqrt{\alpha}\,x\right)}{\sqrt{\alpha}\,x}, \tag{16}$$

for $m=0$, and $r=5$,

$$y_0=1,$$

$$y_1=-\frac{\alpha x^2}{6},$$

$$y_2=\frac{\alpha^2 x^4}{24}-\frac{3\alpha^3 x^6}{70}+\frac{17\alpha^4 x^8}{630}-\frac{59\alpha^5 x^{10}}{11550}+\frac{\alpha^6 x^{12}}{3510},$$

$$\vdots$$

$$y(x)=\left(1+\frac{\alpha x^2}{3}\right)^{-\frac{1}{2}}, \tag{17}$$

Plot of graphically representation for $r=0,1,5.$

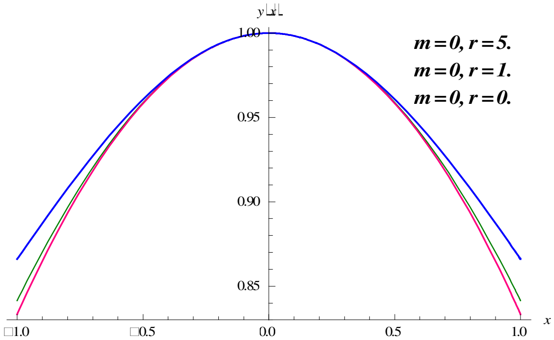

Fig. 1

5.2 Example

Consider the non-linear Emden-Fowler Equation

$$\frac{d^2 y}{d x^2}+\frac{2}{x}\frac{d y}{d x}+\alpha \ln x\, y^r=0, \tag{18}$$

Subject to the conditions

$$y(0)=1,\ y'(0)=0.$$

VIMHP

According to VIM, the correction functional for the Eq. (18) can be written as

$$y_{n+1}(x)=y_n(x)+\int_0^x \lambda(\tau)\left(\frac{d^2 y_n(\tau)}{d\tau^2}+\frac{2}{\tau}\frac{d y_n}{d\tau}+\alpha\left(\tau^m \ln\tau\right)\tilde{y}_n{}^r\right)d\tau, \tag{19}$$

The Lagrange Multiplier can be identified via variational theory.

$$\lambda(x,\tau)=\frac{\tau^2}{x}-\tau,$$

Now Eq. (19) becomes

$$y_{n+1}(x)=y_n(x)+\int_0^x \left(\frac{\tau^2}{x}-\tau\right)\left(\frac{d^2 y_n(\tau)}{d\tau^2}+\frac{2}{\tau}\frac{d y_n}{d\tau}+\alpha\left(\tau^m \ln\tau\right)y_n{}^r\right)d\tau, \tag{20}$$

According to VIMHP, Eq. (20) can be written as,

$$\sum_{n=0}^{\infty} p^{(n)} y_n(x) = y_0 + p\int_0^x \left(\frac{\tau^2}{x}-\tau\right)\left(\sum_{n=0}^{\infty} p^{(n)} \frac{d^2 y_n}{d\tau^2} + \sum_{n=0}^{\infty} p^{(n)} \frac{2}{\tau}\frac{d y_n}{d\tau} + \sum_{n=0}^{\infty} p^{(n)} \alpha\left(\tau^m \ln\tau\right) y_n{}^r\right) d\tau,$$

$$\sum_{n=0}^{\infty} p^{(n)} y_n(x) = 1 + p\int_0^x \left(\frac{\tau}{x}-1\right)\left(\sum_{n=0}^{\infty} p^{(n)} \frac{\tau d^2 y_n}{d\tau^2} + \sum_{n=0}^{\infty} p^{(n)} 2\frac{d y_n}{d\tau} + \sum_{n=0}^{\infty} p^{(n)} \alpha\left(\tau^{m+1} \ln\tau\right) y_n{}^r\right) d\tau,$$

for $r=1$,

$$\sum_{n=0}^{\infty} p^{(n)} y_n(x) = 1 + p\int_0^x \left(\frac{\tau}{x}-1\right)\left(\sum_{n=0}^{\infty} p^{(n)} \frac{\tau d^2 y_n}{d\tau^2} + \sum_{n=0}^{\infty} p^{(n)} 2\frac{d y_n}{d\tau} + \sum_{n=0}^{\infty} p^{(n)} \alpha\left(\tau^{m+1} \ln\tau\right) y_n\right) d\tau,$$

Now, comparing the co-efficient of like powers of p,

$$p^{(0)}: \qquad y_0 = 1,$$

$$p^{(1)}: \qquad y_1 = \int_0^x \left(\frac{\tau}{x}-1\right)\left(\alpha\tau^{m+1}\ln\tau\, y_0\right) d\tau,$$

$$= \frac{\alpha x^{m+2}\left(5+2m-(m+2)(m+3)\ln x\right)}{(m+2)^2(m+3)^2}.$$

$$p^{(2)}: \qquad y_2 = \int_0^x \left(\frac{\tau}{x}-1\right)\left(\frac{\tau\, d^2 y_1(\tau)}{d\tau^2} + 2\frac{d y_1}{d\tau} + \alpha\tau^{m+1}\ln\tau\, y_1\right) d\tau,$$

$$= \frac{\alpha x^{2m+4}}{4}\frac{\lambda-\mu\ln x-\eta(\ln x)^2}{(2m+3)^3(m+3)^2(m+2)^4},$$

$\vdots$,

where

$$\lambda = 408+503m+206m^2+28m^3,$$
$$\mu = 2(2m+5)(m+2)\left(8m^2\ 41m+52\right),$$
$$\eta = 2(2m+5)^2(m+3)(m+2)^2,$$

Therefore,

$$y(x) == 1 + \frac{\alpha x^{m+2}\left(5+2m-(m+2)(m+3)\ln x\right)}{(m+2)^2(m+3)^2} + \frac{\alpha x^{2m+4}}{4}\frac{\lambda-\mu\ln x-\eta(\ln x)^2}{(2m+3)^3(m+3)^2(m+2)^4} + \cdots, \tag{21}$$

where $m \neq -2, -3, -\frac{3}{2}, \cdots$. The exact solutions are exist for $r = 0$ *and* $m = -1, 0, 1, 2, 3$, respectively.

$$
\begin{aligned}
y(x) &= 1 - \frac{\alpha x}{2}\left(\ln x - \frac{3}{2}\right).\\
y(x) &= 1 - \frac{\alpha x^2}{6}\left(\ln x - \frac{5}{6}\right).\\
y(x) &= 1 - \frac{\alpha x^3}{12}\left(\ln x - \frac{7}{12}\right).\\
y(x) &= 1 - \frac{\alpha x^4}{20}\left(\ln x - \frac{9}{20}\right).\\
y(x) &= 1 - \frac{\alpha x^5}{30}\left(\ln x - \frac{11}{30}\right).
\end{aligned}
\tag{22}
$$

VIMAP

According to VIMAP, Eq. (18) can be written in the form,

$$
y_{n+1}(x) = y_0(x) + \int_0^x \left(\frac{\tau^2}{x} - \tau\right)\left(\frac{d^2 y_n(\tau)}{d\tau^2} + \frac{2}{\tau}\frac{d y_n}{d\tau} + \alpha \tau^m \ln \tau \sum_{n=0}^{\infty} A_n(\tau)\right) d\tau. \tag{23}
$$

The initial approximation is reads as

$$
y_0 = y(0) = 1,
$$

Consequently, we have

$$
\begin{aligned}
y_1(x) &= 1 + \frac{\alpha x^{m+2}\left(5 + 2m - (m+2)(m+3)\ln x\right)}{(m+2)^2 (m+3)^2},\\
y_2(x) &= 1 + \frac{\alpha x^{m+2}\left(5 + 2m - (m+2)(m+3)\ln x\right)}{(m+2)^2 (m+3)^2} + \frac{\alpha r x^{2m+4}}{4}\frac{\lambda - \mu \ln x - \eta (\ln x)^2}{(2m+3)^3 (m+3)^2 (m+2)^4},
\end{aligned}
$$

where

$$
\lambda = 408 + 503m + 206m^2 + 28m^3,
$$

$$\mu = 2(2m+5)(m+2)(8m^2\ 41m+52),$$
$$\eta = 2(2m+5)^2(m+3)(m+2)^2,$$

Therefore,

$$y(x) == 1 + \frac{\alpha x^{m+2}\left(5+2m-(m+2)(m+3)\ln x\right)}{(m+2)^2(m+3)^2} + \frac{\alpha r x^{2m+4}}{4} \cdot \frac{\lambda - \mu \ln x - \eta(\ln x)^2}{(2m+3)^3(m+3)^2(m+2)^4} + \cdots, \tag{24}$$

where $m \neq -2, -3, -\frac{3}{2}, \cdots$. the exact solutions are exists for $r=0$ and $m=-1, 0, 1, 2, 3.$ respectively.

$$y(x) = 1 - \frac{\alpha x}{2}\left(\ln x - \frac{3}{2}\right).$$
$$y(x) = 1 - \frac{\alpha x^2}{6}\left(\ln x - \frac{5}{6}\right).$$
$$y(x) = 1 - \frac{\alpha x^3}{12}\left(\ln x - \frac{7}{12}\right). \tag{25}$$
$$y(x) = 1 - \frac{\alpha x^4}{20}\left(\ln x - \frac{9}{20}\right).$$
$$y(x) = 1 - \frac{\alpha x^5}{30}\left(\ln x - \frac{11}{30}\right).$$

Plot of graphical representation of solution (25) are:

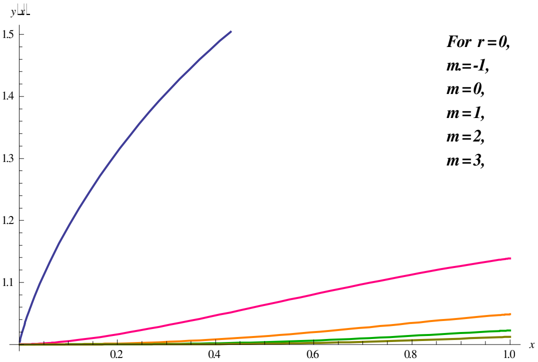

Fig. 2

5.3 Example

Consider a linear homogeneous Lane-Emden Equation

$$\frac{d^2 y}{dx^2}+\frac{2}{x}\frac{dy}{dx}-\left(4x^2+6\right)y=.0, \tag{26}$$

Subject to the conditions

$$y(0)=1,\ \ y'(0)=0.$$

VIMAP

According to VIM, the correction functional for the equation (26) can be written as

$$y_{n+1}(x)=y_n(x)+\int_0^x \lambda(\tau)\left(\frac{d^2 y_n(\tau)}{d\tau^2}+\frac{2}{\tau}\frac{dy_n}{d\tau}-\left(4\tau^2+6\right)\tilde{y}_n\right)d\tau, \tag{27}$$

The Lagrange Multiplier can be identified via variational theory.

$$\lambda(x,\tau)=\frac{\tau^2}{x}-\tau,$$

Now Eq. (27) becomes

$$y_{n+1}(x)=y_n(x)+\int_0^x \left(\frac{\tau^2}{x}-\tau\right)\left(\frac{d^2 y_n(\tau)}{d\tau^2}+\frac{2}{\tau}\frac{dy_n}{d\tau}-\left(4\tau^2+6\right)y_n\right)d\tau, \tag{28}$$

According to VIMHP, equation (28) can be written as,

$$\sum_{n=0}^{\infty} p^{(n)} y_n(x)=y_0+p\int_0^x \left(\frac{\tau^2}{x}-\tau\right)\left(\sum_{n=0}^{\infty} p^{(n)}\frac{d^2 y_n}{d\tau^2}+\sum_{n=0}^{\infty} p^{(n)}\frac{2}{\tau}\frac{dy_n}{d\tau}-\left(4\tau^2+6\right)\sum_{n=0}^{\infty} p^{(n)} y_n\right)d\tau,$$

$$\sum_{n=0}^{\infty} p^{(n)} y_n(x)=1+p\int_0^x \left(\frac{\tau^2}{x}-\tau\right)\left(\sum_{n=0}^{\infty} p^{(n)}\frac{d^2 y_n}{d\tau^2}+\sum_{n=0}^{\infty} p^{(n)}\frac{2}{\tau}\frac{dy_n}{d\tau}-\left(4\tau^2+6\right)\sum_{n=0}^{\infty} p^{(n)} y_n\right)d\tau,$$

Comparing the coefficients of like powers of *p*:

$$p^{(0)}: \qquad y_0=1,$$

$$p^{(1)}: \qquad y_1 = x^2 + \frac{x^4}{5}$$

$$p^{(2)}: \qquad y_2 = \frac{3}{10}x^4 + \frac{13}{105}x^6 + \frac{1}{90}x^8,$$

$$p^{(3)}: \qquad y_3 = \frac{3}{70}x^6 + \frac{17}{630}x^8 + \frac{59}{11550}x^{10} + \frac{1}{3510}x^{12},$$

$$\vdots$$

Therefore the closed form solution is

$$y(x) = e^{x^2}. \tag{29}$$

This is the exact solution of the Lane-Emden equation.

VIMAP

According to VIMAP, Eq. (26) can be written as,

$$y_{n+1}(x) = y_n(x) + \int_0^x \left(\frac{\tau^2}{x} - \tau\right)\left(\frac{d^2 y_n(\tau)}{d\tau^2} + \frac{2}{\tau}\frac{d y_n}{d\tau} - \left(4\tau^2 + 6\right)y_n\right)d\tau, \tag{30}$$

The initial approximation is reads as

$$y_0 = y(0) = 1,$$

Consequently, we have

$$y_1 = 1 + x^2 + \frac{x^4}{5},$$

$$y_2 = 1 + x^2 + \frac{x^4}{5} + \frac{3}{10}x^4 + \frac{13}{105}x^6 + \frac{1}{90}x^8,$$

$$y_3 = 1 + x^2 + \frac{x^4}{5} + \frac{3}{10}x^4 + \frac{13}{105}x^6 + \frac{1}{90}x^8 + \frac{3}{70}x^6 + \frac{17}{630}x^8 + \frac{59}{11550}x^{10} + \frac{1}{3510}x^{12},$$

$$\vdots$$

The closed solution is

$$y(x)=e^{x^2}.$$

This is solution of the Lane-Emden equation; this result is same as obtained [18].

Graphical representation of the solution is:

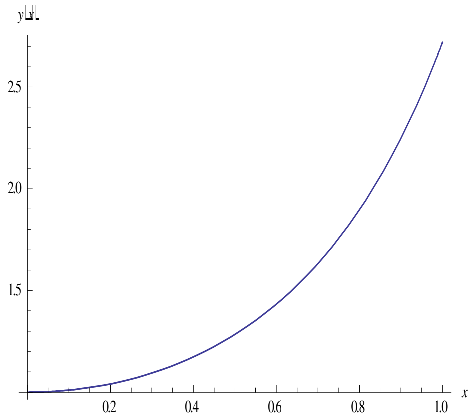

Fig. 3

5.4 Example

Consider a Time-Dependent Lane-Emden equation

$$\frac{\partial^2 y}{\partial x^2}+\frac{2}{x}\frac{\partial y}{\partial x}-\left(6+4x^2-\cos t\right)y=\frac{\partial y}{\partial t} \tag{31}$$

subject to the condition

$$y(0,t)=e^{\sin t},\quad y_x(0,t)=0$$

VIMHP

According to VIM, the correction functional for the equation (31) can be written as

$$y_{n+1}(x,t)=y_n+\int_0^x \lambda\left(\frac{\partial^2 y_n}{\partial \tau^2}+\frac{2}{\tau}\frac{\partial y_n}{\partial \tau}-\left(6+4\tau^2-\cos t\right)\tilde{y}_n-\frac{\partial \tilde{y}_n}{\partial t}\right)d\tau, \tag{32}$$

where λ is called general Lagrange multiplier, which can be identified as

$$\lambda(x,\tau)=\frac{\tau^2}{x}-\tau,$$

Therefore, Eq. (32) becomes

$$y_{n+1}(x,t)=y_n+\int_0^x\left(\frac{\tau^2}{x}-\tau\right)\left(\frac{\partial^2 y_n}{\partial\tau^2}+\frac{2}{\tau}\frac{\partial y_n}{\partial\tau}-\left(6+4\tau^2-\cos t\right)y_n-\frac{\partial y_n}{\partial t}\right)d\tau, \qquad (33)$$

According to VIMHP, Eq. (33) can be written as,

$$\sum_{n=0}^{\infty}p^{(n)}y_n(x,t)=y_0+\int_0^x\left(\frac{\tau^2}{x}-\tau\right)\left(\sum_{n=0}^{\infty}p^{(n)}\frac{\partial^2 y_n}{\partial\tau^2}+\sum_{n=0}^{\infty}p^{(n)}\frac{2}{\tau}\frac{\partial y_n}{\partial\tau}-\left(6+4\tau^2-\cos t\right)\sum_{n=0}^{\infty}p^{(n)}y_n-\sum_{n=0}^{\infty}p^{(n)}\frac{\partial y_n}{\partial t}\right)d\tau,$$

$$\sum_{n=0}^{\infty}p^{(n)}y_n(x,t)=e^{\sin t}+\int_0^x\left(\frac{\tau^2}{x}-\tau\right)\left(\sum_{n=0}^{\infty}p^{(n)}\frac{\partial^2 y_n}{\partial\tau^2}+\sum_{n=0}^{\infty}p^{(n)}\frac{2}{\tau}\frac{\partial y_n}{\partial\tau}-\left(6+4\tau^2-\cos t\right)\sum_{n=0}^{\infty}p^{(n)}y_n-\sum_{n=0}^{\infty}p^{(n)}\frac{\partial y_n}{\partial t}\right)d\tau,$$

Comparing the co-efficient of like powers of p:

$$p^{(0)}: \quad y_0=e^{\sin t},$$

$$p^{(1)}: \quad y_1=e^{\sin t}\left(x^2+\frac{x^4}{5}\right)$$

$$p^{(2)}: \quad y_2=e^{\sin t}\left(\frac{3}{10}x^4+\frac{13}{105}x^6+\frac{x^8}{90}\right),$$

Thus, the closed form solution is

$$\begin{aligned} y(x,t)&=e^{\sin t}\left(1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\frac{x^8}{4!}+\cdots,\right), \\ &=e^{x^2+\sin t}. \end{aligned} \qquad (34)$$

This is the same result as obtained by A. Sami Bataineh, M. S. M. Noorani, and I. Hashim in [19].

VIMAP

According to VIMAP, the Eq. (31) becomes

$$y_{n+1}(x,t)=y_n+\int_0^x\left(\frac{\tau^2}{x}-\tau\right)\left(\frac{\partial^2 y_n}{\partial\tau^2}+\frac{2}{\tau}\frac{\partial y_n}{\partial\tau}-\left(6+4\tau^2-\cos t\right)y_n-\frac{\partial y_n}{\partial t}\right)d\tau. \tag{35}$$

The initial approximate solution is reads as

$$y_0=y(0,t)=e^{\sin t},$$

Consequently, we have

$$y_1=e^{\sin t}+e^{\sin t}\left(x^2+\frac{x^4}{5}\right),$$

$$y_2=e^{\sin t}+e^{\sin t}\left(x^2+\frac{x^4}{5}\right)+e^{\sin t}\left(\frac{3}{10}x^4+\frac{13}{105}x^6+\frac{x^8}{90}\right),$$

$$y_3=e^{\sin t}+e^{\sin t}\left(x^2+\frac{x^4}{5}\right)+e^{\sin t}\left(\frac{3}{10}x^4+\frac{13}{105}x^6+\frac{x^8}{90}\right)+e^{\sin t}\left(\frac{3x^6}{70}+\frac{17x^8}{630}+\frac{54x^{10}}{11550}+\frac{1}{3510}x^{12}\right),$$

$\vdots$

Thus, the close form solution is

$$y(x,t)=e^{x^2+\sin t} \tag{36}$$

This is the same result as obtained by A. Sami Bataineh, M.S.M. Noorani, and I. Hashim [19].

Plot of graphical representation of the solution:

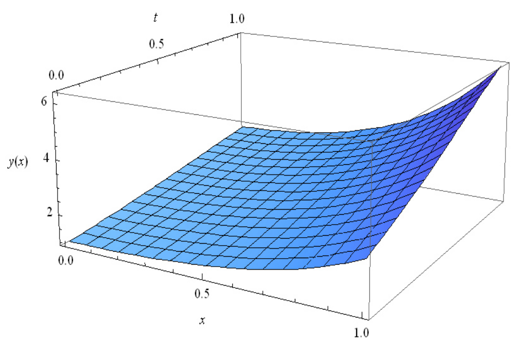

Fig. 4

6 Conclusion

In this paper, two modifications of Variational Iteration Method (VIM) are implemented successfully to obtain the analytical exact and approximate solutions of two nonlinear mathematical problems. The solution procedure is very simple by means of variatonal theory, and only a few steps lead to highly accurate solutions. It has been observed that these modifications are very efficient and reliable for the solution of the non-linear problems.

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