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Augmented Lagrangian Method for Optimal Control of Interconnected Systems

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ABSTRACT

This paper explores the application of the augmented Lagrangian method (ALM) for constructing optimal control of some interconnected systems. The ALM proves to be a robust technique in handling the stability and observability restrictions arising from interconnections among subsystems. By segregating Lagrange multipliers from the solution process, the method effectively solves the optimal control problems in a simpler unconstrained setting. The proposed approach is substantiated through numerical simulation, demonstrating its efficacy in obtaining optimal control strategy for the interconnected networks of a power grid model.

Keywords: Interconnected systems, state-feedback control, augmented Lagrangian method, power grid model.

1 INTRODUCTION

Consider the following control system:

$$\dot{x} = Ax + B_1 d + B_2, \tag{1}$$
$$z = Ex + Du, \tag{2}$$

where $x \in R^n$, $d \in R^q$, $u \in R^m$ and $z \in R^s$, is the state variable, disturbance, control variable, and output, respectively, while $E = \begin{bmatrix} Q^{\frac{1}{2}} & 0 \end{bmatrix}^T \in R^{s\times n}$ and $D = \begin{bmatrix} 0 & R^{\frac{1}{2}} \end{bmatrix}^T \in R^{s\times m}$ where $Q = Q^T \succcurlyeq 0$ and $R = R^T \succ 0$ is the state and performance weights, respectively. Assume that $A \in R^{n\times n}$ is given such that (A, B_2) is stabilizable and $(A, Q^{1/2})$ is detectable.

By letting $u = -Kx$, the corresponding closed-loop system is then given by

$$\dot{x} = (A - B_2 K)u + B_1 d, \tag{3}$$

$$z = \left[Q^{\frac{1}{2}} \quad -R^{\frac{1}{2}}K\right]^T x, \tag{4}$$

where $K \in R^{m\times n}$ denotes the state-feedback matrix. Such control systems arise in the analysis of distributed controllers for interconnected systems [1, 2, 3, 11, 14] where the interconnection structure of the system is usually characterized by A. The embedded structure is then incorporated into the feedback matrix that governs the controllers via some Lyapunov-type equation [3]. The design of optimal control strategies based on the closed-loop system (3) – (4) was well-studied, for example [1, 14]. A typical optimal control problem that could be considered is the standard linear-quadratic-regulator (LQR) control problem in which the stabilizing K is required to minimize the H_2- norm of the communication cost from d to z (see for example, [17]).

$$\min_K tr(\,(P(K)B_1B_1^T), \tag{5}$$

where tr denotes the trace and $P(K) \in R^{n\times n}$ is the observability Gramian of (3) – (4) given by

$$P(K) = \int_0^\infty \exp((A-B_2K)^T t)(Q+K^TRK)\cdot \exp((A-B_2K)t)\,dt. \tag{6}$$

For simplicity, we shall replace $P(K)$ by P only. Hence, after discretization (see for example [1, 14]), P can be obtained by solving the following:

$$L(K,P) = (A-B_2K)^T P + P(A-B_2K) + Q + K^TRK = 0. \tag{7}$$

The solution of (3) subjected to (4) will lead to the desired state-feedback optimal control law.

Traditionally, the optimization problem (5) – (7) is solved directly using constrained optimization techniques such as interior point method [6]. Consider rephrasing to: However, in large networks of dynamical systems controllers based on dense feedback matrix, it may impose expensive computation burden in its solution process. Hence the main aim of this paper is to formulate the constrained optimization problem into an unconstrained optimization problem using augmented Lagrangian method (ALM) that is well-known for handling large number of equality constraints.

2 UNCONSTRAINED OPTIMIZATION MODEL VIA AUGMENTED LAGRANGIAN APPROACH

ALM is a powerful optimization technique widely employed in solving constrained optimization problems. Developed as an extension of the classical Lagrangian method, ALM introduces an augmented term that enhances its convergence properties and performance [4, 5, 7]. Particularly, the method incorporates penalty and Lagrange multiplier terms by introducing an augmented term, which helps address the limitations of traditional Lagrangian methods, especially in cases with non-convex and complex constraints. ALM finds applications in various fields, including engineering, economics, and machine learning. It has proven to be particularly useful in problems involving structural optimization, control system design, and parameter estimation. Additionally, ALM has gained prominence in sparse signal recovery, image processing, and machine learning applications, showcasing its versatility across different domains [12, 13].

By considering ALM on (3) – (4), the associated augmented Lagrangian function is then given by

$$\Phi(K, P, \Lambda, \beta) = tr(P(K)B_1B_1^T) + tr\left(\Lambda^T\ L(K,P)\right) + \frac{\beta}{2}\ \|L(K,P)\|\|_F^2, \tag{8}$$

where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix, $\Lambda \in\ R^{n\times n}$ is the matrix of Lagrange multipliers and $\beta > 0$ is an appropriate penalty parameter. For some given initial K_0, P_0, Λ_0 , and a finite $\beta > 0$, the approach to minimize Φ via ALM takes an alternating iteration between two modules (see e.g. [4]):

$$\begin{cases} (K_{k+1}, P_{k+1}) \coloneqq \arg\min\{\Phi(K, P, \Lambda_k)\}, \\ \Lambda_{k+1} = \Lambda_k + \beta_k \Phi(K_{k+1}, P_{k+1}, \Lambda_k). \end{cases} \tag{9}$$

Since the updating scheme for Λ is done explicitly, then the solution process for obtaining an approximating sequence (K_k, P_k) can be computed independently of Λ and β.

To establish the complete algorithm for constructing optimal control law via ALM, we first give the following convergence result, which is due to [4]:

Theorem 2.1. Consider the following minimization problem:

$$\min_x f(x) \text{ subject to } h(x) = 0, \tag{10}$$

where $f: R^n \to R$ is continuously differentiable and $h: R^n \to R^m$ are component-wise continuously differentiable function. Assume that X^* is an isolated set of local minima of (10), which is compact. Then, there exists a subsequence $\{x_k\}_{k\in K}$ converging to a point $x^* \in X^*$ such that x^* is a local minimum of $\Phi(x, \Lambda_k, \beta_k) = f(x) + \Lambda_k^T h(x) + \frac{\beta_k}{2}\|h(x)\|^2$ for each $k \in K$, where $\{\Lambda_k\}$ is assumed to be bounded and $0 < \beta_k < \beta_{k+1}$, $\beta_k \to \infty$. Furthermore, if problem (10) consists of a single unique minimum point x^*, then there exists a sequence $\{x_k\}$ and an integer $\bar{k} > 0$ such that $x_k \to x^*$, which is also the unique minima of $\Phi(x, \Lambda_k, \beta_k)$ for all $k \geq \bar{k}$.

Based on the convergence conditions as stated in Theorem 2.1, we can now give the following algorithm:

ALOC Algorithm:

Step 0. Input $K_0, P_0, \Lambda_0, \beta_0$. Set $\rho > 1$ and $k = 0$.

Step 1. Employ an unconstrained optimization solver to obtain (K_{k+1}, P_{k+1}) from (K_k, P_k), i.e.

$$(K_{k+1}, P_{k+1}) \coloneqq \text{fmin}_{K,P}\ \{\Phi(K, P, \Lambda_k, \beta_k)\}.$$

Step 2. If the stopping condition is achieved, STOP. Else $\Lambda_{k+1} = \Lambda_k + \beta_k \Phi(K_{k+1}, P_{k+1}, \Lambda_k)$ and

$$\beta_{k+1} = \rho\beta_k.$$

Step 3. Set $k \leftarrow k + 1$ and go to Step 1.

Remark 2.1. In Step 1, the quasi-Newton method is an ideal unconstrained optimization method to handle a small to medium problem. Otherwise, one can utilize the conjugate gradient method if the problem is large.

3 NUMERICAL SIMULATION

3.1 Test Example

To validate the performance of our method, an optimal control problem involving the IEEE 39 New England Power Grid Model is considered. The model consists of 39 buses and 10 generators where generator 10 is an equivalent aggregated model for the part of the network that it is assumed to be uncontrollable; Generator 1 to 9 are equipped with power system stabilizers, which provide good damping of local modes and stabilize otherwise unstable open-loop system. Detailed description of the model can be found in [10, 13, 14, 15, 16]. Inter-area oscillations are associated with the dynamics of power transfer, and they are characterized by groups of coherent machines that swing relative to each other. These oscillations are caused by weakly damped modes of the linearized swing equations, and they physically correspond to active power transfer between different generator groups. In the absence of higher-order generator dynamics, for purely inductive lines and constant current loads, the dynamics of each generator can be represented by the electromechanical swing equations. After the linearization at an operating point the swing equations reduce to

$$M\ddot{\theta} + D\dot{\theta} + L\theta \ = \ 0, \tag{11}$$

where θ is the rotor angles of the synchronous generators, M and D are diagonal matrices of inertia and damping coefficients, and the coupling among generators is entirely described by the weighted graph induced by the Laplacian matrix L. Let the state of the power network be partitioned as $x = [\theta^T \ \ \dot{\theta}^T \ \ x_r^T]$ where x_r are the state variables which correspond to fast electrical dynamics [12, 13]. Hence, the linearized dynamics is given by (3) – (4) where the data set for A, B_1, B_2 are available in [15]. For illustration, we set $R \ = \ I$, and $Q \ = \ cI + \frac{1}{N}ee^T$ are chosen, where the choice of Q is inspired by slow coherency theory with $c > 0$ denoting a small regularization parameter and e is the vector of all ones.

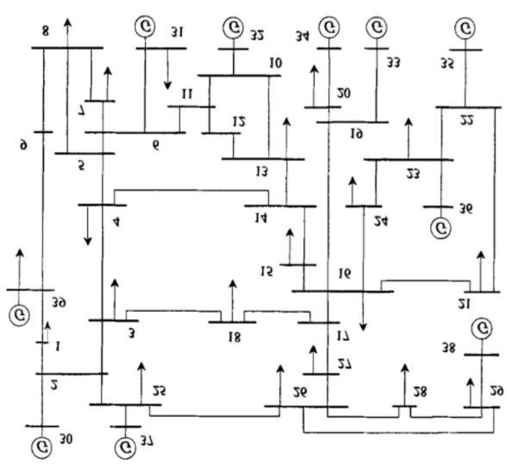

Figure 1: IEEE 39 buses system [15]

3.2 Implementation

Numerical experiments are performed on MATLAB 8.5 programming platform (R2015b) running on a machine with Window 10 operating system. The implementation of ALOC consists of the main loop and an inner loop as described in Step 2 where an unconstrained optimization is employed to obtain an approximate minimizer for a given Δ_k and β_k. Hence in Step 2, a quasi-Newton BFGS [4] method is chosen for the mentioned task. Number of iterations for the main loop and that within the inner loop are denoted by k and $\bar{k}$, respectively. In our experiments, the upper limit for k is 10. As for the inner loop, BFGS method is terminated when the number of iterations, $\bar{k}$ reaches 100 or the following stopping criterion is satisfied:

$$\|\nabla\Phi(K, P, \Lambda_k, \beta_k)\| \leq 10^{-4}. \qquad (12)$$

The upper limit on iterations for both main and inner loop will lead to a total of 1000 iteration. In practice, it is sufficient to ensure convergence. Initial approximate for K, P, Δ is set as follows: , $P_0 = I$, K_0, Δ_0 are matrices with all unit components. Other parameters are: $\beta_0 = 5$ and $\rho = 4$. These give the largest value for the penalty parameter β as 5×4^{10}, which is sufficiently large to guarantee the satisfaction of the constraint in practice.

4 RESULTS AND DISCUSSIONS

The ALOC algorithm is compared to the MATLAB solver fmincon [5, 6, 8, 9], which solves the constrained optimization problem directly using the interior-point algorithm with analytic Hessian. Two performance indicators are used to illustrate the performance of the algorithms, namely

$$r_{Time} = \frac{\text{Execution time for fmincon}}{\text{Execution time for ALOC}}, \quad r_{function} = \frac{\text{Function value at optimal for fmincon}}{\text{Function value at optimal for ALOC}}. \qquad (13)$$

The relative performance of the methods is then given in Table 1.

Table 1 : Performance of ALOC and fmincon on the test example

r_{Time}	$r_{function}$
1.317183	0.981445

We can observe from Table 1 that fmincon utilizes approximately 32% more computation time when compared to ALOC. We believe that the additional computational effort required by fmincon is due to the need to compute and store the analytic Hessian matrix. Moreover, in term of solution's quality, fmincon can obtain a solution with slightly lower function value but the deviation is not more that 2% when compared to ALOC. Hence, in general we can conclude that ALOC is a promising alternative to methods that solve constrained optimization problems directly when computational resource is limited.

5 CONCLUSION

This paper analyzed and developed an optimization method that utilizes the concept of augmented Lagrangian method for optimization problems with equality constraints. We showed the proposed algorithm, namely ALOC performed better with respect to computational time when compared to the well-known optimization solver, fmincon. In terms of quality of solution, the proposed method is also comparable to fmincon. Thus, we can conclude that ALOC would serve as a good alternative when reasonable accurate solution is required urgently.

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