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A Mathematical Model for The Spread of Varroa-Mites in Honeybee Colony with Fractional Order

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ABSTRACT

Honeybees live in colonies with one queen running the whole hive. Worker honeybees are all females and are the only bees most people ever see flying around outside of the hive. They forage for food, build the honeycombs, and protect the hive. This study developed and analyzed the honeybee's transmission dynamics under fractional order derivative via Laplace Adomian Decomposition Method, spread of varroa-mite by analyzing the disease-free equilibrium and global stability of endemic of our formulated model were investigated. Based on the trajectories, it was concluded that the memory index or fractional order could use to control the honeybees infested by varroa-mite carrying virus transmission dynamics.

Keywords: Honeybees, model formulation, disease free equilibrium, Laplace Adomian Decomposition Method

1 INTRODUCTION

Honeybees are the food crops' commonest pollinators in the world and one-third of the food we eat daily dwells on the pollination by bees [1]. Many important crops in the world are being pollinated by honeybees and depend on them for their reproduction [3]. Honeybees pollinate 70% of 1330 cultivated species of crops in tropical crops and 84% of the 264 cultivated species of crops are pollinated by animals. It has been established that the yields of many crops will decline by over 90% without the aid of these pollinators [5]. Bees are of great importance not only for humans but also for all plant species that they pollinate and therefore the economic importance of bees cannot be overemphasized due to their contribution to pollinating many food crops. Honey is a sweet natural substance produced by Honeybees, produced from the nectar of flowers by changing it with the enzymes that are present in the saliva of the worker bees [4]. Despite the enormous importance of bees, a great concern has been raised in the last 20 years due to the reduction in the number of honeybees, which affects the quality of life of all human populations that depend on their product [6, 7]. Also, there is a great concern that with a 50% growth increase in honeybee stocks, the supply fails to keep up with an over 300% increase in agricultural demands [2]. Globally, the number of the colony of honeybee losses has continued to increase rapidly since 2006 [8-10]. Some of the factors that lead to losses include weather conditions, poor diet, and transportation of bees for agricultural practices, pesticides and parasites [11]. One of the major causes of the decline is the parasitic mite called *Varroa destructor* and the viruses it carries [12, 13]. Varroa-mites feed on brood and adult bees and carry and spread viruses from one bee to another [14, 15].

Mathematics has helped to develop models to study the population dynamics of bee colonies. [11] worked on mathematical models of honeybees and the models are divided into three namely: colony, Varroa and foraging models. In the work of [16], the colony of the bees is divided into hive and forager bees. The model was to check the effect of the loss of forager bees on the adaptive early recruitment of hive bees to foraging, and how it affects the overall strength of the colony and survival. [17] studied the mathematical model of the honeybee *Varroa destructor* acute bee paralysis virus system with seasonal effects, where all the parameters are time-periodic to cater for seasonal influence. Their result reveals that the mites have to be controlled, to control the epidemic of the virus. [18] developed a model that shows how the internal demographic processes within a colony interact with the availability of food and brood rearing to change the growth in forager population size.[4] studied the populations of adult and immature honeybees and their honey production using mathematical and statistical modeling approaches. The mathematical approach consists of two models namely: a smooth model and a non-smooth model. In the smooth model, the conditions for the existence and stability of the equilibrium solutions are investigated and in the non-smooth model; the mortality rate of bees is incorporated as a function of the number of adult bees in the population. [20] studied a model to investigate the impact of external stress on social inhibition, forager recruitment rate and the laying rate of the queen.

[22] presented a time-based honeybee colony growth model and the population of Varroa mites was added by [21] to study their dynamics. [23] used a difference equation to model the population of Varroa mites reproducing in a honeybee colony. [25] developed a population model of Varroa that incorporates mortality due to the virus and later modeled the impacts of a constant population of Varroa mites on the brood and adult worker bees, and the result shows that the infestations by a large volume of mites can make hives vulnerable to extinction due to viral epidemics [24]. Some mathematical models of an infectious disease have been developed and analyzed such as in [29-40] and fractional order model of infectious disease were studied by [26, 27, 28] but not in honeybee. However, this study developed a mathematical model of dynamical transmission of varroa-mite in honeybee's colony with bio-control agent population together with fractional order derivative version which was analyzed via Laplace Adomian Decomposition Method to the best of my knowledge this approach has never been exploited especially in the ecology field by any researchers.

1.1 Essentials of fractional calculus

Definition1. The fractional derivative of order α for every α and $n=[\alpha]$ the Rieman-Liouville derivative of order α is defined as

$$D_t^{\alpha} m(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^t (t-u)^{n-\alpha-1} m(u)du$$

Definition 2. The Caputo fractional derivative in the origin is defined as

$$ {}_0^cD_t^\alpha m(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-u)^{n-\alpha-1} m(u)du $$

for a function m(t), m(t) = 0, if t >0 where [α] = n and c is constant, then ${}_0^cD_t^\alpha c = 0$.

2 MODEL FORMULATION

The model consists of different compartments which is divided into the following compartments: susceptible brood B_s, brood infested with varroa-mite B_m, brood infested with varroa-mite carrying virus B_i, susceptible hive bees H_s, hive bees infested with virus-free varroa-mite H_m, hive bees infested with virus-free varroa-mite carrying virus H_i, susceptible forager bees F_s, forager bees infested with virus-free varroa-mite F_m, forager bees infested with varroa mite-mite carrying virus F_i, virus-free varroa-mite population V_f, varroa-mite carrying virus population V_v and bio-control agent population A_B respectively. The parameters used in the model are properly defined in Table1.

Table1.

Parameter	Descriptions
β_1, β_2	Transmission rate of infestation
α_1, α_2	Disinfestation rate for honeybee
δ	Infection induced death rate of honeybee
d	Natural death rate of honeybee
ϕ	Eclosion rate of brood to hive bee
A	Recruitment rate of healthy bee
π	Treatment using thymol powder
μ	Natural death rate for bio-control agent population
μ_v	Natural death rate for varroa-mite population
C_1	Conversation coefficient of virus free varroa-mite to bio-agent
C_2	Conversation coefficient of virus carrying varroa-mite to bio-agent
η_1	Rate at which virus free varroa-mite acquires virus
η_2	Rate at which virus carrying varroa-mite loss it to healthy bee
η_3	Constant recruitment rate of varroa-mite carrying virus into the colony
Q	Environmental carrying capacity for varroa-mite
K	Environmental carrying capacity or bio-control agent
γ	Intrinsic growth rate of varroa-mite
τ	Intrinsic growth rate of bio-control agent
σ_1	Reversion rate of hive bee to forager bee

σ_2	Reversion rate of forager bee to hive bee

Putting all these assumptions together we obtained the model (1)

$$\left.\begin{aligned}
\frac{dB_s}{dt} &= A+(\alpha_1+\pi)B_m+(\alpha_2+\pi)B_i-\beta_1 B_s(V_f+V_v)-(\phi+d)B_s\\
\frac{dB_m}{dt} &= \beta_1 B_s V_f-\beta_2 B_m V_v-(\alpha_1+\pi+\phi+d)B_m\\
\frac{dB_i}{dt} &= (\beta_1 B_s+\beta_2 B_m)V_v-(\alpha_2+\pi+\phi+d+\delta)B_i\\
\frac{dH_s}{dt} &= -\beta_1 H_s(V_f+B_m+V_v+B_i)+\phi B_s+(\alpha_1+\pi)H_m+(\alpha_2+\pi)H_i+R(H_s,F_s)-R(H_s,F_s)H_s-dH_s\\
\frac{dH_m}{dt} &= \beta_1 H_s(V_f+B_m)-\beta_2 H_m(V_v+B_i)+\phi B_m-(\alpha_1+\pi+d)H_m\\
\frac{dH_i}{dt} &= \beta_1 H_s(V_v+B_i)+\beta_2 H_m(V_v+B_i)+\phi B_i-(\alpha_2+\pi+d+\delta)H_i\\
\frac{dF_s}{dt} &= R(H_s,F_s)H_s-R(H_s,F_s)-\beta_1 F_s(V_f+H_m+V_v+H_i)+(\alpha_1+\pi)F_m+(\alpha_2+\pi)F_i-dF_s\\
\frac{dF_m}{dt} &= \beta_1 F_s(V_f+H_m)-\beta_2 F_m(V_v+H_i)-(\alpha_1+\pi+d)F_m\\
\frac{dF_i}{dt} &= \beta_1 F_s(V_v+H_i)+\beta_2 F_m(V_v+H_i)-(\alpha_2+\pi+d+\delta)F_i\\
\frac{dV_f}{dt} &= \gamma V_f\left(1-\frac{V_f}{Q(N)}\right)-(C_1 A_B+\mu_v)V_f\\
\frac{dV_v}{dt} &= \eta_1(N_v-V_v)\frac{N_i}{N_s+N_m+N_i}-\eta_2 V_v\frac{N_s}{N_s+N_m+N_i}+\eta_3-(C_2 A_B+\mu_v)V_v\\
\frac{dA_B}{dt} &= \tau A_B\left(1-\frac{A_B}{K}\right)+(C_1 V_f+C_2 V_v)A_B-\mu A_B
\end{aligned}\right\} \quad (1)$$

But $N=N_s+N_m+N_i$, where $N_s=B_s+N_s+F_s$, $N_m=B_m+N_m+F_m$, $N_i=B_i+N_i+F_i$, $N_v=V_f+V_v$, $V_f=N_v-V_v$.

With initial condition: $B_s\geq 0$, $B_m\geq 0$, $B_i\geq 0$, $H_s\geq 0$, $H_m\geq 0$, $H_i\geq 0$, $F_s\geq 0$, $F_m\geq 0$, $F_i\geq 0$, $V_f\geq 0$, $V_v\geq 0$, $A_B\geq 0$

3 MODEL ANALYSIS

3.1 Disease Free Equilibrium

The system is qualitatively analyzed for disease free equilibrium which is obtained by setting all the derivatives to zero

$$\frac{dB_s}{dt}=\frac{dB_m}{dt}=\frac{dB_i}{dt}=\frac{dH_s}{dt}=\frac{dH_m}{dt}=\frac{dH_i}{dt}=\frac{dF_s}{dt}=\frac{dF_m}{dt}=\frac{dF_i}{dt}\frac{dV_f}{dt}=\frac{dV_v}{dt}=\frac{dA_B}{dt}=0\,.$$

$$E_0=\left(\frac{A}{\phi+d},0,0,\frac{A\phi+R(H_s,F_s)(\phi+d)}{(R(H_s,F_s)+d)(\phi+d)},0,0,\frac{R[A\phi+(\phi+d)+(\phi+d)(R+d)}{(\phi+d)^2(R+d)^2},0,0,0,0,0\right)$$

3.2 Global Stability of Endemic

Theorem 1. The endemic equilibrium E_1^* of the model (1) is globally asymptotically stable if and only if $R>1$.

Proof: We employ Lyapunov functions to demonstrate the endemic equilibrium's global asymptotically stable and unstable if $R<1$. Lyapunov function was considered as follows;

$$F\left(B_s,B_m,B_i,H_s,H_m,H_i,F_s,F_m,F_i,V_f,V_v,A_B\right)=\left(B_S-B_s^*\ln\frac{B_S}{B_s^*}\right)+\left(B_m-B_m^*\ln\frac{B_m}{B_m^*}\right)+\left(B_i-B_i^*\ln\frac{B_i}{B_i^*}\right)$$
$$+\left(H_s-H_s^*\ln\frac{H_s}{H_s^*}\right)+\left(H_m-H_m^*\ln\frac{H_m}{H_m^*}\right)+\left(H_i-H_i^*\ln\frac{H_i}{H_i^*}\right)+\left(F_s-F_s^*\ln\frac{F_s}{F_s^*}\right)+\left(F_m-F_m^*\ln\frac{F_m}{F_m^*}\right)$$
$$+\left(F_i-F_i^*\ln\frac{F_i}{F_i^*}\right)+\left(V_f-V_f^*\ln\frac{V_f}{V_f^*}\right)+\left(V_v-V_v^*\ln\frac{V_v}{V_v^*}\right)+\left(A_B-A_B^*\ln\frac{A_B}{A_B^*}\right)$$

Finding the derivatives of F to obtain the following

$$\frac{dF}{dt}=\left(1-\frac{B_S}{B_s^*}\right)\frac{dB_S}{dt}+\left(1-\frac{B_m}{B_m^*}\right)\frac{dB_m}{dt}+\left(1-\frac{B_i}{B_i^*}\right)\frac{dB_i}{dt}+\left(1-\frac{H_s}{H_s^*}\right)\frac{dH_S}{dt}+\left(1-\frac{H_m}{H_m^*}\right)\frac{dH_m}{dt}+\left(1-\frac{H_i}{H_i^*}\right)\frac{dH_i}{dt}\quad(2)$$
$$+\left(1-\frac{F_s}{F_s^*}\right)\frac{dF_S}{dt}+\left(1-\frac{F_m}{F_m^*}\right)\frac{dF_m}{dt}+\left(1-\frac{F_i}{F_i^*}\right)\frac{dF_i}{dt}+\left(1-\frac{V_f}{V_f^*}\right)\frac{dV_f}{dt}+\left(1-\frac{V_v}{V_v^*}\right)\frac{dV_v}{dt}+\left(1-\frac{A_B}{A_B^*}\right)\frac{dA_B}{dt}$$

Replacing $\frac{dB_S}{dt},\frac{dB_m}{dt},\frac{dB_i}{dt},\frac{dH_S}{dt},\frac{dH_m}{dt},\frac{dH_i}{dt},\frac{dF_S}{dt},\frac{dF_m}{dt},\frac{dF_i}{dt},\frac{dV_f}{dt},\frac{dV_v}{dt},\frac{dA_B}{dt}$.

From (2) to obtain the following;

$$\frac{dF}{dt}=\left(1-\frac{B_S}{B_s^*}\right)\left[A+(\alpha_1+\pi)B_m+(\alpha_2+\pi)B_i-\beta_1 B_s(V_f+V_v)-(\phi+d)B_s\right]$$

$$+\left(1-\frac{B_m}{B_m^*}\right)\left[\beta_1 B_s V_f-\beta_2 B_m V_v-(\alpha_1+\pi+\phi+d)B_m\right]+\left(1-\frac{B_i}{B_i^*}\right)\left[(\beta_1 B_s+\beta_2 B_m)V_v-(\alpha_2+\pi+\phi+d+\delta)B_i\right]$$

$$+\left(1-\frac{H_s}{H_s^*}\right)\left[-\beta_1 H_s(V_f+B_m+V_v+B_i)+\phi B_s+(\alpha_1+\pi)H_m+(\alpha_2+\pi)H_i+R(H_s,F_s)-R(H_s,F_s)H_s-dH_s\right]$$

$$+\left(1-\frac{H_m}{H_m^*}\right)\left[\beta_1 H_s(V_f+B_m)-\beta_2 H_m(V_v+B_i)+\phi B_m-(\alpha_1+\pi+d)H_m\right]$$

$$+\left(1-\frac{H_i}{H_i^*}\right)\left[\beta_1 H_s(V_v+B_i)+\beta_2 H_m(V_v+B_i)+\phi B_i-(\alpha_2+\pi+d+\delta)H_i\right]$$

$$+\left(1-\frac{F_s}{F_s^*}\right)\left[\beta_1 H_s(V_v+B_i)+\beta_2 H_m(V_v+B_i)+\phi B_i-(\alpha_2+\pi+d+\delta)H_i\right]$$

$$+\left(1-\frac{F_m}{F_m^*}\right)\left[\beta_1 F_s(V_f+H_m)-\beta_2 F_m(V_v+H_i)-(\alpha_1+\pi+d)F_m\right]$$

$$+\left(1-\frac{F_i}{F_i^*}\right)\left[\beta_1 F_s(V_v+H_i)+\beta_2 F_m(V_v+H_i)-(\alpha_2+\pi+d+\delta)F_i\right]$$

$$+\left(1-\frac{V_f}{V_f^*}\right)\left[\gamma V_f\left(1-\frac{V_f}{Q(N)}\right)-(C_1 A_B+\mu_v)V_f\right]$$

$$+\left(1-\frac{V_v}{V_v^*}\right)\left[\eta_1(N_v-V_v)\frac{N_i}{N_s+N_m+N_i}-\eta_2 V_v\frac{N_s}{N_s+N_m+N_i}+\eta_3-(C_2 A_B+\mu_v)V_v\right]$$

$$+\left(1-\frac{A_B}{A_B^*}\right)\left[\tau A_B\left(1-\frac{A_B}{K}\right)+(C_1 V_f+C_2 V_v)A_B-\mu A_B\right]$$

The following parameters were obtained at endemic point of the model (1)

$$\phi + d = \frac{A + (\alpha_1 + \pi)B_m^* + (\alpha_2 + \pi)B_i^* - \beta_1 B_i^* (V_f^* + V_v^*)}{B_s^*},$$

$$\alpha_1 + \pi + \phi + d = \frac{\beta_1 B_s^* V_f^* - \beta_2 B_m^* V_v^*}{B_m^*},$$

$$\alpha_2 + \pi + \phi + d + \delta = \frac{(\beta_1 B_S^* + \beta_2 B_m^*) V_v^*}{B_i^*},$$

$$\alpha_1 + \pi + d = \frac{\beta_1 H_s^* (V_f^* + B_m^*) - \beta_2 H_m^* (V_v^* + B_i^*) + \phi B_m^*}{H_m^*},$$

$$\alpha_2 + \pi + d + \delta = \frac{\beta_1 H_s^* (V_v^* + B_i^*) + \beta_2 H_m^* (V_v^* + B_i^*) + \phi B_i^*}{H_i^*},$$

$$\alpha_1 + \pi + d = \frac{\beta_1 F_s^* (V_f^* + H_m^*) - \beta_2 F_m^* (V_v^* + H_i^*)}{F_m^*},$$

$$\alpha_2 + \pi + d + \delta = \frac{\beta_1 F_s^* (V_v^* + H_i^*) + \beta_2 F_m^* (V_v^* + H_i^*)}{F_i^*}$$

then

$$
\left.
\begin{aligned}
\frac{dF}{dt} &= \left(1-\frac{B_S}{B_s^*}\right)\left[A+(\alpha_1+\pi)B_m+(\alpha_2+\pi)B_i-\beta_1 B_s(V_f+V_v)-\left(\frac{A+(\alpha_1+\pi)B_m^*+(\alpha_2+\pi)B_i^*-\beta_1 B_i^*(V_f^*+V_v^*)}{B_s^*}\right)B_s\right]\\
&+\left(1-\frac{B_m}{B_m^*}\right)\left[\beta_1 B_s V_f-\beta_2 B_m V_v-\left(\frac{\beta_1 B_s^* V_f^*-\beta_2 B_m^* V_v^*}{B_m^*}\right)B_m\right]\\
&+\left(1-\frac{B_i}{B_i^*}\right)\left[(\beta_1 B_s+\beta_2 B_m)V_v-\left(\frac{(\beta_1 B_S^*+\beta_2 B_m^*)V_v^*}{B_i^*}\right)B_i\right]\\
&+\left(1-\frac{H_s}{H_s^*}\right)\left[-\beta_1 H_s(V_f+B_m+V_v+B_i)+\phi B_s+(\alpha_1+\pi)H_m+(\alpha_2+\pi)H_i+R(H_s,F_s)-R(H_s,F_s)H_s-dH_s\right]\\
&+\left(1-\frac{H_m}{H_m^*}\right)\left[\beta_1 H_s(V_f+B_m)-\beta_2 H_m(V_v+B_i)+\phi B_m-\left(\frac{\beta_1 H_s^*(V_f^*+B_m^*)-\beta_2 H_m^*(V_v^*+B_i^*)+\phi B_m^*}{H_m^*}\right)H_m\right]\\
&+\left(1-\frac{H_i}{H_i^*}\right)\left[\beta_1 H_s(V_v+B_i)+\beta_2 H_m(V_v+B_i)+\phi B_i-\left(\frac{\beta_1 H_s^*(V_v^*+B_i^*)+\beta_2 H_m^*(V_v^*+B_i^*)+\phi B_i^*}{H_i^*}\right)H_i\right]\\
&+\left(1-\frac{F_s}{F_s^*}\right)\left[\beta_1 H_s(V_v+B_i)+\beta_2 H_m(V_v+B_i)+\phi B_i-(\alpha_2+\pi+d+\delta)H_i\right]\\
&+\left(1-\frac{F_m}{F_m^*}\right)\left[\beta_1 F_s(V_f+H_m)-\beta_2 F_m(V_v+H_i)-\left(\frac{\beta_1 F_s^*(V_f^*+H_m^*)-\beta_2 F_m^*(V_v^*+H_i^*)}{F_m^*}\right)F_m\right]\\
&+\left(1-\frac{F_i}{F_i^*}\right)\left[\beta_1 F_s(V_v+H_i)+\beta_2 F_m(V_v+H_i)-\left(\frac{\beta_1 F_s^*(V_v^*+H_i^*)+\beta_2 F_m^*(V_v^*+H_i^*)}{F_i^*}\right)F_i\right]\\
&+\left(1-\frac{V_f}{V_f^*}\right)\left[\gamma V_f\left(1-\frac{V_f}{Q(N)}\right)-(C_1 A_B+\mu_v)V_f\right]\\
&+\left(1-\frac{V_v}{V_v^*}\right)\left[\eta_1(N_v-V_v)\frac{N_i}{N_s+N_m+N_i}-\eta_2 V_v\frac{N_s}{N_s+N_m+N_i}+\eta_3-(C_2 A_B+\mu_v)V_v\right]\\
&+\left(1-\frac{A_B}{A_B^*}\right)\left[\tau A_B\left(1-\frac{A_B}{K}\right)+(C_1 V_f+C_2 V_v)A_B-\mu A_B\right]
\end{aligned}
\right\} \quad (3)
$$

Without loss of generality, it was assumed that $B_s = B_s^*, B_m = B_m^*, B_i = B_i^*, H_s = H_s^*, H_m = H_m^*, H_i = H_i^*, F_s = F_s^*, F_m = F_m^*, F_i = F_i^*, V_f = V_f^*, V_v = V_v^*, A_B = A_B^*$ then $\frac{dF}{dt} = 0$. Therefore by Lasalle's invariant principle every solution of system (1) together with initial condition in $\Omega = \{(B_s, B_m, B_i, H_s, H_m, H_i, F_s, F_m, F_i, V_f, V_v, A_B) \in \Re_+^{12}\}$ it means that Ω^{**} is globally asymptotically stable.

Caputo's fractional derivatives order of (1) is writing below

$$\left.\begin{aligned}
{}^cD_t^\alpha B_s(t) &= A + (\alpha_1 + \pi)B_m + (\alpha_2 + \pi)B_i - \beta_1 B_s(V_f + V_v) - (\phi + d)B_s \\
{}^cD_t^\alpha B_m(t) &= \beta_1 B_s V_f - \beta_2 B_m V_v - (\alpha_1 + \pi + \phi + d)B_m \\
{}^cD_t^\alpha B_i(t) &= (\beta_1 B_s + \beta_2 B_m)V_v - (\alpha_2 + \pi + \phi + d + \delta)B_i \\
{}^cD_t^\alpha H_s(t) &= -\beta_1 H_s(V_f + B_m + V_v + B_i) + \phi B_s + (\alpha_1 + \pi)H_m + (\alpha_2 + \pi)H_i + R(H_s, F_s) - R(H_s, F_s)H_s - dH_s \\
{}^cD_t^\alpha H_m(t) &= \beta_1 H_s(V_f + B_m) - \beta_2 H_m(V_v + B_i) + \phi B_m - (\alpha_1 + \pi + d)H_m \\
{}^cD_t^\alpha H_i(t) &= \beta_1 H_s(V_v + B_i) + \beta_2 H_m(V_v + B_i) + \phi B_i - (\alpha_2 + \pi + d + \delta)H_i \\
{}^cD_t^\alpha F_s(t) &= R(H_s, F_s)H_s - R(H_s, F_s) - \beta_1 F_s(V_f + H_m + V_v + H_i) + (\alpha_1 + \pi)F_m + (\alpha_2 + \pi)F_i - dF_s \\
{}^cD_t^\alpha F_m(t) &= \beta_1 F_s(V_f + H_m) - \beta_2 F_m(V_v + H_i) - (\alpha_1 + \pi + d)F_m \\
{}^cD_t^\alpha F_i(t) &= \beta_1 F_s(V_v + H_i) + \beta_2 F_m(V_v + H_i) - (\alpha_2 + \pi + d + \delta)F_i \\
{}^cD_t^\alpha V_f(t) &= \gamma V_f\left(1 - \frac{V_f}{Q(N)}\right) - (C_1 A_B + \mu_v)V_f \\
{}^cD_t^\alpha V_v(t) &= \eta_1(N_v - V_v)\frac{N_i}{N_s + N_m + N_i} - \eta_2 V_v \frac{N_s}{N_s + N_m + N_i} + \eta_3 - (C_2 A_B + \mu_v)V_v \\
{}^cD_t^\alpha A_B(t) &= \tau A_B\left(1 - \frac{A_B}{K}\right) + (C_1 V_f + C_2 V_v)A_B - \mu A_B
\end{aligned}\right\} \quad (4)$$

Let

$B_s = n_1, B_m = n_2, B_i = n_3, H_s = n_4, H_m = n_5, H_i = n_6, F_s = n_7, F_m = n_8, F_i = n_9, V_f = n_{10}, V_v = n_{11}, A_B = n_{12}$

Taking Laplace transformation of (4), we obtained the following;

$$
\left.
\begin{aligned}
& sL\{B_s(t)\} - B_s(0) = L\{A + (\alpha_1 + \pi)B_m + (\alpha_2 + \pi)B_i - \beta_1 B_s(V_f + V_v) - (\phi + d)B_s\} \\
& sL\{B_m(t)\} - B_m(0) = L\{\beta_1 B_s V_f - \beta_2 B_m V_v - (\alpha_1 + \pi + \phi + d)B_m\} \\
& sL\{B_i(t)\} - B_i(0) = L\{(\beta_1 B_s + \beta_2 B_m)V_v - (\alpha_2 + \pi + \phi + d + \delta)B_i\} \\
& sL\{H_s(t)\} - H_s(0) = L\{-\beta_1 H_s(V_f + B_m + V_v + B_i) + \phi B_s + (\alpha_1 + \pi)H_m + (\alpha_2 + \pi)H_i + R(H_s, F_s) - R(H_s, F_s)H_s - dH_s\} \\
& sL\{H_m(t)\} - H_m(0) = L\{\beta_1 H_s(V_f + B_m) - \beta_2 H_m(V_v + B_i) + \phi B_m - (\alpha_1 + \pi + d)H_m\} \\
& sL\{H_i(t) - H_i(0)\} = L\{\beta_1 H_s(V_v + B_i) + \beta_2 H_m(V_v + B_i) + \phi B_i - (\alpha_2 + \pi + d + \delta)H_i\} \\
& sL\{F_s(t)\} - F_s(0) = L\{R(H_s, F_s)H_s - R(H_s, F_s) - \beta_1 F_s(V_f + H_m + V_v + H_i) + (\alpha_1 + \pi)F_m + (\alpha_2 + \pi)F_i - dF_s\} \\
& sL\{F_m(t)\} - F_m(0) = L\{\beta_1 F_s(V_f + H_m) - \beta_2 F_m(V_v + H_i) - (\alpha_1 + \pi + d)F_m\} \\
& sL\{F_i(t)\} - F_i(0) = L\{\beta_1 F_s(V_v + H_i) + \beta_2 F_m(V_v + H_i) - (\alpha_2 + \pi + d + \delta)F_i\} \\
& sL\{V_f(t)\} - V_f(0) = L\left\{\gamma V_f\left(1 - \frac{V_f}{Q(N)}\right) - (C_1 A_B + \mu_v)V_f\right\} \\
& sL\{V_v(t)\} - V_v(0) = L\left\{\eta_1(N_v - V_v)\frac{N_i}{N_s + N_m + N_i} - \eta_2 V_v \frac{N_s}{N_s + N_m + N_i} + \eta_3 - (C_2 A_B + \mu_v)V\right\}_v \\
& sL\{A_B(t)\} - A_B(0) = L\left\{\tau A_B\left(1 - \frac{A_B}{K}\right) + (C_1 V_f + C_2 V_v)A_B - \mu A_B\right\}
\end{aligned}
\right\}
$$

From the above equation upon the simplification, we obtained the following;

$$
\left.\begin{array}{l}
B_s(t)=B_s(0)+L^{-1}\left\{\frac{1}{s^{\alpha}}L\left[A+(\alpha_1+\pi)B_m+(\alpha_2+\pi)B_i-\beta_1 B_s(V_f+V_v)-(\phi+d)B_s\right]\right\} \\
B_m(t)=B_m(0)+L^{-1}\left\{\frac{1}{s^{\alpha}}L\left[\beta_1 B_s V_f-\beta_2 B_m V_v-(\alpha_1+\pi+\phi+d)B_m\right]\right\} \\
B_i(t)=B_i(0)+L^{-1}\left\{\frac{1}{s^{\alpha}}L\left[(\beta_1 B_s+\beta_2 B_m)V_v-(\alpha_2+\pi+\phi+d+\delta)B_i\right]\right\} \\
H_s(t)=H_s(0)+L^{-1}\left\{\frac{1}{s^{\alpha}}L\left[-\beta_1 H_s(V_f+B_m+V_v+B_i)+\phi B_s+(\alpha_1+\pi)H_m+(\alpha_2+\pi)H_i+R(H_s,F_s)-R(H_s,F_s)H_s-dH_s\right]\right\} \\
H_m(t)=H_m(0)+L^{-1}\left\{\frac{1}{s^{\alpha}}L\left[\beta_1 H_s(V_f+B_m)-\beta_2 H_m(V_v+B_i)+\phi B_m-(\alpha_1+\pi+d)H_m\right]\right\} \\
H_i(t)=H_i(0)+L^{-1}\left\{\frac{1}{s^{\alpha}}L\left[\beta_1 H_s(V_v+B_i)+\beta_2 H_m(V_v+B_i)+\phi B_i-(\alpha_2+\pi+d+\delta)H_i\right]\right\} \\
F_s(t)=F_s(0)+L^{-1}\left\{\frac{1}{s^{\alpha}}L\left[R(H_s,F_s)H_s-R(H_s,F_s)-\beta_1 F_s(V_f+H_m+V_v+H_i)+(\alpha_1+\pi)F_m+(\alpha_2+\pi)F_i-dF_s\right]\right\} \\
F_m(t)=F_m(0)+L^{-1}\left\{\frac{1}{s^{\alpha}}L\left[\beta_1 F_s(V_f+H_m)-\beta_2 F_m(V_v+H_i)-(\alpha_1+\pi+d)F_m\right]\right\} \\
F_i(t)=F_i(0)+L^{-1}\left\{\frac{1}{s^{\alpha}}L\left[\beta_1 F_s(V_v+H_i)+\beta_2 F_m(V_v+H_i)-(\alpha_2+\pi+d+\delta)F_i\right]\right\} \\
V_f(t)=V_f(0)+L^{-1}\left\{\frac{1}{s^{\alpha}}L\left[\gamma V_f\left(1-\frac{V_f}{Q(N)}\right)-(C_1A_B+\mu_v)V_f\right]\right\} \\
V_v(t)=V_v(0)+L^{-1}\left\{\frac{1}{s^{\alpha}}L\left[\eta_1(N_v-V_v)\frac{N_i}{N_s+N_m+N_i}-\eta_2 V_v\frac{N_s}{N_s+N_m+N_i}+\eta_3-(C_2A_B+\mu_v)V\right]\right\} \\
A_B(t)=A_B(0)+L^{-1}\left\{\frac{1}{s^{\alpha}}L\left[\tau A_B\left(1-\frac{A_B}{K}\right)+(C_1V_f+C_2V_v)A_B-\mu A_B\right]\right\}
\end{array}\right\}
$$

4 NUMERICAL SIMULATION RESULTS AND DISCUSSION

The following initial conditions together with parameters are used for computation of Laplace Adomian Decomposition Method and the results are displayed below.

$B_s(0)=800$, $B_m(0)=750$, $B_i(0)=600$, $H_s(0)=500$, $H_m(0)=400$, $H_i(0)=300$, $F_s(0)=200$, $F_m(0)=150$, $F_i(0)=100$, $V_f(0)=90$, $V_v(0)=70$, $A_B(0)=50$

$\beta_1=0.1984$, $\beta_2=0.05$, $\alpha_1=0.6$, $\alpha_2=0.4$, $d=0.2$, $\delta=0.3$, $\tau=0.6$, $\gamma=0.01656$, $\kappa=2500$, $Q=0.5$, $\eta_1=0.1593$, $\eta_2=0.04959$, $\eta_3=300$, $K=600$, $\pi=0.9$, $A=60000$, $\psi=0.001$, $R=0.56$, $C_1=1.5$, $C_2=1.5$, $\mu_v=0.002$, $\mu=0.005$

$B_s(t)=n_1+B_{s1}+B_{s2}$

$$B_s = 800 + \frac{36349.0000t^{\rho 1}}{\Gamma(\rho_1+1)} + \frac{1}{\Gamma(\rho_1+1)}\left(\left(\begin{array}{l} 59841.2800 - \frac{1.1066453051\,0^6 t^{\rho 1}}{\Gamma(\rho_1+1)} + \frac{16451.24000t^{\rho 3}}{\Gamma(\rho_3+1)} \\ + \frac{1391.973408t^{\rho 10}}{\Gamma(\rho_{10}+1)} + \frac{7.3819434361\,0^5 t^{\rho 11}}{\Gamma(\rho_{10}+1)} + \cdots \end{array}\right)\right)t^{\rho 1}$$

$$B_m = n_2 + B_{m1} + B_{m2},$$

$$B_m = 750 + \frac{10384.0500t^{\rho 2}}{\Gamma(\rho_2+1)} + \frac{1}{\Gamma(\rho_2+1)}\left(\left(\begin{array}{l} \frac{6.4904774401\,0^5 t^{\rho 1}}{\Gamma(\rho_1+1)} - \frac{7.3819434361\,0^5 t^{\rho 11}}{\Gamma(\rho_{11}+1)} + \\ \frac{6.7532788801\,0^6 t^{\rho 4}}{\Gamma(\rho_4+1)} + \frac{1.7539987501\,0^6 t^{\rho 10}}{\Gamma(\rho_{10}+1)} - \\ \frac{17663.26905t^{\rho 2}}{\Gamma(\rho_2+1)} + \cdots \end{array}\right)\right)t^{\rho 2}$$

$$B_i = n_3 + B_{i1} + B_{i2},$$

$$B_i = 600 + \frac{12654.8000t^{\rho 3}}{\Gamma(\rho_3+1)} + \frac{\left(\left(\frac{5.0481491201\,0^5 t^{\rho 1}}{\Gamma(\rho_1+1)} - \frac{7.3819434361\,0^5 t^{\rho 11}}{\Gamma(\rho_{11}+1)} + 0.05 - \frac{22791.29480t^{\rho 3}}{\Gamma(\rho_3+1)} + \right)\right)t^{\rho 3}}{\Gamma(\rho_3+1)} + \cdots$$

,

$$H_s = n_4 + H_{s1} + H_{s2},$$

$$H_s = 500 + \frac{6.7750075001\,0^5 t^{\rho 6}}{\Gamma(\rho_6+1)} + \frac{1}{\Gamma(\rho_6+1)}\left(\left(\begin{array}{l} -\frac{1.9948885671\,0^8 t^{\rho 4}}{\Gamma(\rho_4+1)} - \frac{4.7067330881\,0^6 t^{\rho 11}}{\Gamma(\rho_{11}+1)} + \\ \frac{1.2806657601\,0^7 t^{\rho 3}}{\Gamma(\rho_3+1)} + \frac{2.7443225121\,0^7 t^{\rho 5}}{\Gamma(\rho_5+1)} + 0.750 - \\ \frac{1.2195013501\,0^6 t^{\rho 6}}{\Gamma(\rho_6+1)} + \cdots \end{array}\right)\right)t^{\rho 6}$$

$$H_m = n_5 + H_{m1} + H_{m2},$$

$$H_m = 400 + \frac{8.1920075001\,0^5 t^{\rho 5}}{\Gamma(\rho_5 + 1)} + \frac{1}{\Gamma(\rho_5 + 1)} \left(\left(\begin{array}{l} -\frac{2.5010543231\,0^8 t^{\rho 4}}{\Gamma(\rho_4 + 1)} - \frac{6.9598670401\,0^6 t^{\rho 10}}{\Gamma(\rho_{10} + 1)} \\ + \frac{1.0300987981\,0^7 t^{\rho 2}}{\Gamma(\rho_2 + 1)} - \frac{3.1703069021\,0^7 t^{\rho 5}}{\Gamma(\rho_5 + 1)} \\ - \frac{2.5309600001\,0^5 t^{\rho 3}}{\Gamma(\rho_3 + 1)} + \cdots \end{array} \right) \right) t^{\rho 5}$$

$$H_i = n_6 + H_{i1} + H_{i2},$$

$$H_i = 300 + \frac{6.7750075001\,0^5 t^{\rho 6}}{\Gamma(\rho_6 + 1)} + \frac{1}{\Gamma(\rho_6 + 1)} \left(\left(\begin{array}{l} -\frac{1.9948885671\,0^8 t^{\rho 4}}{\Gamma(\rho_4 + 1)} - \frac{4.7067330881\,0^6 t^{\rho 11}}{\Gamma(\rho_{11} + 1)} + \\ \frac{1.2806657601\,0^7 t^{\rho 3}}{\Gamma(\rho_3 + 1)} + \frac{2.7443225121\,0^7 t^{\rho 5}}{\Gamma(\rho_5 + 1)} + 0.750 - \\ \frac{1.2195013501\,0^6 t^{\rho 6}}{\Gamma(\rho_6 + 1)} + \cdots \end{array} \right) \right) t^{\rho 6}$$

$$F_s = n_7 + F_{i1} + F_{i2},$$

$$F_s = 200 + \frac{17256.6000 t^{\rho 9}}{\Gamma(\rho_9 + 1)} + \frac{1}{\Gamma(\rho_9 + 1)} \left(\left(\begin{array}{l} -\frac{2.2764085071\,0^6 t^{\rho 7}}{\Gamma(\rho_7 + 1)} - \frac{2.1943050111\,0^5 t^{\rho 11}}{\Gamma(\rho_{11} + 1)} + \\ \frac{3.1964485381\,0^7 t^{\rho 6}}{\Gamma(\rho_6 + 1)} + \frac{3.0364420001\,0^5 t^{\rho 8}}{\Gamma(\rho_8 + 1)} \\ - \frac{34513.20000 t^{\rho 9}}{\Gamma(\rho_9 + 1)} + \cdots \end{array} \right) \right) t^{\rho 9}$$

$$F_m = n_8 + F_{m1} + F_{m2},$$

$$F_m = 150 + \frac{16413.2000 t^{\rho 8}}{\Gamma(\rho_8 + 1)} + \frac{1}{\Gamma(\rho_8 + 1)} \left(\left(\begin{array}{l} -\frac{3.0147031581\,0^6 t^{\rho 7}}{\Gamma(\rho_7 + 1)} - \frac{2.7839468161\,0^5 t^{\rho 10}}{\Gamma(\rho_{10} + 1)} + \\ \frac{3.2505885761\,0^7 t^{\rho 5}}{\Gamma(\rho_5 + 1)} - \frac{3.3154664001\,0^5 t^{\rho 8}}{\Gamma(\rho_8 + 1)} + \\ \frac{34881.9151810^5 t^{\rho 11}}{\Gamma(\rho_{11} + 1)} - \frac{5.0812556251\,0^6 t^{\rho 6}}{\Gamma(\rho_6 + 1)} + \cdots \end{array} \right) \right) t^{\rho 8}$$

$$F_i = n_8 + F_{i1} + F_{i2},$$

$$F_i = n_9 + \frac{17256.6000 t^{\rho 9}}{\Gamma(\rho_9 + 1)} + \frac{1}{\Gamma(\rho_9 + 1)} \left(\left(\begin{array}{l} -\frac{2.2764085071\,0^{6} t^{\rho 7}}{\Gamma(\rho_7 + 1)} - \frac{2.1943050111\,0^{5} t^{\rho 11}}{\Gamma(\rho_{11} + 1)} + \\ \frac{3.1964485381\,0^{7} t^{\rho 6}}{\Gamma(\rho_6 + 1)} + \frac{3.0364420001\,0^{5} t^{\rho 8}}{\Gamma(\rho_8 + 1)} - \\ \frac{34513.20000 t^{\rho 9}}{\Gamma(\rho_9 + 1)} + \cdots \end{array} \right) \right) t^{\rho 9}$$

$$V_f = n_{10} + V_{f1} + V_{f2},$$

$$V_f = 90 - \frac{7015.995000 t^{\rho 10}}{\Gamma(\rho_{10} + 1)} + \frac{\left(-\frac{101.7319275 t^{\rho 10}}{\Gamma(\rho_{10} + 1)} - \frac{1.6243981331\,0^{6} (t^{\rho 10})^2}{\Gamma(\rho_{10} + 1)^2} - 1.5 \right) t^{\rho 10}}{\Gamma(\rho_{10} + 1)} + \cdots$$

$$V_v = n_{11} + V_{v1} + V_{v2}$$

$$V_v = 70 - \frac{4950.922024 t^{\rho 11}}{\Gamma(\rho_{11} + 1)} + \frac{\left(299.2179759 - \frac{1.2631165311\,0^{6} t^{\rho 12}}{\Gamma(\rho_{12} + 1)} + \frac{2.3255540301\,0^{5} t^{\rho 11}}{\Gamma(\rho_{11} + 1)} \right) t^{\rho 11}}{\Gamma(\rho_{11} + 1)} + \cdots$$

$$A_b = n_{12} + A_{b1} + A_{b2},$$

$$A_b = 50 - \frac{12027.25000 t^{\rho 12}}{\Gamma(\rho_{12} + 1)} + \frac{\left(\frac{3.2551715971\,0^{6} t^{\rho 12}}{\Gamma(\rho_{12} + 1)} - \frac{1.4471323101\,0^{5} (t^{\rho 12})^2}{\Gamma(\rho_{12} + 1)^2} - \frac{1.0523992501\,0^{6} t^{\rho 10}}{\Gamma(\rho_{10} + 1)} \right) t^{\rho 12}}{\Gamma(\rho_{12} + 1)} + \cdots$$

Consider numerical simulations of the suggested approach utilizing the fractal fractional technique for fractional order in this section. For a spread of varroa-mites in honeybee model was used. The various numerical methods identify the mechanical features of the fractional-order model with the time-fractional parameters. The results of the nonlinear system memory were also detected with the help of fractional values. Figure.1 to Figure 5 represents the simulations obtained by fractal fractional method and easily observed that all the compartments start decreasing by increasing the fractional values which converge to steady state. Also, susceptible brood, brood infested by virus free varroa-mite and brood infested by varroa-mite carrying virus were successfully compared in Fig.6 and it was discovered that susceptible brood keep increasing while brood infested by virus-free varroa-mite and brood infested by varroa-mite carrying virus decreasing as time increasing. Similarly, susceptible hive bees, hive bees infested by virus free varroa-mite and hive bees infested by varroa-mite carrying virus were successfully compared in Figure 7 and it was noted that hive bees infested by varroa mite carrying virus increased to a peak before it starts declining gradually as time increasing. Susceptible forager bees, forager bees infested by virus free varroa mite and forager bees infested by varroa mite carrying virus were successfully compared in Figure 8 and it was observed that forager bees infested by varroa-mite carrying virus increased to a peak before it starts declining gradually as time increasing.

Finally, virus free varroa mite, varroa mite carrying virus and bio-control agent were successfully compared in Figure 9 and bio-control agent was increasing indefinitely as time increasing.

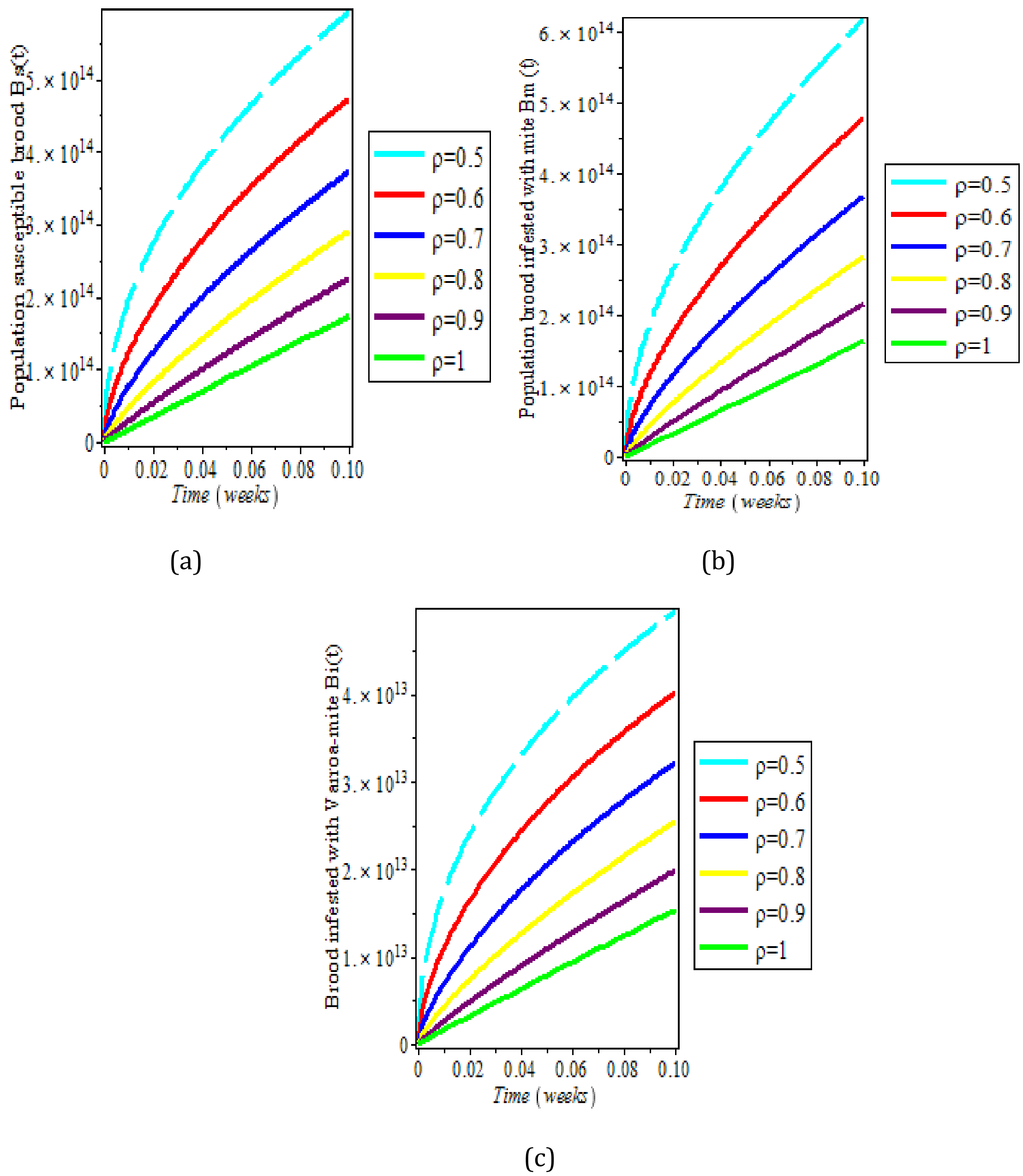
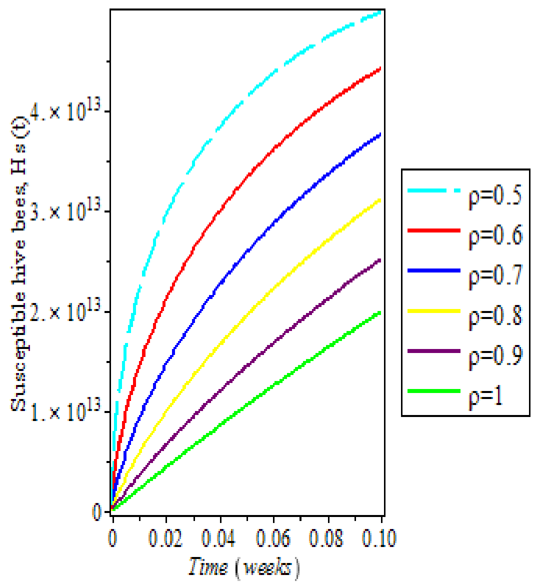
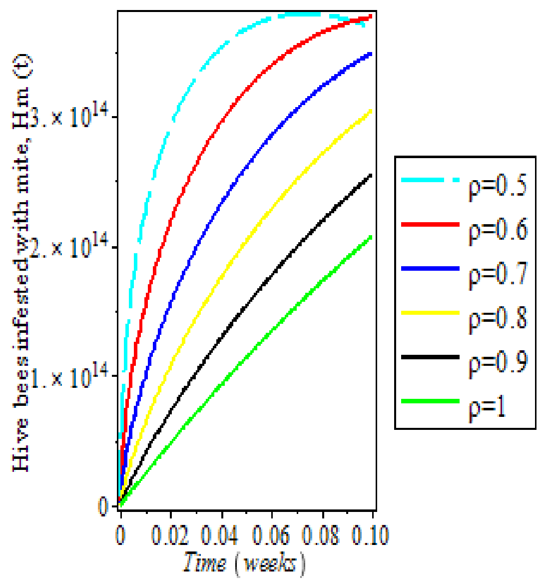


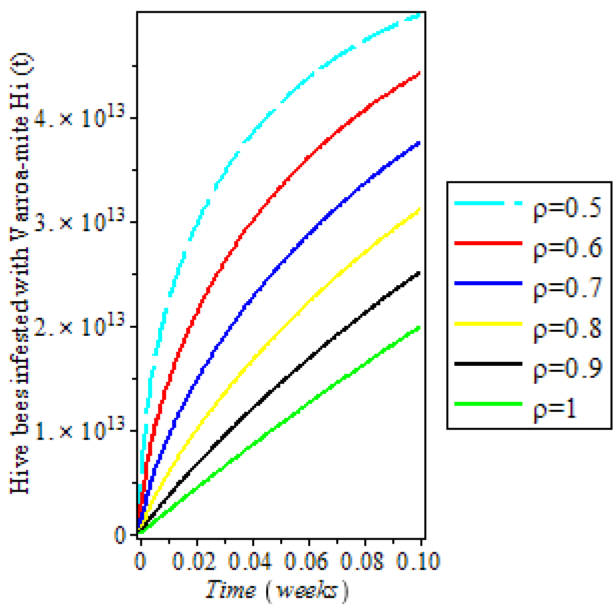
Figure 1: Behavior of numerical trajectory of brood population $B(t)$ at different values of fractional order ρ



(a)

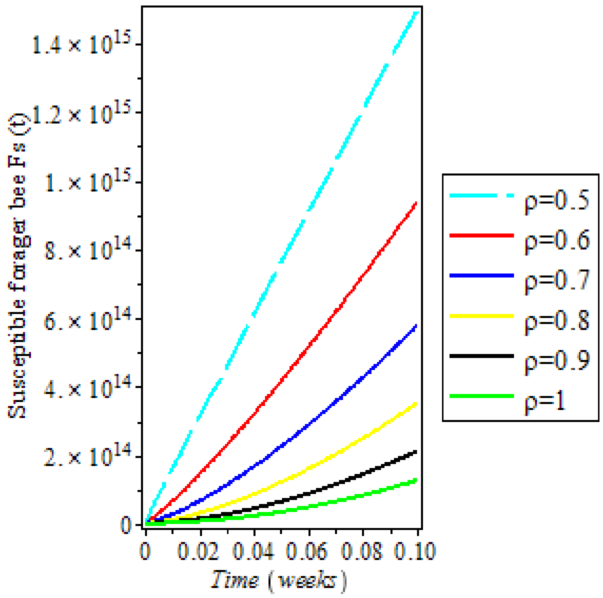


(b)

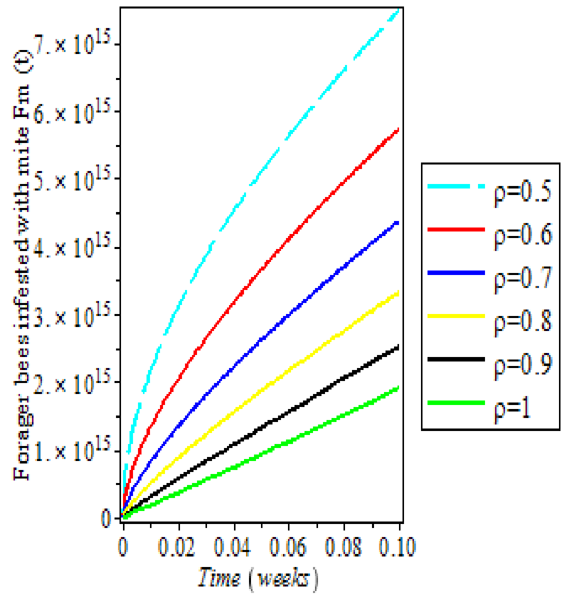


(c)

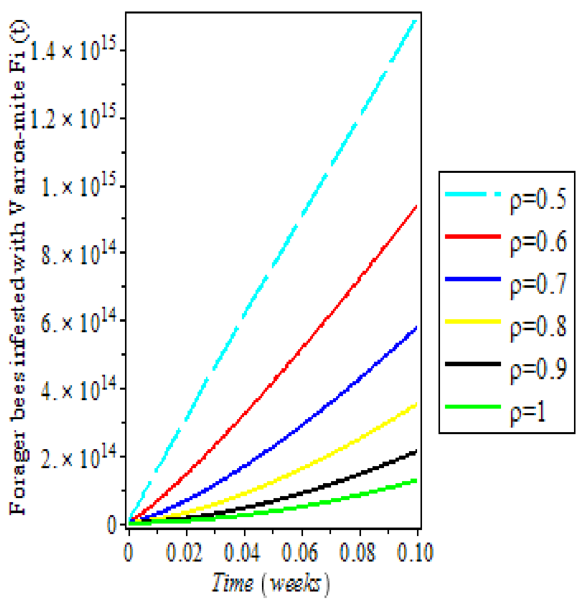
Figure 2: Behavior of numerical trajectory of Hive $H(t)$ at different values of fractional order ρ



(a)

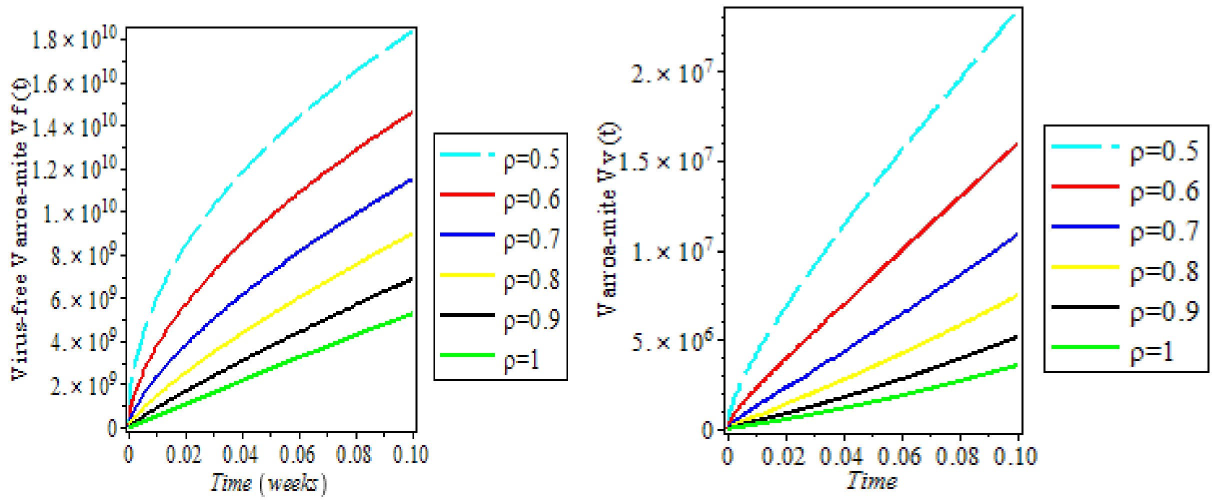


(b)



(c)

Figure 3: Behavior of numerical trajectory of forager population $F(t)$ at different values of fractional order ρ



(a) (b)

Figure 4: Behavior of numerical trajectory of varroa-mite $V_f(t)$ population at different values of fractional order ρ

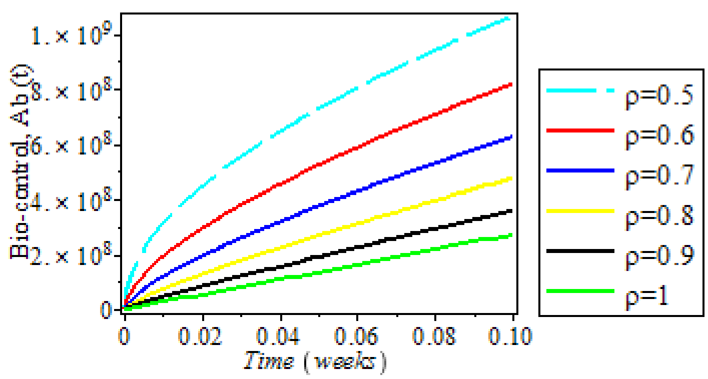


Figure 5: Behavior of numerical trajectory of $A_b(t)$ at different values of fractional order ρ

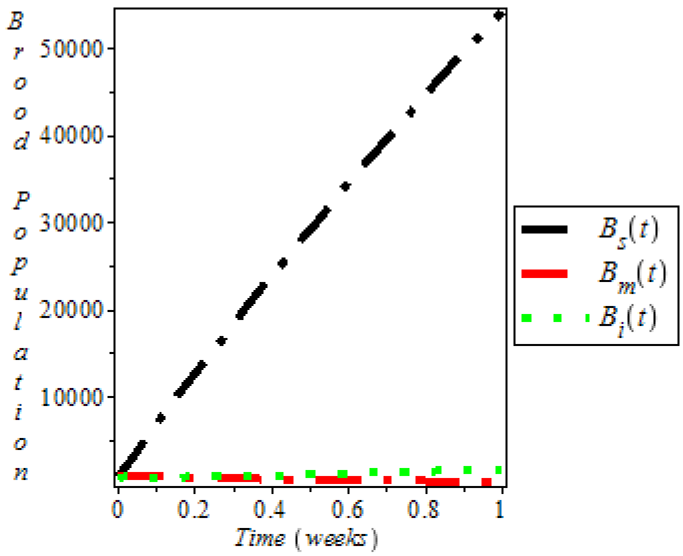


Figure 6: Comparison of behavior among brood population at different values of time

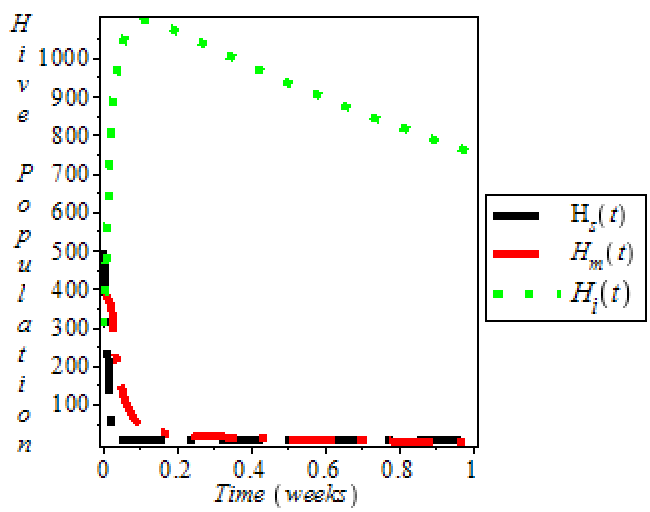


Figure 7: Comparison of behavior among hive population at different values of time

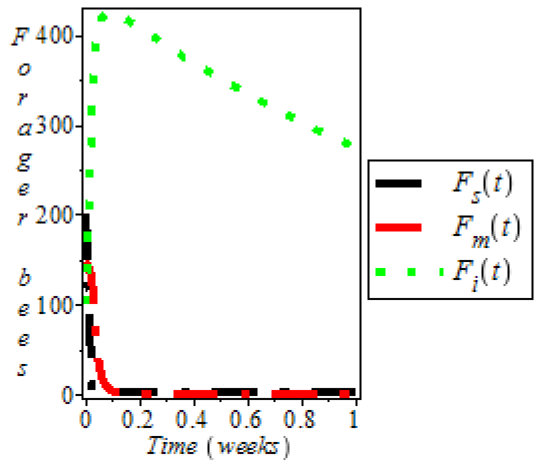


Figure 8: Comparison of behavior among forager population at different values of time

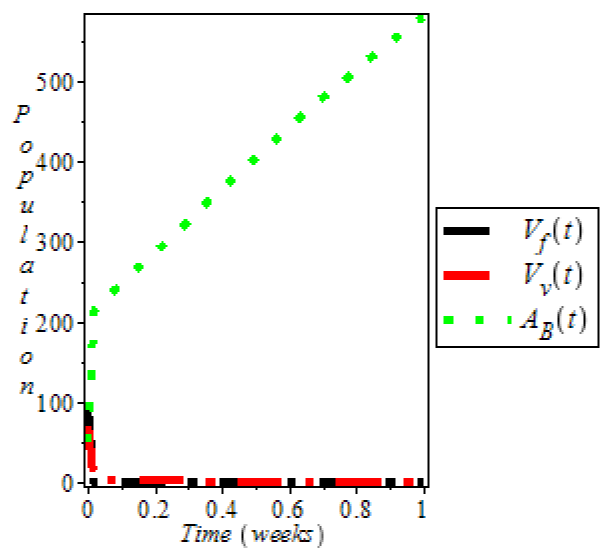


Figure 9: Comparison of behavior between varroa-mite population and bio-control agent at different values of time

Theorem 2. Let X be a Banach space and P : X→ X be a contractive nonlinear operator such that for all x, $x^1 \in X$, $\|\|T(X) - T(X^1)\|\| \le k\|\| x - x^1\|\|$, $0 < k < 1$. Then T has a unique point x such that T x = x, where $x = S, V_1, V_2, E, U, D, T, J, R$.

The series given in (10) can be written by applying Adomian decomposition method as:

$x_m = T\, x_{m-1}$, $x_{m-1} = \sum x_i$ $m = 1, 2, 3, \ldots$, and assume that $x_0 \in B_r(x)$ where $B_r(x) = \{x^1 \in X :$
(i) $x_m \in B_r(x)\}$; (ii) $\lim_{n\to\infty} x_m = x$

Proof.

For (i), using mathematical induction for $m = 1$, we have

$\|\|\|x_0 - x\|\|\| = \|\|\|T(x_0) - T(x)\|\|\| \leq k\|\|\|x_0 - x\|\|\|$.

Let the result is true for $n = 1$, then $\|\|\|x_0 - x\|\|\| \leq k^{n-1} \|\|\|x_0 - x\|\|\|$.

We have $\|\|\|x_n - x\|\|\| = \|\|\|T(x_{n-1}) - T(x)\|\|\| \leq k \|\|\|x_{n-1} - x\|\|\| \leq k_n \|\|\|x_n - x\|\|\|$

i.e. $\|\|\|x_n - x\|\|\| \leq k_n \|\|\|x_0 - x\|\|\| \leq k_n r <$ which implies that $x_m \in B$.

(ii) Since $\|\|\|x_n - x\|\|\| \leq k_n \|\|\|x_0 - x\|\|\|$ and as $\lim_{n\to\infty} k_n = 0$, therefore,

we have $\lim_{n\to\infty} \|\|\|x_n - x\|\|\| = 0 \Rightarrow \lim_{n\to\infty} x_m = x$

5 CONCLUSION

We have comprehensively analyzed the honeybee's transmission dynamics under fractional order derivative via Laplace Adomian Decomposition Method (LADM) in the form of infinite series that converge quickly to its exact value. The results obtained with different values of ρ are compared, and it is determined that the results obtained with ρ equal to one are stronger. The qualitative aspect of the spread of varroa-mite by analyzing the disease-free equilibrium and global stability of endemic of our proposed model are investigated. Numerical trajectories are obtained for twelve compartments in the fractional order model. Based on the trajectories, we hypothesized that the memory index or fractional order could be used to control the honeybees infested by varroa-mite carrying virus transmission dynamics. Based on this, we think that the research presented in this study will help the honey's bee practitioners. Future versions of the model could be created by combining appropriate time-dependent control actions and cost effectiveness analysis will be carried out.

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