Applied Mathematics and Computational Intelligence
Volume 14, No. 1, 2025 [1-11]

Visualization of Monotonic Shaped Data by a Rational Cubic Ball

Wan Nurhadani Wan Jaafar[1*], Siti Jasmida Jamil[2], Shakila Saad[3], Nooraihan Abdullah[4]

[1,2,3,4]Institute of Engineering Mathematics, Universiti Malaysia Perlis, Kampus Pauh Putra, 02600 Arau, Perlis

[*] Corresponding author : hadani@unimap.edu.my

Received: 26 January 2024
Revised: 25 September 2024
Accepted: 22 December 2024

ABSTRACT

This paper discusses the monotonicity-preserving curve interpolation of 2D monotone data. A piecewise rational cubic Ball function in form of (cubic numerator/cubic denominator), with four shape parameters is presented. The rational cubic Ball spline has four shape parameters in its descriptions where two of them are constrained shape parameters and the remaining two of them provide the freedom to user to easily control the shape of the curve by simply changing their values. The sufficient data dependent conditions are derived for two shape parameters to ensure monotonicity everywhere. Numerical results show that the Ball interpolation scheme is quite efficient and well tested for monotone data.

Keywords: Rational cubic Ball function, shape parameters, monotone data, parametric continuity.

1 INTRODUCTION

Interpolation is one of the important methods in area of scientific visualization. It is the process of transforming numerical data into visual form and shape preserving is one of the main applications in the field of Computer Aided Geometric Design (CAGD). The monotonicity of data can be seen in many communications applications such as detecting the level of uric acid in patients and the study of tensile strength. The most important aspects to study the underlying the feature of this paper is monotonicity-preserving curve through monotone data. Several methods of spline have been discussed in CAGD for designing the curves and surfaces. The aim is to satisfy the lots of demands for getting more effective tools in designing and to grip strength model complexity. One such attempt is introducing the effective Ball spline with free shape parameters without inserting knots [1]. The main advantage of Ball functions is that lower degree curves and surfaces can be obtained by coalescing interior control points [2-4]. Normally, rational cubic and bi-cubic Bezier functions [5-8] or trigonometric functions were used to achieve the monotonicity-preserving curves with C^1 or C^2 degree of smoothness. In this paper, a new monotonicity-preserving scheme for curves is presented based on Ball basis functions. The proposed scheme produces the C^2 monotonicity-preserving curve with four shape parameters.

Recently, Jaafar et al. [9] developed a rational cubic Ball function with three shape parameters to preserve the monotonicity-preserving curves through monotone data with C^2 degree of smoothness. In [10-11], the authors developed a C^1 and GC^1 rational cubic Ball interpolant to preserve the monotonicity of monotone data. Wang and Tan [12] developed a C^2 piecewise rational quartic spline (quartic/ linear) interpolant with two shape parameters. The authors constructed the necessary and sufficient conditions for monotonicity through monotone data. Piah and Unsworth [13] have improved the monotonicity-preserving conditions by using same rational Bezier quartic function with quartic numerator and linear denominator. They improved the sufficient conditions for the derivative parameters and monotonicity region proposed by [12]. Karim [14] studied the problem of monotonicity preserving interpolation and the scheme was used Ball cubic interpolant. This motivates the advancement, improvement and modification of rational Ball functions in monotonicity-preserving curves. Fiorot and Tabka [15] have solved the problem of shape preserving through monotonic and convex data using C^2 piecewise cubic polynomial interpolation spline. The authors also solved the three systems of linear equations for solution of derivative parameters.

In this paper, the new curve scheme using piecewise rational cubic Ball function is presented to achieve the monotonicity-preserving curves with C^2 continuity. The rational cubic Ball functions involving four shape parameters u_i, v_i, α_i and β_i to preserve the shape of monotone data. Shape parameters are divided into two parts: i) u_i, v_i are free shape parameters and play a role to control the shape of the curve and ii) α_i and β_i are constrained parameters which are calculated automatically to preserve the shape of monotone data. The proposed scheme has the following useful features:

i) The present scheme produces C^2 continuity.

ii) No extra knots are inserted in the interval, while in (1), knots were inserted in the interval where it lost the shape features.

iii) This scheme works well for uniform and non-uniform spaced data.

2 RATIONAL CUBIC BALL CURVES

Rational cubic Ball curve is defined by [3-4] as follows:

$$z(t) = \frac{\sum_{i=0}^{3} S_i^3(t)\xi_i Z_i}{\sum_{i=0}^{3} S_i^3(t)\xi_i} \qquad (1)$$

where, $\xi_i,\ i = 0,\ 1,\ 2,\ 3$ are related to weights and $Z_i,\ i = 0,\ 1,\ 2,\ 3$ are related to control points on the plane. According to [4], the cubic Ball basis function of degree three is defined as:

$$
\begin{aligned}
S_0^3(t) &= (1-t)^2 \\
S_1^3(t) &= 2t(1-t)^2 \\
S_2^3(t) &= 2t^2(1-t) \\
S_3^3(t) &= t^2
\end{aligned} \tag{2}
$$

2.1 Rational cubic Ball Interpolation Representation

Let $(x_i, y_i), i=1,\ 2,\ \ldots, n$ be given set of monotone data points. It is defined over the interval $[a,b]$ such that $a = x_1 < x_2 < \ldots < x_n = b$. Consider a piecewise rational cubic Ball function (cubic numerator and cubic denominator) with four shape parameters and it can be defined over the interval $I_i = [x_i,\ x_{i+1}], i = 1,2,\ldots,n-1$ as:

$$
U(x) = U_i(\vartheta) = \frac{W_i(\vartheta)}{T_i(\vartheta)} \tag{3}
$$

where

$$
\begin{aligned}
W_i(\vartheta) &= s_0(1-\vartheta)^2 + s_1(1-\vartheta)^2\vartheta + s_2(1-\vartheta)\vartheta^2 + s_3\vartheta^2 \\
T_i(\vartheta) &= u_i(1-\vartheta)^2 + \alpha_i(1-\vartheta)^2\vartheta + \beta_i(1-\vartheta)\vartheta^2 + v_i\vartheta^2
\end{aligned} \tag{4}
$$

with

$h_i = x_{i+1} - x_i$, $\Delta_i = (y_{i+1} - y_i)/h_i$, $i = 1,2,\ldots,n-1$ and $\vartheta_i = \left(x - x_i\right)/h_i$, $\vartheta \in [0,\ 1]$.

A rational cubic Ball function **Error! Reference source not found.**, satisfies the following conditions to ensure the C^1 continuity.

$$
\begin{aligned}
U(x_i) &= y_i, \qquad & U(x_{i+1}) &= y_{i+1} \\
U'(x_i) &= d_i, \qquad & U'(x_{i+1}) &= d_{i+1}
\end{aligned} \tag{5}
$$

Let $U'(x_i)$ denote, the first derivative of U with respect to x_i and d_i and d_{i+1} represents the values of derivative (tangent) at the knots x_i and x_{i+1}. The results which provide the following coefficients of piecewise cubic Ball interpolant (3),

$$
\begin{aligned}
s_0 &= u_i y_i \\
s_1 &= \alpha_i y_i + u_i h_i d_i \\
s_2 &= \beta_i y_{i+1} - v_i h_i d_{i+1} \\
s_3 &= v_i y_{i+1}
\end{aligned} \tag{6}
$$

To make rational function **Error! Reference source not found.** is C^2 continuity, it needs to improve interpolatory condition as:

$$U''(x_i^+) = U''(x_i^-) \tag{7}$$

where $U''(x_i)$ represents the second derivative with respect to x and the results which provide the following tri-diagonal system of equations for piecewise cubic Ball interpolant $U(x) \in C^2[x_i, x_{i+1}]$

$$\mu_i d_{i-1} + \kappa_i d_i + \eta_i d_{i+1} = \xi_i \tag{8}$$

where

$$\begin{aligned}
\mu_i &= u_{i-1} u_i h_i, \\
\kappa_i &= \beta_{i-1} u_i h_i + \alpha_i v_{i-1} h_{i-1}, \\
\eta_i &= v_{i-1} v_i h_{i-1}, \\
\zeta_i &= (\alpha_{i-1} + u_{i-1}) u_i h_i \Delta_{i-1} + (\beta_i + v_i) v_{i-1} h_{i-1} \Delta_i.
\end{aligned} \tag{9}$$

The Equation (8) becomes the tri-diagonal system as:

$$\begin{aligned}
&\{u_{i-1} u_i h_i\} d_{i-1} + \{\beta_{i-1} u_i h_i + \alpha_i v_{i-1} h_{i-1}\} d_i + \{v_{i-1} v_i h_{i-1}\} d_{i+1} \\
&\qquad = (\alpha_{i-1} + u_{i-1}) u_i h_i \Delta_{i-1} + (\beta_i + v_i) v_{i-1} h_{i-1} \Delta_i
\end{aligned} \tag{10}$$

Remark 1: When $u_i = v_i = 1$ and $\alpha_i = \beta_i = 2$ in each subinterval $I_i = [x_i, x_{i+1}], i = 0,1,2,\ldots,n-1$, the rational cubic Ball function clearly becomes the non-rational cubic Ball function like standard cubic Hermite spline.

2.2 C^2 Monotonicity-Preserving Rational Cubic Ball Interpolation

The monotonicity-preserving curves of monotone data are achieved by assigning a suitable constraint for shape parameters. The aim to construct C^2 piecewise rational cubic Ball interpolant for monotone curves with following mathematical treatment.

Theorem 1: The C^2 piecewise rational cubic Ball function defined in **Error! Reference source not found.** preserves the monotonicity-preserving curves through monotone data if the shape parameters satisfy the following sufficient conditions, in each subinterval $I =_i [x_i, x_{i+1}], i = 1,2,\ldots,n$.

$$\begin{aligned}
&u_i > 0, v_i > 0 \\
&\alpha_i = k_i + \max\left\{0, \frac{u_i d_i}{\Delta_i}\right\}, k_i > 0 \\
&\beta_i = l_i + \max\left\{0, \frac{v_i \alpha_i d_{i+1}}{\alpha_i \Delta_i - u_i d_i}\right\}, l_i > 0
\end{aligned}$$

Proof: Let us assume the data points $(x_i, y_i), i = 1,2,\ldots,n$ given as an increasing set of monotonic data such that $x_1 < x_2 < \ldots < x_n$

$$y_i \le y_{i+1}, \quad i = 1, 2, \ldots, n-1 \tag{11}$$

Similarly, for monotonically increasing data, or equivalently.

$$\Delta_i \ge 0, \quad i = 1, 2, \ldots, n-1 \tag{12}$$

The derivative parameters $d_i \ge 0$ specifically for monotonically increasing data and $d_i \le 0$ specifically for monotonically decreasing data. For increasing data, we discuss the following two cases of monotonicity.

Case 1: When $\Delta_i = 0$, the value of derivatives should be $d_i = d_{i+1} = 0$. It is shown that $U_i(x) = y_i, \forall \in [x_i, x_{i+1}], i = 1, 2, \ldots, n-1$.So the interpolant is automatically monotonic.

Case 2: When $\Delta_i \ge 0$, the interpolant defined in (3) preserves the monotonicity if $U_i'(x) > 0$ for all $x \in [x_i, x_{i+1}]$. So $U_i'(x) > 0$ can be defined as:

$$U'(x) = \frac{\begin{array}{l} B_{0,i}(1-\vartheta)^4 + B_{1,i}(1-\vartheta)^3\vartheta^2 + B_{2,i}(1-\vartheta)^3\vartheta + B_{3,i}(1-\vartheta)^2\vartheta^3 + B_{4,i}(1-\vartheta)^2\vartheta^2 + B_{5,i}(1-\vartheta)^2\vartheta + \\ B_{6,i}(1-\vartheta)\vartheta^3 + B_{7,i}(1-\vartheta)\vartheta^2 + B_{8,i}\vartheta^4 \end{array}}{(T_i(\vartheta))^2} \tag{13}$$

with

$$\begin{cases} B_{0,i} = u_i^2 d_i \\ B_{1,i} = \alpha\beta_i\Delta_i - v_i\alpha_i d_{i+1} - \beta_i u_i d_i \\ B_{2,i} = 2u_i(\beta_i\Delta_i - v_i d_{i+1}) \\ B_{3,i} = \alpha_i\beta_i\Delta_i - v_i\alpha_i d_{i+1} - \beta_i u_i d_i \\ B_{4,i} = (u_i\beta_i + \alpha_i v_i)\Delta_i - u_i v_i(d_i + d_{i+1}) \\ B_{5,i} = 2u_i v_i\Delta_i \\ B_{6,i} = 2v_i(\alpha_i\Delta_i - u_i d_i) \\ B_{7,i} = 2u_i v_i\Delta_i \\ B_{8,i} = v_i^2 d_{i+1} \end{cases}$$

So $U_i'(x) \ge 0, \forall \in [x_i, x_{i+1}]$ if $B_{0,i}, B_{1,i}, B_{2,i}, B_{3,i}, B_{4,i}, B_{5,i}, B_{6,i}, B_{7,i}, B_{8,i} \ge 0$. The necessary conditions for monotonicity preserving curve are

$$\begin{cases} d_i \ge 0,\ d_{i+1} \ge 0, \\ u_i \ge 0,\ v_i \ge 0 \end{cases} \tag{14}$$

From the condition given in (14), it is clear that $B_{0,i}, B_{1,i}, B_{2,i}, B_{3,i}, B_{4,i}, B_{5,i}, B_{6,i}, B_{7,i}, B_{8,i} \geq 0$. For $B_{1,i}$ & $B_{3,i} \geq 0$ if

$$\beta_i > \frac{v_i \alpha_i d_{i+1}}{\alpha_i \Delta_i - u_i d_i} \qquad (15)$$

and $B_{6,i} \geq 0$ if

$$\alpha_i > \frac{u_i d_i}{\Delta_i} \qquad (16)$$

So conditions on α_i and β_i given in Equations (15)-(16) are reasonable and efficient for monotonicity-preserving curves. The above monotonicity-preserving conditions can be summarized as:

$$\begin{aligned} & u_i > 0, v_i > 0 \\ & \alpha_i > \frac{u_i d_i}{\Delta_i} \\ & \beta_i > \frac{v_i \alpha_i d_{i+1}}{\alpha_i \Delta_i - u_i d_i} \end{aligned} \qquad (17)$$

The above results can be summarized as

$$\begin{aligned} & u_i > 0, v_i > 0 \\ & \alpha_i = k_i + \max\left\{0, \frac{u_i d_i}{\Delta_i}\right\}, k_i > 0 \\ & \beta_i = l_i + \max\left\{0, \frac{v_i \alpha_i d_{i+1}}{\alpha_i \Delta_i - u_i d_i}\right\}, l_i > 0 \end{aligned} \qquad (18)$$

3 RESULTS AND DISCUSSION

In this section, a numerical result of shape preserving monotonicity based on the scheme of rational cubic Ball in Section (2) is demonstrated.

Example 1: Consider a monotone data which can be given in Table 1. Figure 1 shows the curve draw by non-rational cubic Ball spline with the values of shape parameters as $u_i = v_i = 1$ and $\alpha_i = \beta_i = 2$. It shows that the curve is not monotone, even though the data is monotone. Figure 2 and Figure 3 shows the monotone curve with different values of shape parameters. On the other hand, a PCHIP (Piecewise Cubic Hermite Interpolating Polynomial) curve in Figure 4 for monotone data set is drawn by using built in function in MATLAB. It is a shape preserving piecewise cubic Hermite interpolation. The characteristic of PCHIP which is the curve is differentiable at every point shows that this curve seems to preserve the monotonicity but not smooth and a bit tight as compared to curves in Figure 2 and 3. Numerical results of parameters for Figure 2 tabulated in Table 2. In general, rational cubic Ball function are smoother than PCHIP curves, however.

Table 1 : A monotone data set

i	1	2	3	4	5	6	7	8	9
x_i	0	3	6	7	11.5	18	26	27	30
y_i	11	11	11.5	16	19	51	56	71	91

Table 2 : Numerical results of Figure 2

i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
d_i	0	0	3.7782	3.9680	2.2800	0.7797	13.7675	14.1601	0.4167
Δ_i	0	0.1667	4.5000	0.6667	4.9231	0.6250	15	6.6667	-
α_i	0.1000	0.1000	0.1415	0.3976	0.1232	0.1624	0.1459	0.2062	-
β_i	0.1000	1.2215	0.1624	0.7799	0.1098	1.8884	0.1689	0.1064	-

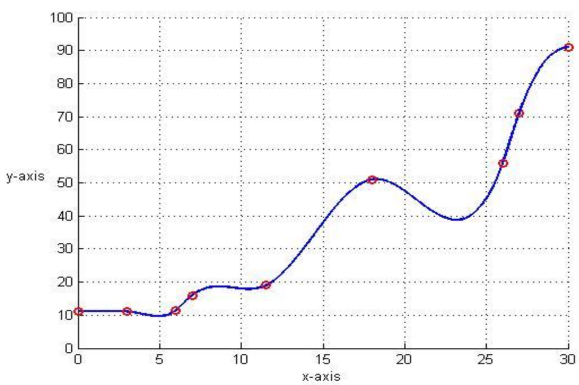

Figure 1 : The piecewise non-rational cubic Ball function with shape parameters $u_i = v_i = 1$ and $\alpha_i = \beta_i = 2$.

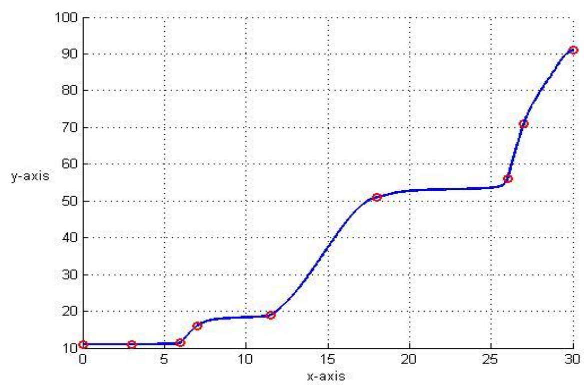

Figure 2 : The piecewise rational cubic Ball function with shape parameters $u_i = v_i = 0.05$.

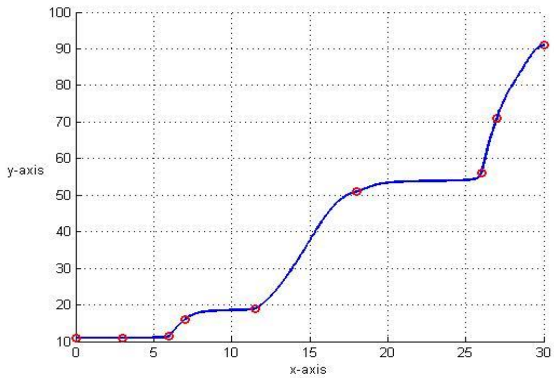

Figure 3 : The piecewise rational cubic Ball function with shape parameters $u_i = v_i = 0.1$.

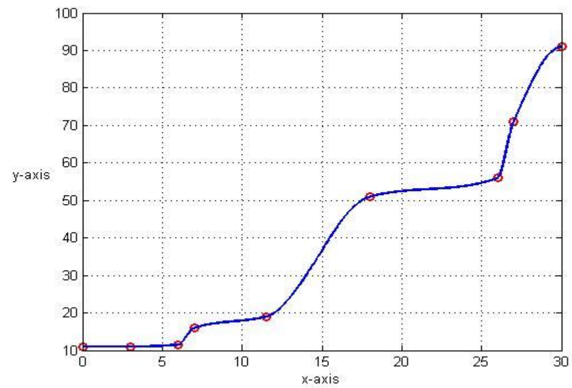

Figure 4 : PCHIP curve.

Example 2: A monotonic cricket data set in Table 3. A great amount of monotone data has been collected on a cricket match where the total score of the team at different number of balls was recorded. The *x*-values represent the number of ball and *y*-values represent the number of scores. Figure 5 is produced by the non-rational cubic Ball (3) by assuming the values of shape parameters as $u_i = v_i = 1$ and $\alpha_i = \beta_i = 2$. The curve shows the non-monotonicity-preserving curves of monotone. Figure 6 and 7 are demonstrated with different set values of shape parameters using the proposed scheme. Figure 8 is generated through PCHIP (Piecewise Cubic Hermite Interpolating Polynomial) that is a built-in program in MATLAB.

Table 3 : Monotone data set obtained from cricket match

i	**1**	**2**	**3**	**4**	**5**	**6**
x_i	15	16	21	22	27	31
y_i	5	6	10	40	75	85

Table 4 : Numerical results of Figure 6

i	**2**	**3**	**4**	**5**	**6**	**7**
d_i	1.033	0	25.9581	28.4093	0.2979	0.5000
Δ_i	1	0.8000	30	7	2.5000	-
α_i	0.3583	0.1000	0.3163	1.1146	0.1298	-
β_i	0.1000	8.2119	0.8489	0.2186	0.1649	-

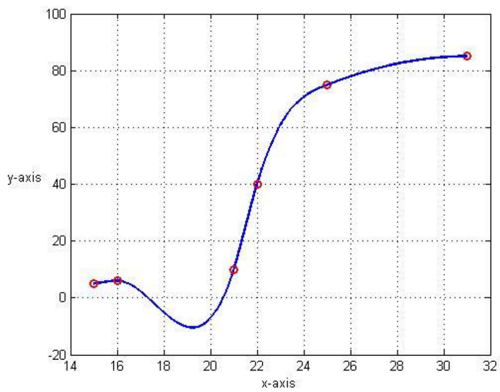

Figure 5 : The piecewise non-rational cubic Ball function with shape parameters $u_i = v_i = 1$ and $\alpha_i = \beta_i = 2$.

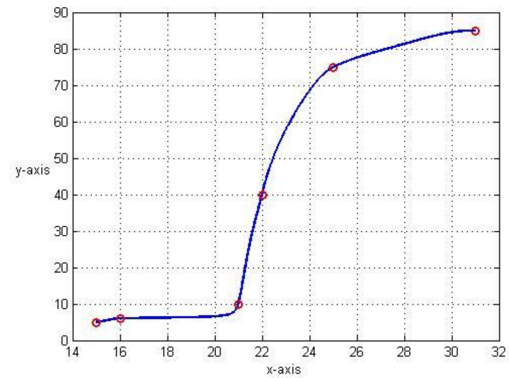

Figure 6 : The piecewise rational cubic Ball function with shape parameters $u_i = v_i = 0.25$.

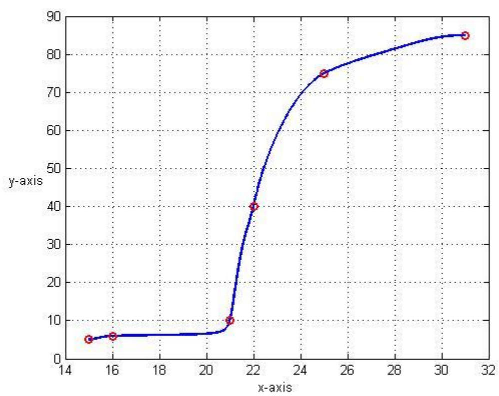

Figure 7 : The piecewise rational cubic Ball function with shape parameters $u_i = v_i = 0.5$.

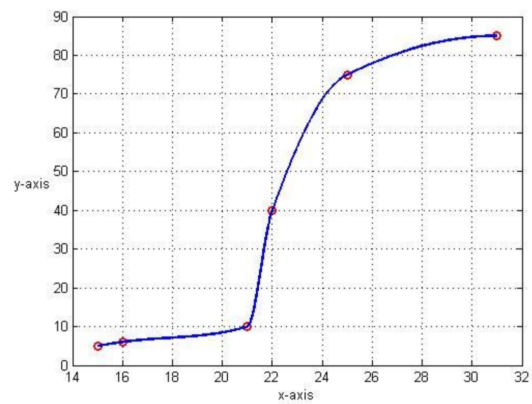

Figure 8 : PCHIP curve.

Example 3: The monotone data are given in Table 5. Figure 9 is the default curve for the monotonicity data. The default curve is generated by considering the value of shape parameters with $u_i = v_i = 1$ and $\alpha_i = \beta_i = 2$. Figure 10 and 11 have been tested with different set of shape parameters for monotonicity curve, respectively. The curves in figures indicate to satisfy the property of shape preserving. The user has been provided with a facility to visualize monotone data sets through PCHIP which is built in function in MATLAB. Figure 12 represented the demonstration of PCHIP. The curve in Figure 10 and 11 is smoother than Figure 12.

Table 5 : Monotone data set

i	**1**	**2**	**3**	**4**
x_i	710	850	960	1271
y_i	500	1360	2940	3090

Table 6 : Numerical results of Figure 14

i	1	2	3	4
d_i	1.5392	12.5719	11.4615	0
Δ_i	6.1429	14.3636	0.4823	-
α_i	0.1376	0.2313	3.6645	-
β_i	0.5224	0.3768	0.1000	-

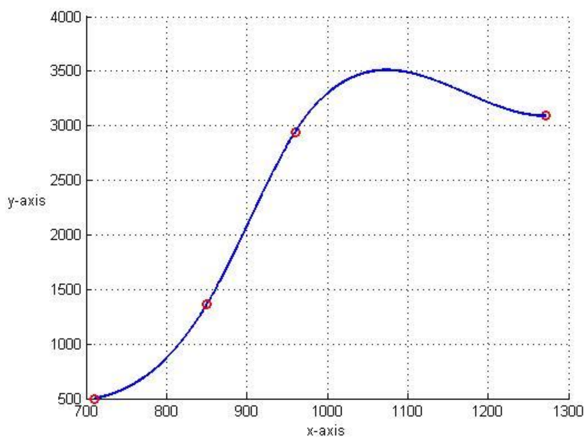

Figure 9 : The piecewise non-rational cubic Ball function with shape parameters $u_i = v_i = 1$ and $\alpha_i = \beta_i = 2$.

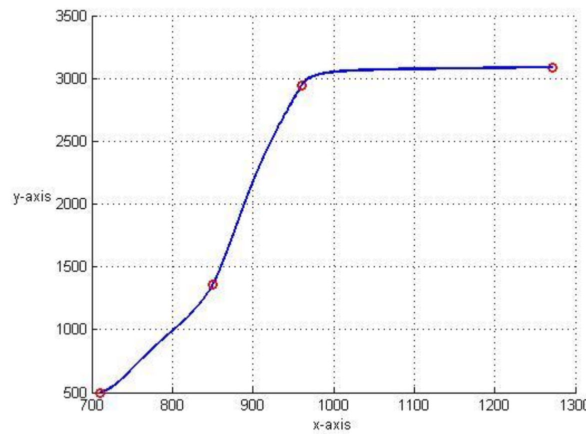

Figure 10 : The piecewise rational cubic Ball function with shape parameters $u_i = v_i = 0.05$.

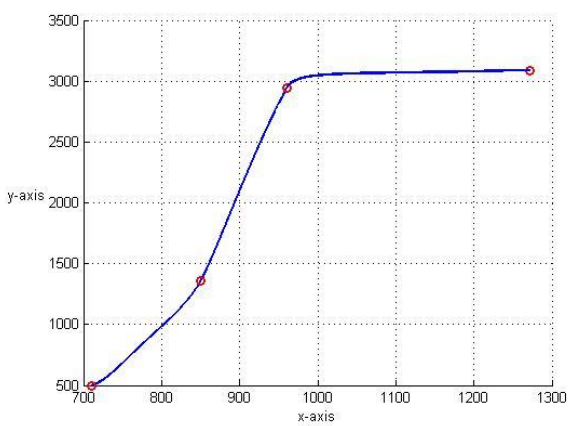

Figure 11 : The piecewise rational cubic Ball function with shape parameters $u_i = v_i = 0.15$.

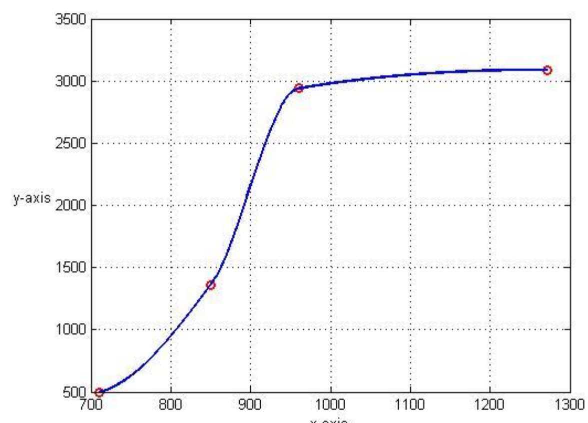

Figure 12 : PCHIP curve.

4 CONCLUSION

A rational cubic Ball interpolant with four shape parameters u_i, v_i, α_i and β_i has been applied in order to obtain C^2 monotonicity-preserving interpolating spline. Monotone interpolant can be controlled and preserved by considering the sufficient condition for α_i and β_i. In developed scheme, no extra knots are inserted in the interval when the curve loses the monotonicity. The developed monotonicity curve scheme has been tested through different numerical examples and it is proved that the rational cubic Ball scheme is not only C^2 continuity, local but also visually pleasing curve of monotonicity.

REFERENCES

[1] S. Butt, " Shape Preserving Curves and Surface for Computer Graphics," The University of Leeds, UK, 1991.

[2] H. Said "A Generalized Ball Curve and Its Recursive Algorithm," *ACM Transactions on Graphics*, vol. 8, no. 4, pp. 360–371, 1990.

[3] Q. Hu and G. Wang, "Rational cubic/quartic Said-Ball conics," *Appl. Math. J. Chinese Univ*., vol. 26, no. 2, pp. 198–212, Jun. 2011.

[4] H. L. Tien, L. Stein, "Rational Ball curves," *Int. J. Math. Educ. Sci. Technol.*, vol. 30, no. 2, pp. 243–257, Mar. 1999 .

[5] M. Abbas, A.A. Majid M.N.H. Awang, J.M. Ali, "Monotonicity preserving interpolation using rational spline", Lecture Notes in Engineering and Computer Science 2188 (1) (2011) 278–282.

[6] M. Abbas, A. A. Majid, and J. M. Ali, "Monotonicity-preserving rational cubic spline for monotone data," *Appl. Math. Comput*., vol. 219, no. 6, pp. 2885–2895, Nov. 2012.

[7] M. Abbas, "Shape preserving data visualization for curves and surfaces using rational cubic functions". PhD thesis, School of Mathematical Sciences, 2012, Universiti Sains Malaysia, Malaysia.

[8] M. Abbas, A.A. Majid, M.N.H. Awang, J.M. Ali, "Shape-preserving rational bi-cubic spline for monotone surface data". *WSEAS Trans Math* 7, pp. 660–73, 2012.

[9] W. N. W. Jaafar, M. Abbas, and A. R. M. Piah, " Shape preserving visualization of monotone data using a rational cubic ball function," *ScienceAsia* vol. 40S, no. 1, pp.40, 2014.

[10] S. A. A. Karim, "Monotonic Interpolating Curves by Using Rational Cubic Ball Interpolation," *Applied Mathematical Sciences*, vol. 8, no. 146, pp. 7259–7276, 2014.

[11] S. A. A. Karim, "GC1 Monotonicity Preserving using Cubic Ball Interpolation," *Aust. J. Basic Appl. Sci*., vol. 7, no. 2, pp. 780–790, 2012.

[12] Q. Wang and J. Tan, "Rational Quartic Spline Involving Shape Parameters The Rational Quartic Interpolant, " *Journal of Information and Computational Science*, vol. 1, no. 1, pp. 127-130, 2004.

[13] A. R. M. Piah and K. Unsworth, " Improved Sufficient Conditions for Monotonic Piecewise Rational Quartic Interpolation, " *Sains Malaysiana*, vol. 40, no. 10, pp. 1173-1178, 2011.

[14] S. Ariffin and A. Karim, "GC1 Monotonicity Preserving using Ball Cubic Interpolation, " *Aust. J. Basic Appl. Sci.,* vol. 7, no. 2, pp. 780-790, 2013.

[15] J. C. Fiorot and J. Tabka, "Shape-Preserving C^2 Cubic Polynomial Interpolating Splines, " *Mathematics of Computation*, vol. 57, no. 195, pp. 291-298, 1991.