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C^1 Rational Cubic Ball Triangular Patch for Positivity Preserving Scattered Data Interpolation

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ABSTRACT

Shape preservation and interpolation are a fundamental process in scientific visualization for graphical presentation of data. The known data represents only a sample and may not be sufficient to let one visualize the entire entity accurately. Therefore, a sufficiently smooth univariate or bivariate function that interpolates or approximates these data preserving the same characteristic features should be constructed. There are some special characteristic features in the data that are most often used in shape preserving interpolation such as positivity, monotonicity and convexity. At the beginning of the previous study, a number of the positivity scheme have been proposed as univariate positivity-preserving interpolants, for example, quadratic spline [1]*, cubic spline, quartic spline, quintic spline, rational function. The positivity-preserving interpolation could be achieved by inserting one or two extra knots where the shape of the curves is not preserved, or by modifying the given derivative values to ensure that the condition acquired are satisfied. Other than that, positivity-preserving could be achieved by introducing the weight functions of the rational spline as a free shape parameter that is used to generate the desired curves as required. This paper presents the construction of a C^1 rational interpolant which is positive everywhere when given data are positive. A local positivity preservation scheme is developed using rational cubic Ball basis functions, which involves weights as free parameters. Sufficient conditions are derived on Ball ordinates to ensure generated surface comprising of triangular patches is always positive. Each triangular patch of the interpolating surface is represented by a convex combination of three adjoining triangular patches. Gradients at data sites are calculated and modified (if necessary). The values of weights (free parameters) may also be changed to generate a positive surface. The weights (free parameters) provide extra freedom to users for them to modify the shape of surface as desired. Graphical examples are illustrated.*

Keywords: Positivity preserving, Rational cubic Ball, Triangular surface, Weights (free parameters).

1 INTRODUCTION

Shape preserving interpolation is an important area for graphical presentation of data. The problems of shape preserving interpolation have been overviewed by several authors. In many interpolation problems, it is essential that interpolant conserves some inherited shape features of the data, such as positivity, monotonicity, and convexity. In this work, our concern is to preserve the positivity of data.

Some work ([1], [2], [3], [4], [5], [6], [7]) on positivity preservation in surface have been published in recent year. Although most of them have solved the problems of non-rational interpolation surface, very few authors have considered the rational of scattered data interpolation, see for details ([4], [5]). These previous effort ([4], [5]) are motivated by an earlier work of the second author, Piah et. al [1] are derived the sufficient conditions on Bézier points. The derivative estimations at the data points are stated to be consistent with these conditions in each triangle to guarantee preservation of positivity. In [2], where sufficient conditions for the non-negativity of a cubic Bézier triangle were gained and used as lower bounds on the Bézier ordinates. They described a local scheme for range restricted C^1 interpolation of scattered data. The main difference between this work and that of [1] is the way in which the Bézier ordinates are constrained. Compared to this study, [1] offered more relaxed sufficient conditions that are easier to compute. An approach similar to [2] is adopted in [3], where they subdivided each triangle in the triangulated domain into three mini triangles and the interpolating surface on each mini triangle is a cubic Bézier triangular patch. In ([1], [2], [3]) positivity preserving conditions were derived either on derivatives at the data points and then modified if necessary to ensure that these conditions are gratified. Since all these schemes are local, they will work if the data are given without the derivatives. [4] is interested with positivity preserving triangle-based interpolation of scattered data using a C^1 continuity and local side-vertex method that is applicable to data both with and without derivatives. They used rational function with parameters to preserve the positive shape of irregular surface data. Positivity is accomplished by deriving simple sufficient data dependent constraints on these free parameters. [5] derived the positivity preserving conditions for rational cubic Bernstein-Bézier function with weights (free parameters). In contrast to ([1], [2], [3]), if the Bézier ordinates do not fulfill the designated lower bounds, then these are modified by the weights (free parameters). In 2014, Hussain et al. [6] continued their work on the rational quartic Bernstein-Bézier interpolation scheme for positive scattered data. This scheme has 15 Bézier ordinates and 15 weight functions compared to the previous scheme in [5]. Due to the high number of weight functions in this scheme, it gives advantage to the user by providing degrees of freedom for refinement of the surface shape, if required. There are three free parameters (weight functions at each vertex of triangle) in [6], while the remaining parameter are constrained by these free parameters. This contrasts with [5] where all the weight functions are defined as a free shape parameter to enhance the resulting shape of surface and preserve the shape of positive data. Both developed schemes in [5] and [6] are local with continuity and applicable for data accompanied with derivatives or not. [7] used quartic triangular basis initiated by [8] which only requires ten control points to construct one triangular patch. In this study, they show that the proposed scheme for positivity-preserving scattered data interpolation is significant used in visualizing large and irregular scattered data sets. [9] have proposed the construction of scattered data interpolation scheme based on rational quartic triangular patches with C^1 continuity. They also tested the proposed scheme by using 36, 65, and 100 data points that uniformly (and irregularly) with three free parameters for shape modification.

This paper is concerned with C^1 positivity preserving interpolation of scattered data using rational cubic Ball triangular patch. Sufficient conditions are imposed on Ball ordinates to ensure generated surface comprising of triangular patches is always positive. The rational function possesses weights as free parameters for each triangular patch. Positivity is accomplished by imposing a lower bound on Ball ordinates, then same as [5], these are modified by the values of weights (free parameters) to ensure that surfaces consisting of cubic Ball triangular patches and satisfy C^1 continuity conditions.

These free parameters provide the advantage to the designer to refine the shape of surfaces without modifying the data.

This paper is arranged as follows. In Section 2, we derived sufficient positivity condition on the Ball ordinates to ensure positivity for a rational cubic Ball triangular patch. The summary of the surface construction process is given in Section 3, while some numerical examples are presented to show the performance of the method in Section 4. Finally, the conclusion of this work is discussed in Section 5.

2 SUFFICIENT POSITIVITY CONDITIONS FOR A RATIONAL CUBIC BALL TRIANGULAR PATCH

Consider T be the triangle on the *xy*-plane with vertices V_1, V_2, V_3 and barycentric coordinates u, v and w with respect to the vertices V_1, V_2, V_3 are $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ respectively. Any point V on the triangle can be expressed as

$$\boldsymbol{V} = uV_1 + vV_2 + wV_3, \quad u + v + w = 1 \text{ and } u,v,w \geq 0.$$

A rational cubic Ball triangular patch $\boldsymbol{W}$ on T is defined as

$$\boldsymbol{W}(u,v,w) = \frac{W_a(u,v,w)}{W_b(u,v,w)}. \tag{1}$$

where

$$\begin{aligned} W_a(u,v,w) = {} & u^2\beta_{3,0,0}b_{3,0,0} + v^2\beta_{0,3,0}b_{0,3,0} + w^2\beta_{0,0,3}b_{0,0,3} + 2u^2v\beta_{2,1,0}b_{2,1,0} + 2u^2w\beta_{2,0,1}b_{2,0,1} \\ & + 2uv^2\beta_{1,2,0}b_{1,2,0} + 2v^2w\beta_{0,2,1}b_{0,2,1} + 2uw^2\beta_{1,0,2}b_{1,0,2} + 2vw^2\beta_{0,1,2}b_{0,1,2} \\ & + 6uvw\beta_{1,1,1}b_{1,1,1} \end{aligned} \tag{2}$$

$$\begin{aligned} W_b(u,v,w) = {} & u^2\beta_{3,0,0} + v^2\beta_{0,3,0} + w^2\beta_{0,0,3} + 2u^2v\beta_{2,1,0} + 2u^2w\beta_{2,0,1} + 2uv^2\beta_{1,2,0} + 2v^2w\beta_{0,2,1} \\ & + 2uw^2\beta_{1,0,2} + 2vw^2\beta_{0,1,2} + 6uvw\beta_{1,1,1} \end{aligned} \tag{3}$$

Let $\beta_{i,j,k}$ is the weight functions attached with $b_{i,j,k}$ denoting Ball ordinates of $\boldsymbol{W}$. Note that $\boldsymbol{W}$ interpolated the Ball ordinates $b_{3,0,0}$, $b_{0,3,0}$, $b_{0,0,3}$ at the vertices V_1, V_2, V_3 of T respectively. The rest Ball ordinates $b_{i,j,k}$ are referred as a boundary ball ordinates for $i \neq j \neq k$ and $b_{1,1,1}$ is referred as the inner Ball ordinate of the cubic Ball triangle when $i = j = k,\ i + j + k = 3$ (see Figure 1). From (1), it can be easily observed that $\boldsymbol{W}(u,v,w) > 0$ if $W_a(u,v,w) > 0$ and $W_b(u,v,w) > 0$.

We assume that $b_{3,0,0} = A$, $b_{0,3,0} = B$, $b_{0,0,3} = C$ and $A,B,C > 0$ are strictly positive and we derive sufficient conditions on the remaining ordinates of rational cubic Ball functions determined in (1) to preserve the shape of positive data. Our approach is to find the lower bounds of the remaining Ball ordinates, so that $\boldsymbol{W}(u,v,w) \geq 0$. We also assume all the boundaries and inner Ball ordinates, $b_{i,j,k}$ have the same value $-r < 0$ (where $r > 0$), i.e.

$$b_{2,1,0} = b_{1,2,0} = b_{2,0,1} = b_{1,0,2} = b_{0,2,1} = b_{0,1,2} = b_{1,1,1} = -r \tag{4}$$

From (3), $W_b(u,v,w) > 0$, where $u,v,w \geq 0$ if we set the following values to the weight functions,

$$\begin{aligned} &\beta_{3,0,0} = a,\ \beta_{0,3,0} = b,\ \beta_{0,0,3} = c \\ &\beta_{2,1,0} = \beta_{1,2,0} = \beta_{2,0,1} = \beta_{1,0,2} = \beta_{0,2,1} = \beta_{0,1,2} = \beta_{1,1,1} = \omega \end{aligned} \tag{5}$$

where $a,b,c,\omega > 0$. Since $W_b(u,v,w)$ is always positive, hence we only consider $W_a(u,v,w)$ to ensure that $\boldsymbol{W}(u,v,w) > 0$. From (2), we know that positivity of $W_a(u,v,w)$ depends upon the Ball ordinates $b_{i,j,k}$. By taking relation $(u+v+w)^2 = 1$, $W_1(u,v,w)$ is rewritten as

$$\begin{aligned} W_a(u,v,w) &= u^2 aA + v^2 bB + w^2 cC - r\omega(1 - u^2 - v^2 - w^2) \\ &= u^2(aA + r\omega) + v^2(bB + r\omega) + w^2(cC + r\omega) - r\omega \end{aligned} \tag{6}$$

From (6) clearly that when $r = 0$, $W_a(u,v,w) > 0$. As $r\omega$ increase, $W_a(u,v,w)$ decreases. We are concerned to find the value $r = r_0$ when the minimum value of $W_a(u,v,w)$ equal to zero. The derivative of W_a in (6) with respect to u, v and w are given by,

$$\frac{\partial W_a}{\partial u} = 2u(aA + r\omega),\quad \frac{\partial W_a}{\partial v} = 2v(bB + r\omega),\quad \frac{\partial W_a}{\partial w} = 2w(cC + r\omega). \tag{7}$$

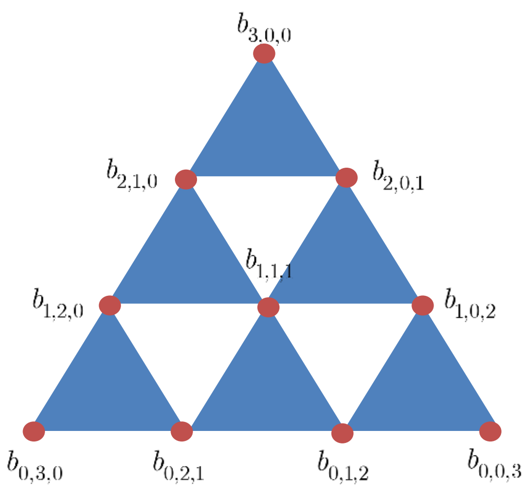


Figure 1 : Ball ordinates for $\boldsymbol{W}(u,v,w)$.

Assume that a, b, c, ω and t are fixed. At the minimum value of $W_a(u,v,w)$ we know that:

$$\frac{\partial W_a}{\partial u} - \frac{\partial W_a}{\partial v} = 0 \quad \text{and} \quad \frac{\partial W_a}{\partial u} - \frac{\partial W_a}{\partial w} = 0. \tag{8}$$

By substituting (7) into (8), we obtain $\frac{u}{v} = \frac{bB + r\omega}{aA_1 + r\omega}$ and $\frac{u}{w} = \frac{cC + r\omega}{aA + r\omega}$.

Hence, $u : v : w = \dfrac{1}{aA + r\omega} : \dfrac{1}{bB + r\omega} : \dfrac{1}{cC_3 + r\omega}$.

Since, we have obtained:

$$u = \frac{\dfrac{1}{aA + r\omega}}{\dfrac{1}{aA + r\omega} + \dfrac{1}{bB + r\omega} + \dfrac{1}{cC + r\omega}}$$
$$v = \frac{\dfrac{1}{bB + r\omega}}{\dfrac{1}{aA + r\omega} + \dfrac{1}{bB + r\omega} + \dfrac{1}{cC + r\omega}} \quad (9)$$
$$w = \frac{\dfrac{1}{cC + r\omega}}{\dfrac{1}{aA + r\omega} + \dfrac{1}{bB + r\omega} + \dfrac{1}{cC + r\omega}}$$

Substituting u, v and w from (9) into (6), we obtain the minimum values of $W_a\ u, v, w$,

$$W_a(u, v, w) = \frac{1}{\dfrac{1}{aA + r\omega} + \dfrac{1}{bB + r\omega} + \dfrac{1}{cC + r\omega}} - r\omega \quad (10)$$

We choose a value $r = r_0$ so that this minimum value of W_a being zero. From (10) and $r > 0,$ we know that, $W_a\ u, v, w\ = 0$ when

$$\frac{1}{\dfrac{aA}{r\omega} + 1} + \frac{1}{\dfrac{bB}{r\omega} + 1} + \frac{1}{\dfrac{cC}{r\omega} + 1} = 1 \quad (11)$$

Put $s = \dfrac{1}{r}$, $s > 0$. Therefore, if s increase, then r decreases. Thus, the left-hand side of (10) can be rewritten as

$$G\ s\ = \frac{1}{\dfrac{saA}{\omega} + 1} + \frac{1}{\dfrac{sbB}{\omega} + 1} + \frac{1}{\dfrac{scC_3}{\omega} + 1} \quad (12)$$

If A, B and C are strictly positive, then $s_0 = \dfrac{1}{r_0}$ is the solution of $G\ s\ = 1.$ The first and second derivative of G in (12) with respect to s are given by,

$$G'(s) = -\frac{1}{\omega}\left(\frac{aA}{\left(\frac{saA}{\omega}+1\right)^2} + \frac{bB}{\left(\frac{sbB}{\omega}+1\right)^2} + \frac{cC}{\left(\frac{scC}{\omega}+1\right)^2} \right)$$
$$G''(s) = \frac{2}{\omega^2}\left(\frac{a^2(A)^2}{\left(\frac{saA}{\omega}+1\right)^3} + \frac{b^2(B)^2}{\left(\frac{sbB}{\omega}+1\right)^3} + \frac{c^2(C)^2}{\left(\frac{scC}{\omega}+1\right)^3} \right) \tag{13}$$

Since $A,B,C > 0$, $a,b,c,\omega > 0$ and $r > 0$, so from (13) it is easy to show that for $s \geq 0$, $G'(s) < 0$ and $G''(s) > 0$. From above, clearly that $G(s)$ is a convex function, thus it must have a local minimum. So let $M = \max\left(\frac{aA}{\omega}, \frac{bB}{\omega}, \frac{cC}{\omega}\right)$ and $N = \min\left(\frac{aA}{\omega}, \frac{bB}{\omega}, \frac{cC}{\omega}\right)$. It is observed that $\frac{3}{Ms+1} \leq G(s) \leq \frac{3}{Ns+1}$. We have

$$G\left(\frac{2}{M}\right) \geq 1 \text{ and } G\left(\frac{2}{M}\right) \leq 1. \tag{14}$$

From (14), the roots of (12) would lie in the interval $\left(\frac{2}{M}, \frac{2}{N}\right)$. Figure 2 shows the form of $G(s)$, $s \geq 0$ and also shown relative locations of $\frac{2}{M}, \frac{2}{N}$ and s_0. The value s_0 can be find by using the method false-position [10]. This value can be solved with estimate for the root would be the value of s for which the line joining $\frac{2}{M}$ and $\frac{2}{N}$ has the value 1.

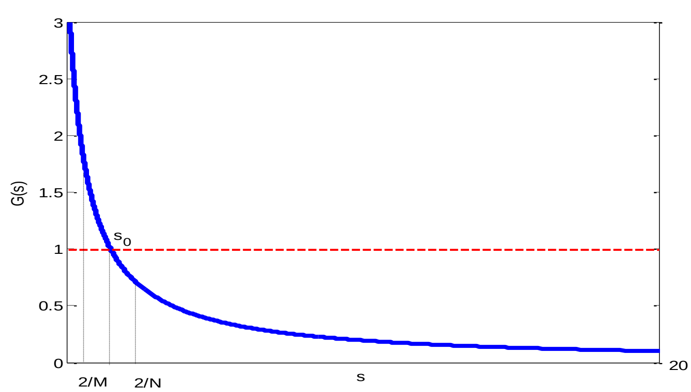


Figure 2 : Function $G(s)$ for $s \geq 0$

This iterative scheme is simple than calculating the roots of the cubic expression as done in paper [2]. All this discussion is summarized as below:

Proposition 1. Consider the rational cubic Ball triangular patch $\boldsymbol{W}(u,v,w)$ with $b_{3,0,0}=A,\ b_{0,3,0}=B,\ b_{0,0,3}=C$, and $A,B,C>0$. If $b_{2,1,0},b_{1,2,0},b_{2,0,1},b_{1,0,2},b_{0,2,1},b_{0,1,2},b_{1,1,1}\geq -r_0$, where $r_0=\dfrac{1}{s_0}$ is the unique solution of (12) and $G(s)=1$ then $W_a(u,v,w)\geq 0$, $\forall\ u,v,w\geq 0$, $u+v+w=1$.

Note that, if any of the values of $A,\ B$ or C are zero, r_0 is assigned the value zero for that triangle.

Remark 1. *It is important to observe that the weight functions (free parameters) are used to refine the shape of surface.*

3 RESULTS AND DISCUSSION CONSTRUCTION POSITIVITY-PRESERVING INTERPOLATING SURFACE

In this section, we discuss the construction of the positivity preserving interpolation surface by using rational cubic Ball triangular patch. Given a positive scattered data $(x_i,y_i),i=1,\ldots,N$, we describe the construction of a C^1 positivity preserving function $F(x,y)$ with $F(x_i,y_i)=z_i,i=1,\ldots,N$. We solve the problem following these three steps:

3.1 Model and Data Triangulation

Let *D* be the convex hull of $\{V_i=(x_i,y_i):i=1,\ldots,N\}$. We use Delaunay triangulation [11] to triangulate *D* where the data points are arranged at the vertices $V_i,i=1,2,\ldots,N$ of the triangles.

3.2 Derivative Estimation

Estimation of first order partial derivative, *i.e.* F_x and F_y at each vertex $V_i(x_i,y_i)$ in the triangulated domain *D* for surface *F*, is obtained by using the method proposed in [12]. These partial derivatives are used to find derivative in the direction of an edge of the triangle. Let $\Delta V_1V_2V_3$ be the given triangle whose edges are $e_i,\ i=1,2,3$ opposite to the vertices $V_i,\ i=1,2,3$ respectively. For each triangular patch *W* as in (1), the derivative in the direction of the triangle edge e_3 and e_2 at V_1 [13] is given by

$$\begin{aligned}\frac{\partial \boldsymbol{W}}{\partial e_3}(V_1) &= (x_2-x_1)F_x(V_1)+(y_2-y_1)F_y(V_1)\\ \frac{\partial \boldsymbol{W}}{\partial e_3}(V_1) &= (x_1-x_3)F_x(V_1)+(y_1-y_3)F_y(V_1).\end{aligned}\tag{15}$$

By considering (15), the directional derivative along edges e_1, e_3 at V_2 and along edges e_1, e_2 at V_3 are defined in similar way.

3.3 Calculation of Boundary Ball Ordinates

From the given data together with the estimated derivatives values at x_i, y_i are used to assign initial values to all the Ball ordinates $b_{i,j,k}$ except for b_{111}. For example, we would have:

$$b_{3,0,0} = F \ V_1 \ , b_{2,1,0} = F \ V_1 \ + \frac{\beta_{3,0,0}}{2\beta_{2,1,0}} \left(\frac{\partial \boldsymbol{W}}{\partial e_3} \ V_1 \ \right), b_{2,0,1} = F \ V_1 \ - \frac{\beta_{3,0,0}}{2\beta_{2,0,1}} \left(\frac{\partial \boldsymbol{W}}{\partial e_2} \ V_1 \ \right) \tag{16}$$

Similarly, by considering the directional derivatives along the edges e_1, e_3 at V_2 and along the edges e_1, e_2 at V_3, we obtain

$$b_{0,3,0} = F \ V_2 \ , \ b_{0,2,1} = F \ V_2 \ + \frac{\beta_{0,3,0}}{2\beta_{0,2,1}} \left(\frac{\partial \boldsymbol{W}}{\partial e_1} \ V_2 \ \right), \ b_{1,2,0} = F \ V_2 \ - \frac{\beta_{0,3,0}}{2\beta_{1,2,0}} \left(\frac{\partial \boldsymbol{W}}{\partial e_3} \ V_2 \ \right) \tag{17}$$

$$b_{0,0,3} = F \ V_3 \ , b_{1,0,2} = F \ V_3 \ + \frac{\beta_{0,0,3}}{2\beta_{1,0,2}} \left(\frac{\partial \boldsymbol{W}}{\partial e_2} \ V_3 \ \right), \ b_{0,1,2} = F \ V_3 \ - \frac{\beta_{0,0,3}}{2\beta_{0,1,2}} \left(\frac{\partial \boldsymbol{W}}{\partial e_{21}} \ V_3 \ \right). \tag{18}$$

However, the initial estimate for each edge ordinate may not satisfy the positivity condition for $\boldsymbol{W}$ developed in Proposition 1. If they are not fulfilled, then the ordinates are modified by increasing the values of the free parameters ω for any values of the free parameters a, b and c. In such a way, the derived positivity conditions are satisfied: i.e., so that, for example,

$$b_{2,1,0} = F \ V_1 \ + \frac{\beta_{3,0,0}}{2\beta_{2,1,0}} \frac{\partial \boldsymbol{W}}{\partial e_3} \ V_1 \ \geq -r_0 \text{ and } b_{2,0,1} = F \ V_1 \ - \frac{\beta_{3,0,0}}{2\beta_{2,0,1}} \frac{\partial \boldsymbol{W}}{\partial e_2} \ V_1 \ \geq -r_0.$$

Having adjusted these derivatives, if necessary, the Ball ordinates are recalculated using the formulae above. The free parameters ω are local to each triangle, so that any changing value of parameters would not affect the C^1 continuity at vertices.

3.4 Calculation of Inner Ball Ordinates

Next, the inner Ball ordinated for each triangle remains to be calculated to ensure C^1 continuity across boundaries and to preserve positivity. We use similar methods in [10] where the calculation of inner ordinates is determined by local scheme proceeds as follows:

(i) We determine b_{111}^{ℓ}, $\ell = 1,2,3$, so that the C^1 condition on the along a triangle edge boundary is satisfied and a local patch $\boldsymbol{W}_\ell, \ell = 1,2,3$ is defined by replacing b_{111} in (1) with b_{111}^{ℓ}.

(ii) Convex combination is then used to blend these three local patches $\boldsymbol{W}_\ell, \ell = 1,2,3$, so that conditions on all sides of the triangle are satisfied.

From the given data together with the estimated derivatives values at x_i, y_i are used to assign initial values to all the Ball ordinates.

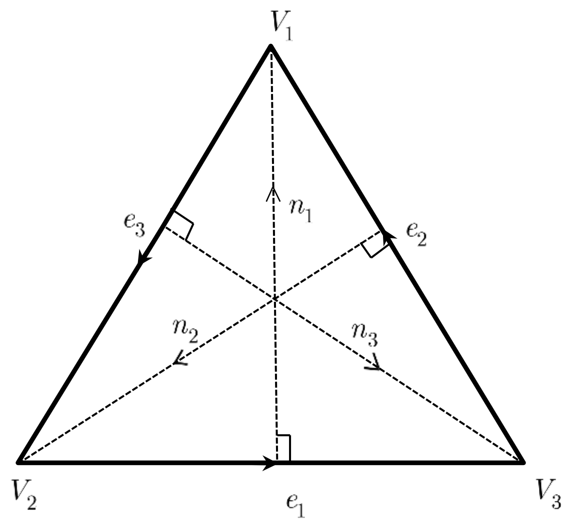


Figure 3 : Inward normal direction to the edges of triangle T

Let $T_1 \;\; \Delta V_1V_2V_3$ and $T_2 \;\; \Delta U_1U_2U_3$ be two adjacent cubic Ball triangular patches with a common boundary curve. Assume that, $\boldsymbol{n}_i,\ i = 1,2,3$ be the barycentric inward vectors to the edges V_2V_3, V_3V_1, V_1V_2. Let e_i denoted as the side opposite the vertex V_i from V_{i+1} to V_{i-1} (where indices are always taken modulo 3) as shown in Figure 3 [11] which are given by

$$\boldsymbol{n}_1 = \; 1, h_1 - 1, -h_1 \;, \; \boldsymbol{n}_2 = \; -h_2, 1, h_2 - 1 \;, \; \boldsymbol{n}_3 = \; h_3 - 1, -h_3, 1 \; . \tag{19}$$

where

$$h_1 = \frac{-e_3 \cdot e_1}{\left|e_1\right|^2}, \;\; h_2 = \frac{-e_1 \cdot e_2}{\left|e_2\right|^2}, \;\; h_3 = \frac{-e_2 \cdot e_3}{\left|e_3\right|^2}.$$

If $\boldsymbol{d} = (\lambda_1, \lambda_2, \lambda_3)$ is a barycentric vector, the directional derivative of rational cubic Ball surface $\boldsymbol{W}$ with respect to d is given by

$$\frac{\partial \boldsymbol{W}}{\partial \boldsymbol{d}} \;\; u, v, w \;\; = \lambda_1 \frac{\partial \boldsymbol{W}}{\partial u} + \lambda_2 \frac{\partial \boldsymbol{W}}{\partial v} + \lambda_3 \frac{\partial \boldsymbol{W}}{\partial w} \tag{20}$$

Hence, by using Equations (1), (19) and (20) we can define the normal derivative of local scheme $\boldsymbol{W}$ at edge $e_1 \;\; (u = 0, v + w = 1)$ along $\boldsymbol{n}_1$ is

$$\frac{\partial \boldsymbol{W}}{\partial \boldsymbol{n}_1}(0,v,w) = (1)\frac{\partial \boldsymbol{W}}{\partial u} + (h_1 - 1)\frac{\partial \boldsymbol{W}}{\partial v} + (-h_1)\frac{\partial \boldsymbol{W}}{\partial w}$$
$$= \frac{\left(\begin{gathered}2A_1v^4 + 4B_1v^4w + 2C_1v^3w + 4D_1v^3w^2 + 2E_1v^2w + 2F_1v^2w^2 + 2G_1vw^2 + \\ 4H_1v^2w^3 + 2I_1vw^3 + 4J_1vw^4 + 2K_1w^4\end{gathered}\right)}{\left(\beta_{0,3,0}v^2 + 2\beta_{0,2,1}v^2w + 2\beta_{0,1,2}vw^2 + \beta_{0,0,3}w^2\right)^2} \tag{21}$$

where

$$A_1 = \beta_{0,3,0}\left((b_{1,2,0} - b_{0,3,0})\beta_{1,2,0} - h_1(b_{0,2,1} - b_{0,3,0})\beta_{0,2,1}\right),$$
$$B_1 = \beta_{0,2,1}\beta_{1,2,0}(b_{1,2,0} - b_{0,2,1}),$$
$$C_1 = \beta_{0,3,0}\left(3(b_{1,1,1} - b_{0,3,0})\beta_{1,1,1} - h_1(2b_{0,1,2} - 2b_{0,3,0})\beta_{0,1,2}\right),$$
$$D_1 = 3(b_{1,1,1} - b_{0,2,1})\beta_{1,1,1}\beta_{0,2,1} + (b_{1,2,0} - b_{0,1,2})\beta_{1,2,0}\beta_{0,1,2} - h_1(b_{0,1,2} - b_{0,2,1})\beta_{0,1,2}\beta_{0,2,1},$$
$$E_1 = -h_1(b_{0,0,3} - b_{0,3,0})\beta_{0,0,3}\beta_{0,3,0},$$
$$\begin{aligned}F_1 = {} & (b_{0,1,2} - b_{0,3,0})\beta_{0,3,0}\beta_{0,1,2} + (b_{1,0,2} - b_{0,3,0})\beta_{1,0,2}\beta_{0,3,0} + (b_{1,2,0} - b_{0,0,3})\beta_{1,2,0}\beta_{0,0,3} \\ & - h_1\left((b_{0,0,3} - b_{0,2,1})\beta_{0,0,3}\beta_{0,2,1} + (b_{0,1,2} - b_{0,3,0})\beta_{0,1,2}\beta_{0,3,0}\right),\end{aligned}$$
$$G_1 = \beta_{0,0,3}\beta_{0,3,0}\left((b_{0,0,3} - b_{0,3,0}) - h_1(b_{0,0,3} - b_{0,3,0})\right)$$
$$\begin{aligned}H_1 = {} & 3(b_{1,1,1} - b_{0,1,2})\beta_{1,1,1}\beta_{0,1,2} + (b_{0,1,2} - b_{0,2,1})\beta_{0,1,2}\beta_{0,2,1} + (b_{1,0,2} - b_{0,2,1})\beta_{1,0,2}\beta_{0,2,1} \\ & - h_1(b_{0,1,2} - b_{0,2,1})\beta_{0,1,2}\beta_{0,2,1},\end{aligned}$$
$$I_1 = \beta_{0,0,3}\left(2(b_{0,0,3} - b_{0,2,1})\beta_{0,2,1} + 3(b_{1,1,1} - b_{0,0,3})\beta_{1,1,1} - h_1(2b_{0,0,3} - 2b_{0,2,1})\beta_{0,2,1}\right),$$
$$J_1 = \beta_{1,0,2}\beta_{0,1,2}(b_{1,0,2} - b_{0,1,2}),$$
$$K_1 = \beta_{0,0,3}\left((b_{0,0,3} - b_{0,1,2})\beta_{0,1,2} + (b_{1,0,2} - b_{0,0,3})\beta_{1,0,2} - h_1(b_{0,0,3} - b_{0,1,2})\beta_{0,1,2}\right).$$

By substituting barycentric coordinates $(0,1,0)$ on V_2, $(0,0,1)$ on V_3 and $(0,0.5,0.5)$ between V_2V_3 into (21), we get

$$\frac{\partial \boldsymbol{W}}{\partial \boldsymbol{n}_1}(0,1,0) = \frac{2A_1}{(\beta_{0,3,0})^2}, \tag{22}$$

$$\frac{\partial \boldsymbol{W}}{\partial \boldsymbol{n}_1}(0,0,1) = \frac{2K_1}{(\beta_{0,0,3})^2}, \tag{23}$$

$$\frac{\partial \boldsymbol{W}}{\partial \boldsymbol{n}_1}\left(0,0.5,0.5\right)=\frac{2\left(A_1+B_1+C_1+D_1+2E_1+F_1+2G_1+H_1+I_1+J_1+K_1\right)}{\left(\beta_{0,3,0}+\beta_{0,2,1}+\beta_{0,1,2}+\beta_{0,0,3}\right)^2}. \tag{24}$$

From Equations (22) – (24), this inward normal derivative varies linearly along the edge in order to ensure C^1 continuity on e_1. This is true when the following relation is satisfied:

$$\frac{2\left(A_1+B_1+C_1+D_1+2E_1+F_1+2G_1+H_1+I_1+J_1+K_1\right)}{\left(\beta_{0,3,0}+\beta_{0,2,1}+\beta_{0,1,2}+\beta_{0,0,3}\right)^2}=\frac{A_1}{\left(\beta_{0,3,0}\right)^2}+\frac{K_1}{\left(\beta_{0,0,3}\right)^2}. \tag{25}$$

Hence, the value of $b^1_{1,1,1}$ from (25) can be determined as

$$b^1_{1,1,1}=\frac{\left(\frac{1}{2}\left(\mu_1\right)^2\left(\frac{A_1}{\beta_{0,3,0}}+\frac{K_1}{\beta_{0,0,3}}\right)-\mu_1\left(b_{1,2,0}\beta_{1,2,0}+b_{1,0,2}\beta_{1,0,2}\right)+\mu_2 b_{0,3,0}\beta_{0,3,0}+\mu_3 b_{0,2,1}\beta_{0,2,1}+\mu_4 b_{0,1,2}\beta_{0,1,2}+\mu_5 b_{0,0,3}\beta_{0,0,3}\right)}{\mu_1\left(3\beta_{1,1,1}\right)} \tag{26}$$

where

$\mu_1=\beta_{0,3,0}+\beta_{0,2,1}+\beta_{0,1,2}+\beta_{0,0,3}$,

$\mu_2=\beta_{0,1,2}+\beta_{1,0,2}+\beta_{1,2,0}+2\beta_{0,3,0}+3\beta_{1,1,1}-h_1\left(\beta_{0,2,1}+3\beta_{0,1,2}+4\beta_{0,0,3}\right)$,

$\mu_3=\beta_{0,1,2}+\beta_{1,0,2}+\beta_{1,2,0}+2\beta_{0,0,3}+3\beta_{1,1,1}-h_1\left(3\beta_{0,0,3}+2\beta_{0,1,2}-\beta_{0,3,0}\right)$,

$\mu_4=\beta_{0,0,3}+\beta_{1,0,2}+\beta_{1,2,0}+3\beta_{1,1,1}-\beta_{0,2,1}-\beta_{0,3,0}-h_1\left(\beta_{0,0,3}-2\beta_{0,2,1}-3\beta_{0,3,0}\right)$,

$\mu_5=\beta_{1,0,2}+\beta_{1,2,0}+3\beta_{1,1,1}-\beta_{0,1,2}-2\beta_{0,2,1}-2\beta_{0,3,0}+h_1\left(\beta_{0,1,2}+3\beta_{0,2,1}+4\beta_{0,3,0}\right)$

Similarly, b^2_{111} and b^3_{111} are defined by the same way for local schemes $\boldsymbol{W}_2$ and $\boldsymbol{W}_3$. If an inner Ball point b^{ℓ}_{111}, $\ell=1,2,3$ fails to satisfy the positivity condition, it is modified according to Proposition 1 by changing the value of free parameter β_{111}. The value β_{111} is obtained by considering the minimum value for which, $\ell=1,2,3$ that would guarantee satisfaction of the positivity conditions for all triangles.

3.5 C^1 Continuity Across Patch Boundaries

Consider two adjacent rational cubic Ball triangular patches with the same boundary curve along the common edge of the domain triangles. Let $T_1\left(\Delta V_1V_2V_3\right)$ and $T_2\left(\Delta U_1U_2U_3\right)$ be two adjacent domain triangles with $V_2=U_2$ and $V_3=U_3$. Figure 4 shows two rational cubic Ball triangular patches, $\boldsymbol{W}_1\left(u,v,w\right)$ and $\boldsymbol{W}_2\left(u,v,w\right)$, joined at the common boundary curve $\boldsymbol{W}_1\left(0,v,w\right)=\boldsymbol{W}_2\left(0,v,w\right)$ where $u=0$.

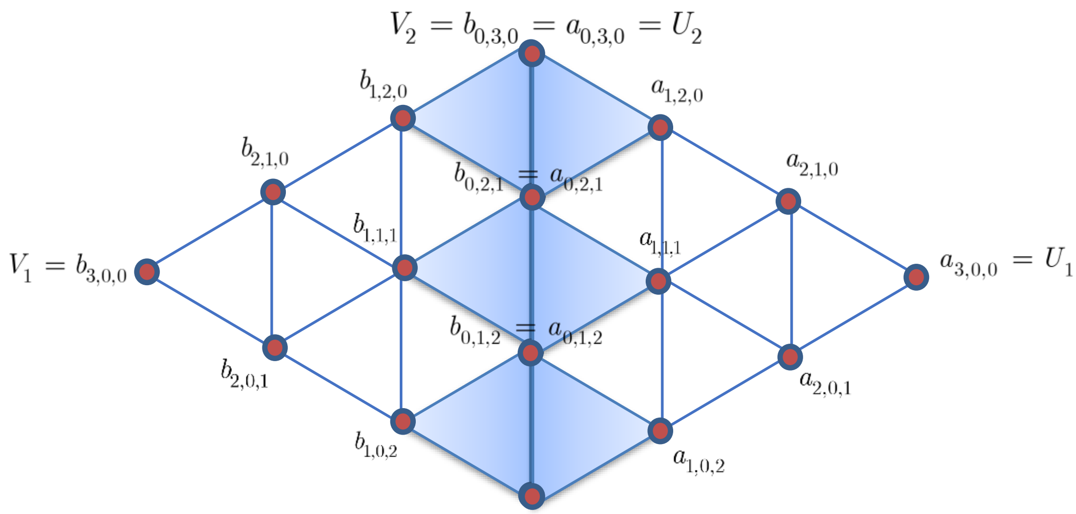


Figure 4 : Two adjoining domain triangles

Suppose that two rational cubic Ball triangular patches $\boldsymbol{W}_1\ u,v,w$ and $\boldsymbol{W}_2\ u,v,w$ on T_1 and T_2 have weights $\beta_{i,j,k}$ and $\alpha_{i,j,k}$ attached with Ball ordinates $b_{i,j,k}$ and $a_{i,j,k}$, respectively, as shown in Figure 4. The condition for the C^1 continuity along the common boundary of two adjacent Ball patches should be expressed in terms of constraints on the control points. We use similar method as in [12] to obtain the conditions for smooth joining of these surface patches. The idea is to calculate three vectors that are tangent to the surface at the common boundary curve. Therefore, for two Ball patches to achieve C^1 continuously along the common edge $V_2V_3\ (u=0)$ we require

$$\boldsymbol{W}_1\ 0,v,w\ = \boldsymbol{W}_2\ 0,v,w$$

and

$$D_{\boldsymbol{d}_l}\boldsymbol{W}_1\ 0,v,w\ = D_{\boldsymbol{d}_m}\boldsymbol{W}_2\ 0,v,w$$

where $\boldsymbol{d}_l$ and $\boldsymbol{d}_m$ are the barycentric vectors of a direction $\boldsymbol{d}$ with respect to $\Delta V_1V_2V_3$ and $\Delta U_1U_2U_3$, respectively. These yields

$$a_{1,0,2} = \frac{1}{\alpha_{1,0,2}} \left(\bar{u}\ \beta_{1,0,2} b_{1,0,2} + \bar{v}\ \beta_{0,0,3} b_{0,0,3} + \bar{w}\ \beta_{0,1,2} b_{0,1,2} \right),$$
$$a_{1,1,1} = \frac{1}{\alpha_{1,1,1}} \left(\bar{u}\ \beta_{1,1,1} b_{1,1,1} + \bar{v}\ \beta_{0,1,2} b_{0,1,2} + \bar{w}\ \beta_{0,2,1} b_{0,2,1} \right), \tag{27}$$
$$a_{1,2,0} = \frac{1}{\alpha_{1,2,0}} \left(\bar{u}\ \beta_{1,2,0} b_{1,2,0} + \bar{v} \beta_{0,2,1} b_{0,2,1} + \bar{w}\ \beta_{0,3,0} b_{0,3,0} \right),$$

and

$$\alpha_{1,0,2} = \bar{u}\ \beta_{1,0,2} + \bar{v}\ \beta_{0,0,3} + \bar{w}\ \beta_{0,1,2},$$
$$\alpha_{1,1,1} = \bar{u}\ \beta_{1,1,1} + \bar{v}\ \beta_{0,1,2} + \bar{w}\ \beta_{0,2,1}, \tag{28}$$
$$\alpha_{1,2,0} = \bar{u}\ \beta_{1,2,0} + \bar{v} \beta_{0,2,1} + \bar{w}\ \beta_{0,3,0},$$

where $\bar{u}$, $\bar{v}$ and $\bar{w}$ are the barycentric coordinates of U_1 with respect to V_1, V_2 and V_3, i.e., $U_1 = \bar{u} V_1 + \bar{v} V_2 + \bar{w} V_3$.

3.6 Interpolating Triangular Surface

The interpolating surface $\boldsymbol{W}$ on triangle T is defined as a convex combination of all the three patches $\boldsymbol{W}_\ell,\ \boldsymbol{\ell} = 1,2,3,$ such that sufficient conditions on all sides of the triangles are satisfied as below:

$$\boldsymbol{W}_T = c_1 \boldsymbol{W}_1 + c_2 \boldsymbol{W}_2 + c_3 \boldsymbol{W}_3$$

with

$$c_1 = \frac{v^2 w^2}{v^2 w^2 + u^2 v^2 + u^2 w},\ \ c_2 = \frac{u^2 w^2}{v^2 w^2 + u^2 v^2 + u^2 w^2},\ \ c_3 = \frac{u^2 v^2}{v^2 w^2 + u^2 v^2 + u^2 w^2}$$

where u, v and w are the barycentric coordinates.

4 NUMERICAL EXAMPLE AND DISCUSSION

In this section, we would construct a positivity preserving interpolating scheme developed in Section 2 through several numerical examples are presented. Let a, b and c denote the free parameters of Ball ordinates at vertices V_1, V_2, V_3 and ω refer to as the free parameters at the other boundary Ball ordinates. Here, we test with the various values of free parameters that is $0 < a,b,c,\omega < 1$ in order to generate an interpolating surface. We illustrate the effect of free parameters, a,b,c and ω on the shape of a surface, graphically. When all free parameters are set as equal to 1, from Equation (1), we obtain a non-rational cubic Ball surface.

Example 4.1: 36 points samples is generated by the following function were given in Table A.1 (Appendix A).

$$F_1(x,y) = \frac{1}{3}\exp\left(-20.25\left((x-0.5)^2+(y-0.5)^2\right)\right), \qquad (x,y)\in[0,1]\times[0,1]$$

This function $F_1(x,y)$ is taken from [16]. We generate the positive data points in the domain $[0,1]\times[0,1]$. The triangulation of the domain is illustrated in Figure 5 (a). The linear interpolant to the data is shown in Figure 5 (b). As a comparison, Figure 6 (a) shows the C^1 interpolating surface generated without applying the positivity conditions, while Figure 6 (b) presents xz-view of 6 (a). We observe that interpolating surface does not preserve the positive surface. Figure 7 (a) – 7 (d) are produced by using a C^1 positivity preserving rational cubic Ball triangular patch. This scheme generates an interpolating C^1 triangular surface by assigning four different values to free parameters, a, b, c and ω. By satisfying Proposition 1 in Equation (29), the positivity is preserved.

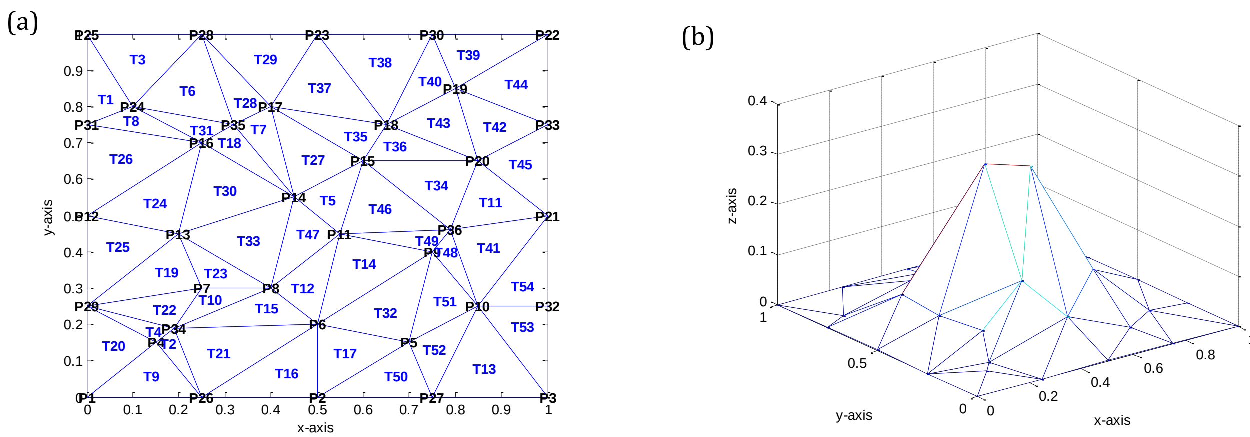


Figure 5 : (a) Delaunay triangulation of domain for $F_1(x,y)$ (b) Linear interpolant for data

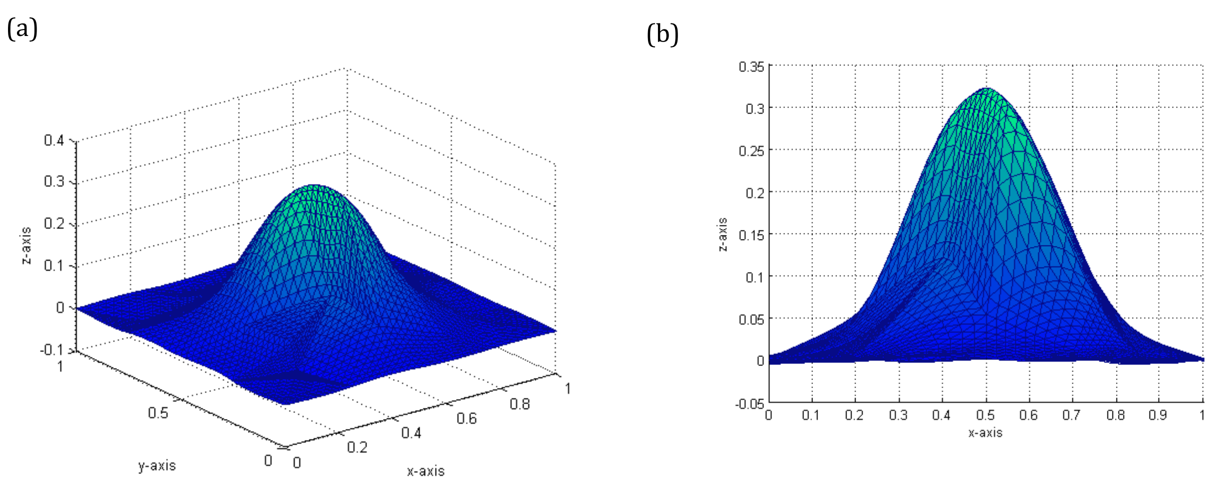
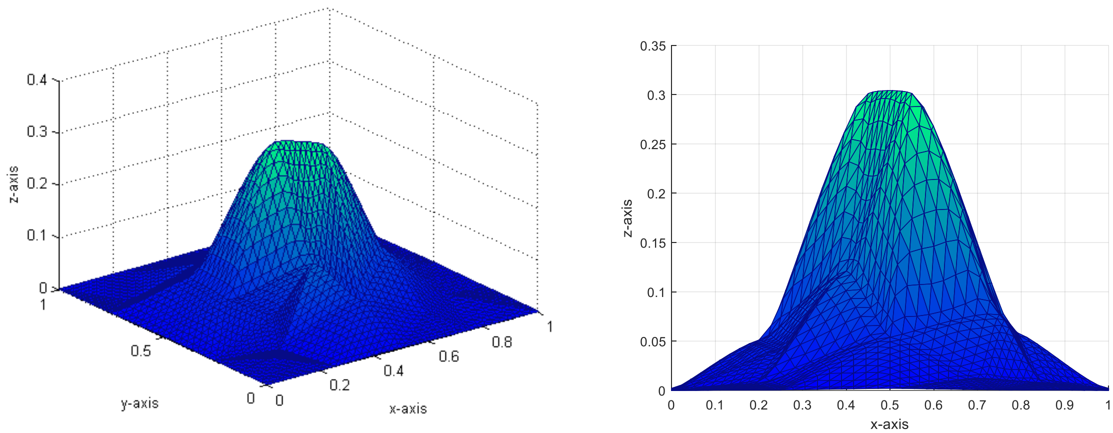
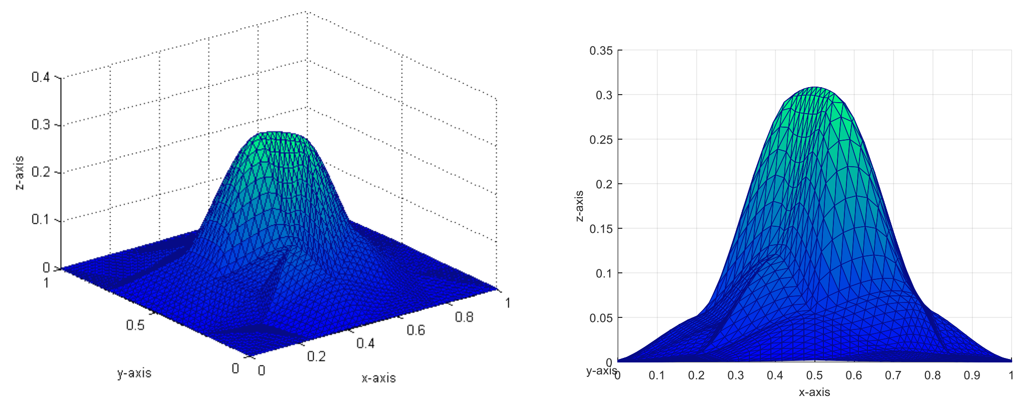


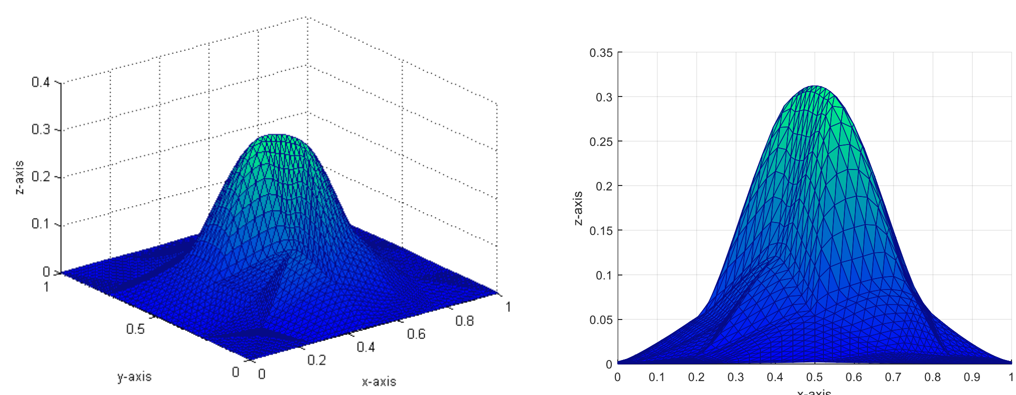
Figure 6 : (a) Interpolating surface without positivity conditions (b) xz-view



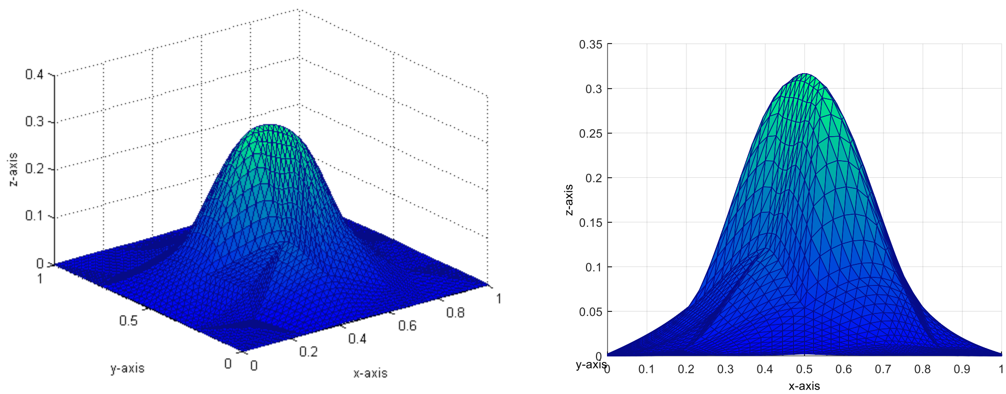
(a) $a = b = c = 0.2$ and $\omega = 0.5$



(b) $a = b = c = 0.4$ and $\omega = 0.6$



(c) $a = b = c = 0.6$ and $\omega = 0.8$



(d) $a = b = c = 0.8$ and $\omega = 0.9$

Figure 7 : Positivity preserving surface using rational cubic Ball interpolation for $F_1(x,y)$ with different values of free parameters.

Example 4.2: The function comprises 36 data points sampled from data set taken from [17] is given in Table A.2 (Appendix A).

$$F_2(x,y) = \begin{cases} 1.0 & \textit{if } (y-x) \geq 0.5 \\ 2(y-x) & \textit{if } 0.5 \geq (y-x) \geq 0.0 \\ \dfrac{\cos\left(4\pi\sqrt{(x-1.5)^2+(y-0.5)^2}\right)+1}{2} & \textit{if } (x-1.5)^2+(y-0.5)^2 \leq \dfrac{1}{16} \\ 0 & \textit{otherwise} \end{cases},$$

The positive data points are generated from the above positive function with the domain as $(x,y) \in [0,2]\times[0,1]$. Example 4.2 shows similar behavior as in Example 4.1. Figure 8 (a) shows the triangulation of the domain and the linear interpolant to the data is shown in Figure 8 (b). Figure 9 (a) gives an interpolating surface that has negative value. It can be seen from xz -view as shown in Figure 9 (b). To overcome these flaws, the surfaces in Figure 10 (a) – 10 (d) are generated by the proposed positivity scheme with four different values of free parameters, a, b, c and ω to preserve the positivity of data everywhere.

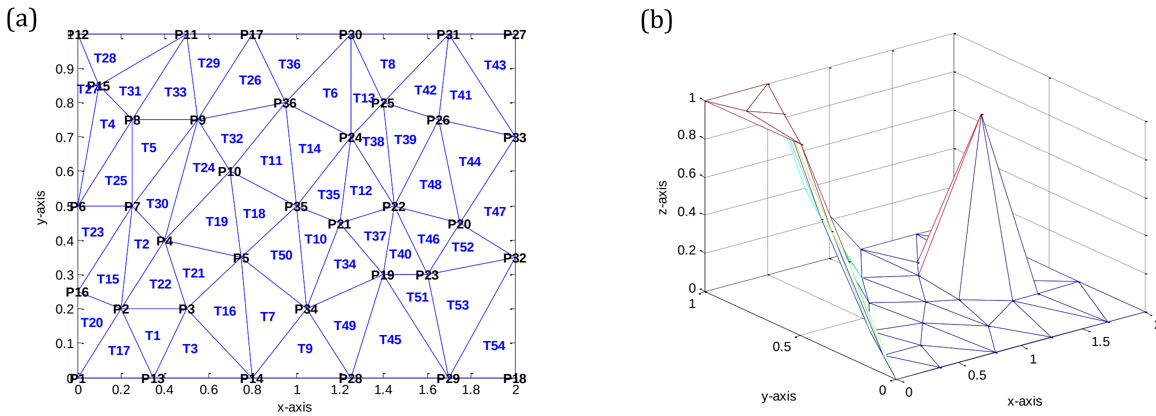


Figure 8 : (a) Delaunay triangulation of domain for $F_2(x,y)$ (b) Linear interpolant for data

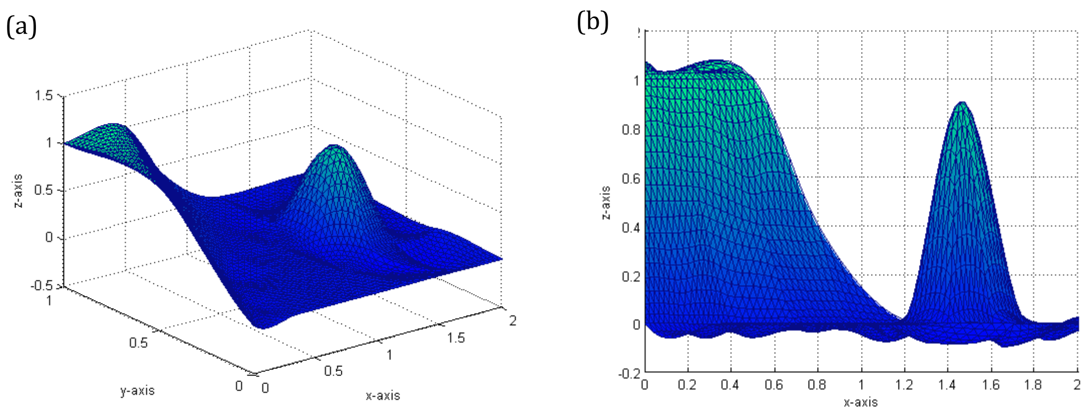
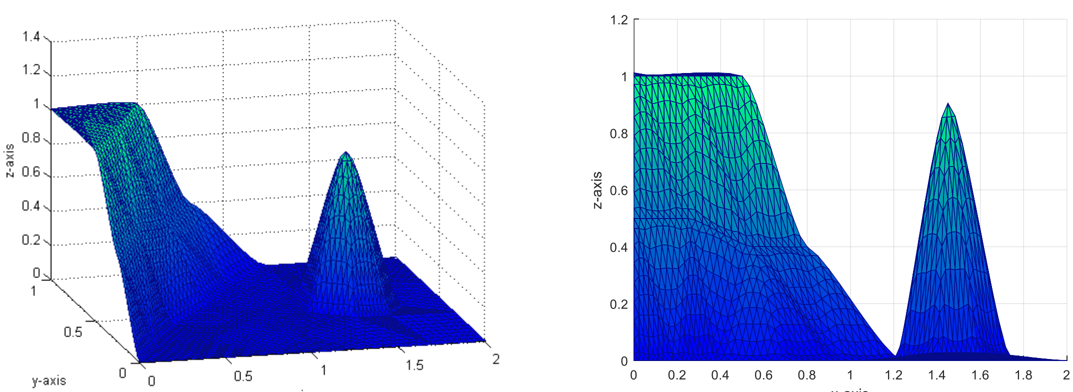
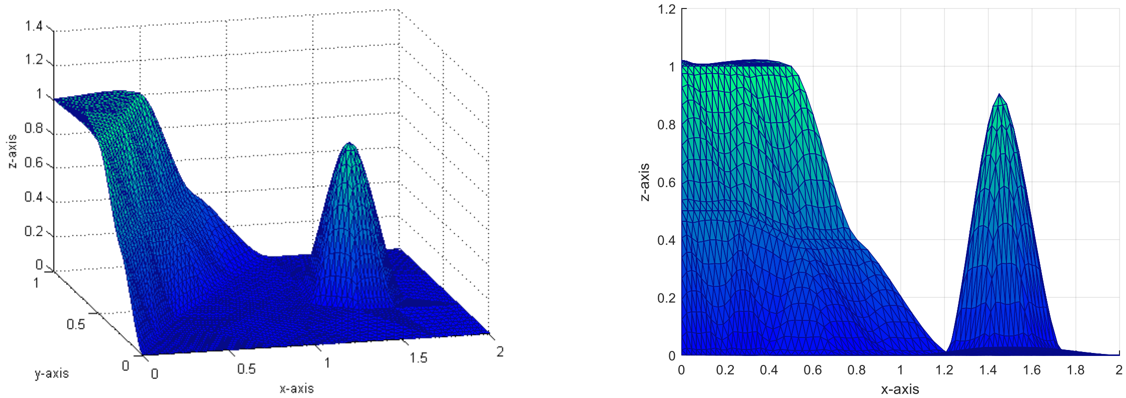


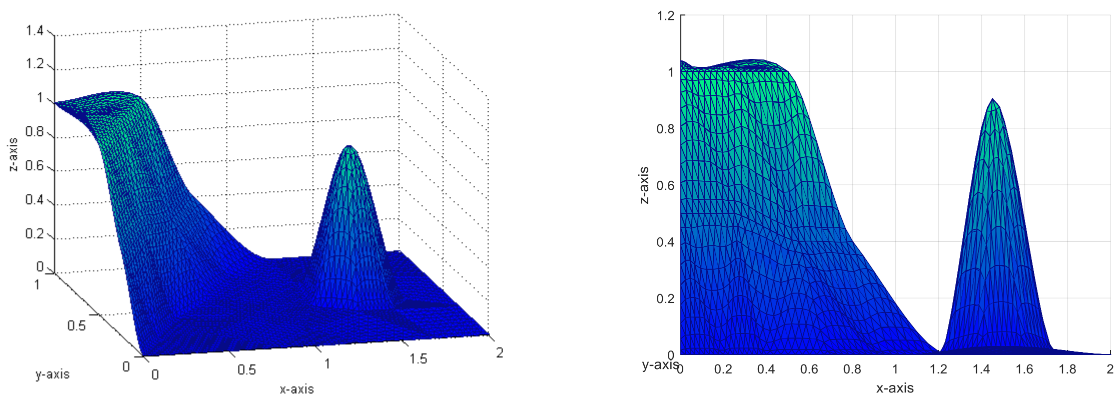
Figure 9 : (a) Interpolating surface without positivity conditions (b) xz -view



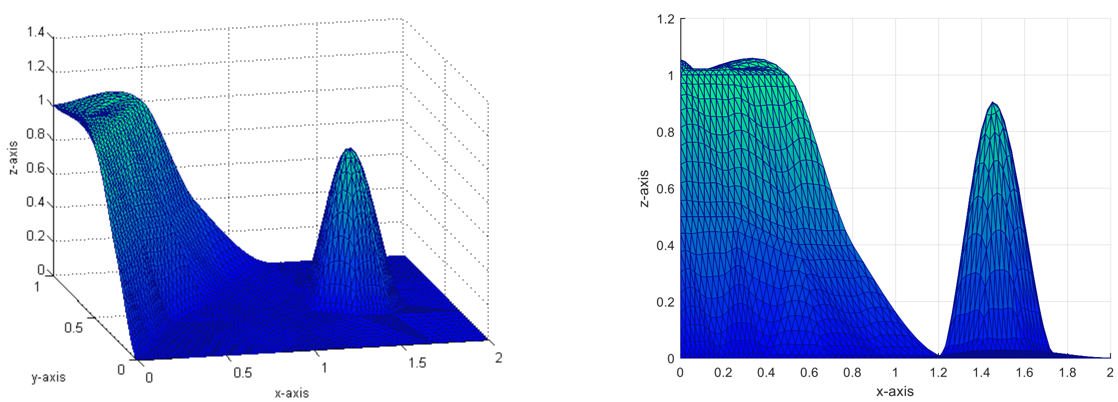
(a) $a = b = c = 0.2$ and $\omega = 0.5$



(b) $a = b = c = 0.4$ and $\omega = 0.6$



(c) $a = b = c = 0.6$ and $\omega = 0.8$



(d) $a = b = c = 0.8$ and $\omega = 0.9$

Figure 10 : Positivity preserving surface using rational cubic Ball interpolation for $F_2(x, y)$ with different values of free parameters.

The result of the experiment is tested by choosing the values of free parameter within interval 0 to 1 to ensure that surface comprising of triangular patch is always positive. From the Figures 6 and 9 are illustrate different parameter's values in the free parameters at vertex a, b, c and at boundary ω. These surfaces satisfy the positivity of interpolating function when given positive data. Figure 7 (c) – 7 (d) and Figure 10 (c) – 10 (d) show a smoother surface compared to Figure 7 (a) – 7 (b) and Figure 10 (a) – 10 (b), respectively. A positivity preserving surface of rational cubic Ball can be obtained.

5 CONCLUSION

A C^1 rational cubic Ball triangular patch that involves weights (free parameters) has been constructed in this paper to preserve the positivity of triangular surface. The given examples suggest that the proposed interpolating scheme produces smooth graphical results and preserves the shape of positive data. In this study, we tested various values of free parameters, a, b, c and ω on the regions of the surface. The selection of these values parameters is dependent on designer's choice for the refinement of positive surface as desired. Here, we conclude that the proposed scheme has advantages compared to existing scheme such as [1] and [3] as the proposed scheme involved the weight functions (free parameters). In any triangular patch if the Ball ordinates do not satisfy the derived conditions of positivity, then they can be modified by changing the free parameters to assure the positivity of triangular patches is achieved. The shape violations are found and the positivity preserving interpolants are also achieved. In the future, we would try to extend this study to optimize the weights (free parameters) of the surface by using optimization method to find the best values of the weights.

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APPENDIX

Table A.1: Positive surface data set of 36 points

x	y	$F_1(x,y)$	x	y	$F_1(x,y)$
0	0	0	0.8000	0.8500	0.00450
0.5000	0	0.0021	0.8500	0.6500	0.0177
1	0	0	1	0.5000	0.0021
0.1500	0.1500	0.0023	1	1	0
0.7000	0.1500	0.0124	0.5000	1	0.0021
0.5000	0.2000	0.0539	0.1000	0.8000	0.0021
0.2500	0.3000	0.0418	0	1	0
0.4000	0.3000	0.1211	0.2500	0	0.0006
0.7500	0.4000	0.0768	0.7500	0	0.0006
0.8500	0.2500	0.0079	0.2500	1	0.0006
0.5500	0.4500	0.3012	0	0.2500	0.0006
0	0.5000	0.0021	0.7500	1	0.0006
0.2000	0.450	0.0512	0	0.7500	0.0006
0.4500	0.550	0.3012	1	0.2500	0.0006
0.6000	0.650	0.1726	1	0.7500	0.0006
0.2500	0.700	0.0418	0.1900	0.1900	0.0068
0.4000	0.8000	0.0434	0.3200	0.7500	0.0488
0.6500	0.7500	0.0596	0.7900	0.4600	0.0588

Table A.2: Positive surface data set of 36 points

x	y	$F_2(x,y)$	x	y	$F_2(x,y)$
0	0	0	1.4000	0.3000	0.0272
0.2000	0.2000	0	1.7500	0.4500	0
0.5000	0.2000	0	1.2000	0.4500	0
0.4000	0.4000	0	1.4500	0.5000	0.9045
0.7500	0.3500	0	1.6000	0.3000	0.02729
0	0.5000	1	1.2500	0.7000	0
0.2500	0.5000	0.5000	1.4000	0.8000	0
0.2500	0.7500	1	1.6500	0.7500	0
0.5500	0.7500	0.4000	2	1	0
0.7000	0.6000	0	1.2500	0	0
0.5000	1	1	1.7000	0	0
0	1	1	1.2500	1	0
0.3500	0	0	1.7000	1	0
0.8000	0	0	2	0.3500	0
0.1000	0.8500	1	2	0.7000	0
0	0.2500	0.5000	1.0500	0.2000	0
0.8000	1	0.4000	1	0.5000	0
2	0	0	0.9500	0.8000	0