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Likelihood and Bayesian Intervals in The Stress-Strength Model Using Records from the Pareto Distribution of the Second Kind

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ABSTRACT

We consider likelihood and Bayesian inference in the stress-strength model using records from the Pareto distribution. Confidence intervals, including percentile intervals and intervals based on the maximum likelihood estimator are derived. Bayesian credible sets are also considered. Simulations are conducted to explore and compare the intervals in terms of their length and coverage probability.

***Keywords*:** confidence interval, Pareto distribution, records, stress-strength reliability

1 INTRODUCTION

Record data arise when only data values that are more extreme the current extreme value are recorded. This arise in many fields including industrial life testing and sports. More details and examples are presented in [1] and [2]. Records were introduced by [3]. He studied some of their properties. An account of records and their applications is provided by [4] and [5].

The density and distribution functions of the one parameter Pareto distribution $Pa(\theta)$ are as follows

$$f(x) = \frac{\theta}{(1+x)^{\theta+1}}, x > 0, \theta > 0,$$

$$F(x) = 1 - \frac{1}{(1+x)^{\theta}}, x > 0, \theta > 0. \quad (1)$$

Usually, this distribution involves another "scale" parameter. The two – parameter distribution has received considerable attention in the literature, see [6]. However, the one – parameter case has a simpler mathematical structure which allows for relatively simple solutions to many inference problems, classical and Bayesian as well. The one-parameter Lomax distribution has been considered by several authors including [7] and [8] for the stress-strength problem.

Let $X_1, X_2, \ldots$be an infinite sequence of iid random variables. An observation X_j is called an upper record if its value is greater than all previous observations. That is $X_j > X_i$ for every $i < j$. Our interest is in developing inference procedures for $Pr(X < Y)$. Several applications and motivations for the stress-strength model were presented by [9]. We consider the Pareto case with record data. Related problems were considered by [10] and [11] for the generalized exponential and the two parameter exponential distributions. More recent work on the stress-strength model based on records is given by [12], [13], and [14]. In Section 2, we consider likelihood inference. In Section 3 we consider Bayesian inference. Bootstrap methods are considered in Section 4. A simulation experiment is designed in Section 5. The final section concludes the paper.

2 LIKELIHOOD INFERENCE

Let $X \sim Pa(\theta_1)$ and $Y \sim Pa(\theta_2)$ be independent and define $R = Pr(X < Y)$ to be the stress strength reliability. It is straightforward to find that $R = \frac{\theta_1}{\theta_1+\theta_2}$. We want to estimate R based on lower records on both variables. Let $r_0, \dots, r_n$be the first n records from Pa(θ_1) and let $s_0, \dots, s_m$be a sequence of independent records from $Pa(\theta_2)$. The likelihood functions are

$$L_1(\theta_1|r_0, \dots, r_n) = f(r_n) \prod_{i=0}^{n-1} \quad f(r_i)/(1-F(r_i)),$$

$$L_2(\theta_2|s_0, \dots, s_m) = g(s_m) \prod_{i=0}^{m-1} \quad g(s_i)/(1-G(s_i)). \quad (2)$$

where $f$and F are the pdf and cdf of $Pa(\theta_1)$ and g and G are the corresponding functions for $Pa(\theta_2)$. Substituting in the likelihood functions we obtain,

$$L_1(\theta_1|r_0, \dots, r_n) = \frac{\theta_1^n}{(1+r_n)^{\theta_1+1}} \coprod_{i=0}^{n-1} \quad (1+r_i)^{-1},$$

$$L_2(\theta_2|s_0, \dots, s_m) = \frac{\theta_2^m}{(1+s_m)^{\theta_2+1}} \coprod_{i=0}^{m-1} \quad (1+s_i)^{-1}. \quad (3)$$

The log likelihood functions are given by,

$$l_1(\theta_1|r_0, \dots, r_n) = nln(\theta_1) - (\theta_1+1)ln(1+r_n) + ln \coprod_{i=0}^{n-1} \quad (1+r_i)^{-1},$$

$$l_2(\theta_2|s_0, \dots, s_n) = mln(\theta_2) - (\theta_2+1)ln(1+s_m) + ln \coprod_{i=0}^{m-1} \quad (1+s_i)^{-1}.$$

The MLEs of $\theta_1$and θ_2 based on the records can be obtained by solving the likelihood equations

$$\frac{d}{d\theta_1} l_1(\theta_1) = n/\theta_1 - ln(1+r_n),$$

$$\frac{d}{d\theta_2} l_2(\theta_2) = m/\theta_2 - ln(1+s_m).$$

therefore the MLEs are $\hat{\theta}_1 = \frac{n}{ln(1+r_n)}$, $\hat{\theta}_2 = \frac{m}{ln(1+s_m)}$. It follows that the MLE of R is

$$\hat{R} = \frac{\hat{\theta}_1}{\hat{\theta}_1+\hat{\theta}_2} \quad (4)$$

Consider $\hat{\theta}_1 = \frac{n}{ln(1+r_n)}$, the pdf of $R_n$is given by [4]

$$f_{R_n}(r_n) = f(r_n)\left[-ln(1-F(r_n))\right]^{n-1}/(n-1)! \qquad = \frac{\theta_1}{(1+r_n)^{\theta_1+1}\Gamma(n)} \left(ln(1+r_n)\right)^n, r_n > 0.$$

Note that $ln(1+r_n)$ is distributed as $Gamma(n, 1/\theta_1)$. Similarly, $ln(1+s_m) \sim Gamma(m, 1/\theta_2)$. Note that $2\theta_1 ln(1+r_n) \sim \chi^2_{2n}$ and $2\theta_2 ln(1+s_m) \sim \chi^2_{2m}$and they are independent, it follows that $\hat{R} = \frac{1}{1+\hat{\theta}_2/\hat{\theta}_1} \sim \frac{1}{1+\theta_1/\theta_2 F_{2m,2n}}$ where $F_{2m,2n}$ denotes a Snedecor's F random variable with $(2m, 2n)$ degrees of freedom. Therefore, a $(1-\alpha)\%$ confidence interval for R is

$$\left\{\left(1+\left(\frac{1}{\hat{R}}-1\right)/F_{1-\alpha/2,2m,2n}\right)^{-1}, \left(1+\left(\frac{1}{\hat{R}}-1\right)/F_{\alpha/2,2m,2n}\right)^{-1}\right\}.$$

Another confidence interval for R can be obtained based on the MLEs. Note that

$$\sqrt{n}\left(\hat{\theta}_1-\theta_1\right) D \rightarrow N(0, v_1^2),$$

where v_1^2 is the asymptotic variance. The second order derivatives are $\frac{d^2}{d\theta_1^2} l_1(\theta_1) = -n/\theta_1^2$ and $\frac{d}{d\theta_2} l_2(\theta_2) = -m/\theta_2^2$, hence

$$v_1^2 = \left[-E\left(\frac{\partial^2 lnL(\theta|r_0, \dots r_n)}{\partial\theta_1^2}\right)\right]^{-1} = \frac{\theta_1^2}{n},$$

$$\sqrt{m}\left(\hat{\theta}_2-\theta_2\right) D \rightarrow (0, v_2^2) \text{ as } m \rightarrow \infty,$$

where

$$v_2^2 = \left[-E\left(\frac{\partial^2 lnL(\theta|s_0, \dots s_m)}{\partial\theta_2^2}\right)\right]^{-1} = \frac{\theta_2^2}{m}.$$

Let $n \rightarrow \infty, m \rightarrow \infty$ such that $m/n \rightarrow p$ where $0 < p < 1$, we have

$$\sqrt{n}\left(\hat{\theta}_2-\theta_2\right) D \rightarrow N(0, v_2^2/p).$$

since $R = \frac{\theta_1}{\theta_1+\theta_2} = h(\theta_1, \theta_2)$ say, and $\hat{R} = \frac{\hat{\theta}_1}{\hat{\theta}_1+\hat{\theta}_2} = h\left(\hat{\theta}_1, \hat{\theta}_2\right)$

$$\sqrt{n}\left(\hat{R}-R\right) = \sqrt{n}\left[h\left(\hat{\theta}_1, \hat{\theta}_2\right) - h(\theta_1, \theta_2)\right] D \rightarrow N(0, \eta^2)$$

where $\eta^2 = \left(\frac{\partial h(\theta_1,\theta_2)}{\partial\theta_1}\right)^2 v_1^2 + \left(\frac{\partial h(\theta_1,\theta_2)}{\partial\theta_2}\right)^2 v_2^2/p$, see [15]. An asymptotic $(1-\alpha)\%$ confidence interval for R is

$$\left\{\hat{R} - z_{1-\alpha/2}\hat{\eta}, \hat{R} + z_{1-\alpha/2}\hat{\eta}\right\} \quad (5)$$

where $\hat{\eta}$ is obtained by substituting m/n for p and the MLEs in η.

3 BAYESIAN INFERENCE

The conjugate prior distributions for θ_1 and θ_2 are the following Gamma distributions;

$$\psi_1(\theta_1) = \frac{\beta_1^{\delta_1}\theta_1^{\delta_1-1}e^{-\beta\theta_1}}{\Gamma(\delta_1)}, \qquad \theta_1 > 0,$$

where β_1 and δ_1 are the parameters of the prior distribution of θ_1 and

$$\psi_2(\theta_2) = \frac{\beta_2^{\delta_2}\theta_2^{\delta_2-1}e^{-\beta_2\theta_2}}{\Gamma(\delta_2)}, \qquad \theta_2 > 0,$$

where β_2 and δ_2 are parameters of the prior distribution of θ_2, see [16]. It can be shown that the posterior distribution of θ_1 given $r_0,\ldots,r_n$ is

$$\pi_1(\theta_1|r_0,\ldots,r_n) = \frac{(\beta_1+ln(1+r_n))^{(n+\delta_1)}}{\Gamma(n+\delta_1)}{\theta_1}^{n+\delta_1-1}e^{-\theta_1(\beta_1+ln(1+r_n))}, \theta_1 > 0. \qquad (6)$$

Similarly the posterior distribution of θ_2 is given by

$$\pi_1(\theta_2|s_0,\ldots,s_m) = \frac{(\beta_2+ln(1+s_m))^{(m+\delta_2)}}{\Gamma(m+\delta_2)}{\theta_2}^{m+\delta_2-1}e^{-\theta_2(\beta_2+ln(1+s_m))}, \theta_2 > 0,$$

that is,

$$\theta_1|r_0,\ldots,r_n \sim Gamma \quad \left(n+\delta_1, \left(\beta_1 + ln(1+r_n)\right)^{-1}\right),$$

$$\theta_2|s_0,\ldots,s_m \sim Gamma \quad \left(m+\delta_2, \left(\beta_2 + ln(1+s_m)\right)^{-1}\right).$$

It follows that

$$2(\beta_1 + ln(1+r_n))\theta_1|r_0,\ldots,r_n \sim \chi^2_{2(n+\delta_1)},$$

$$2(\beta_2 + ln(1+s_m))\theta_2|s_0,\ldots,s_m \sim \chi^2_{2(m+\delta_2)}.$$

Therefore, $\frac{(\beta_2+ln(1+s_m))\theta_2/(m+\delta_2)}{(\beta_1+ln(1+r_n))\theta_1/(n+\delta_1)}|r_0,\ldots,r_n,s_0,\ldots,s_m \sim F_{2(m+\delta_2),2(n+\delta_1)}$.

The posterior distribution of R is $\left(1+\frac{(m+\delta_2)/(\beta_2+ln(1+s_m))}{(n+\delta_1)/(\beta_1+ln(1+r_n))}W\right)^{-1}$ where $W \sim F_{2(m+\delta_2),2(n+\delta_1)}$. The Bayes estimator is the mean of this posterior distribution,

$$R\sim = \int_0^1 R\pi(R|r_0,\ldots,r_n,s_0,\ldots,s_m)dR. \qquad (7)$$

This estimator may be approximated numerically. A $(1-\alpha)$ probability interval for R is,

$$\left(AF_{1-\alpha/2,2(m+\delta_2),2(n+\delta_1)}+1\right)^{-1}, \left(AF_{\alpha/2,2(m+\delta_2),2(n+\delta_1)}+1\right)^{-1}, \qquad (8)$$

where $A = \frac{(m+\delta_2)(\beta_1+ln(1+r_n))}{(n+\delta_1)(\beta_2+ln(1+s_m))}$. The Jefferey priors for θ_1 and θ_2 are proportional to $\frac{1}{\theta_1}$ and $\frac{1}{\theta_2}$ respectively. It can be easily shown that the posterior of R is distributed as $\left(1 + \frac{m/ln(1+s_m)}{n/ln(1+r_n)}W\right)^{-1}$ where $W \sim F_{2m,2n}$. Therefore a $(1-\alpha)$ probability interval for R is,

$$\left(\frac{mln(1+r_n)}{nln(1+s_m)}F_{1-\alpha/2,2m,2n} + 1\right)^{-1}, \left(\frac{mln(1+r_n)}{nln(1+s_m)}F_{\alpha/2,2m,2n} + 1\right)^{-1}. \tag{9}$$

The highest posterior density (HPD) regions is defined as;

$$BR(\pi_\alpha) = \{\theta: \pi(\theta|r_0, \dots, r_n, s_0, \dots, s_n) \geq \pi_\alpha\}, \tag{10}$$

where π_α is the largest constant such that $Pr(\theta \in BR(\pi_\alpha)) \geq 1-\alpha$. This is often done numerically. A simulation algorithm to approximate the bounds of the HPD interval was developed by [17].

4 BOOTSTRAP INTERVALS

Bootstrap methods can be used to obtain intervals using resampling with replacement from the original data or from the parametric distribution of the data with its parameters replaced by their estimates. The bootstrap-t interval and the percentile interval are among the most widely used bootstrap intervals, [18,19]. Bootstrap methods were used in a variety of problems recently, see [20].

Let $\hat{R}$ be the MLE of δ based on the original sample and let $\hat{R}^*$ be the MLE based on the bootstrap sample. Let z^*_α be the α quantile of $Z^* = (\hat{R}^* - \hat{R}\frac{)}{\hat{\eta}^*}$, where $\hat{\eta}^*$ is estimated standard error of $\hat{R}$ based on the bootstrap sample. The bootstrap-t interval for R is,

$$\left(\hat{R} - z^*_{1-\alpha/2}\hat{\eta}, \hat{R} - z^*_{\alpha/2}\hat{\eta}\right), \tag{11}$$

where $\hat{\eta}$ is the estimated standard deviation obtained from the original sample. Another bootstrap-t interval is based on the quantiles of $\varepsilon^* = (\hat{R}^* - \hat{R}\frac{)}{sd^*}(\hat{R}^*)$ where $sd^*(\hat{R}^*)$ is the estimated standard error of $\hat{R}^*$ obtained from a second stage bootstrap sample. The second bootstrap-t interval is,

$$\left(\hat{R} - \varepsilon^*_{1-\alpha/2}\hat{\eta}, \hat{R} - \varepsilon^*_{\alpha/2}\hat{\eta}\right). \tag{12}$$

where ε^*_α is found using simulation.

The percentile interval may be described as follows; Let $\hat{R}^*$be the estimate of the stress-strength probability calculated from the bootstrap sample. The bootstrap distribution of $\hat{R}^*$ is obtained by resampling from the original distribution and calculating $\hat{R}^*_i, i = 1, \dots, B$, where B is the number of bootstrap samples. The $1-\alpha$ interval is given by,

$$\left(\widehat{H}^{-1}\left(\frac{\alpha}{2}\right), \widehat{H}^{-1}\left(\frac{1-\alpha}{2}\right)\right). \tag{13}$$

where $\widehat{H}^{-1}(.)$ Is the inverse of the estimated bootstrap distribution function.

5 A SIMULATION STUDY

A simulation study is conducted to explore and compare the intervals presented in this paper. In the simulation design we used $(n, m) = (5, 5), (5, 10), (5, 15), (10, 10), (10, 15)$ and $(15, 15)$. We used $\theta_1 = 1$, R =0.1, 0.3, 0.5, 0.7, and 0.9. The confidence level taken is $(1 - \alpha) = 0.95$. We used 2000 replications and calculated the following intervals;

1) ML: The interval based on the asymptotic normality of the MLE (5).

2) AHPD: The approximate HPD interval proposed by [17].

3) Boot1: The bootstrap-t interval (11).

4) Boot2: The bootstrap-t interval using bootstrap variance estimate (12).

5) Perc: The percentile interval (13).

We estimated the coverage probabilities and expected lengths of the intervals. In bootstrap calculations we used 500 bootstrap replicates. The bootstrap variance estimate is based on 25 second stage bootstrap samples. We used 1000 Monte Carlo samples from the posterior density of R to approximate the endpoints of the HPD interval. The results are given in Table 1.

Table 1: Estimated Lengths and Coverage Probabilities of the Intervals

			ML		AHPD		Boot1		Boot2		Perc	
n	m	R	L	CV	L	CV	L	CV	L	CV	L	CV
5	5	0.10	0.229	0.910	0.259	0.930	0.267	0.889	0.293	0.877	0.250	0.934
5	5	0.30	0.485	0.890	0.456	0.922	0.599	0.903	0.576	0.891	0.494	0.915
5	5	0.50	0.579	0.870	0.496	0.912	0.746	0.863	0.665	0.877	0.569	0.908
5	5	0.70	0.520	0.871	0.434	0.915	0.700	0.835	0.602	0.867	0.497	0.928
5	5	0.90	0.263	0.869	0.232	0.930	0.382	0.782	0.331	0.848	0.253	0.954
5	10	0.10	0.190	0.901	0.238	0.911	0.209	0.889	0.240	0.901	0.211	0.920
5	10	0.30	0.401	0.898	0.418	0.920	0.475	0.898	0.496	0.909	0.443	0.931
5	10	0.50	0.462	0.895	0.446	0.919	0.592	0.873	0.588	0.885	0.522	0.937
5	10	0.70	0.402	0.877	0.372	0.915	0.546	0.823	0.530	0.857	0.451	0.953
5	10	0.90	0.192	0.865	0.180	0.899	0.280	0.777	0.274	0.817	0.222	0.946
5	15	0.10	0.183	0.931	0.231	0.897	0.195	0.918	0.226	0.931	0.200	0.939
5	15	0.30	0.381	0.905	0.404	0.908	0.444	0.896	0.472	0.908	0.425	0.933
5	15	0.50	0.436	0.885	0.423	0.902	0.546	0.855	0.557	0.870	0.499	0.938
5	15	0.70	0.374	0.876	0.349	0.898	0.502	0.831	0.503	0.842	0.430	0.943
5	15	0.90	0.171	0.861	0.162	0.890	0.248	0.772	0.254	0.800	0.206	0.945

10	10	0.10	0.162	0.924	0.170	0.923	0.167	0.910	0.180	0.895	0.176	0.935
10	10	0.30	0.356	0.914	0.333	0.919	0.378	0.909	0.381	0.896	0.380	0.921
10	10	0.50	0.422	0.911	0.372	0.922	0.462	0.900	0.448	0.903	0.452	0.935
10	10	0.70	0.365	0.902	0.318	0.932	0.416	0.858	0.395	0.893	0.382	0.944
10	10	0.90	0.170	0.902	0.152	0.928	0.201	0.817	0.192	0.866	0.173	0.951
10	15	0.10	0.138	0.927	0.155	0.925	0.142	0.924	0.156	0.915	0.157	0.940
10	15	0.30	0.307	0.909	0.310	0.915	0.332	0.912	0.345	0.915	0.353	0.929
10	15	0.50	0.361	0.903	0.345	0.920	0.405	0.891	0.409	0.901	0.422	0.943
10	15	0.70	0.309	0.892	0.289	0.912	0.361	0.847	0.359	0.876	0.353	0.945
10	15	0.90	0.143	0.889	0.135	0.916	0.174	0.813	0.174	0.851	0.161	0.950
15	15	0.10	0.130	0.932	0.131	0.919	0.127	0.916	0.136	0.901	0.141	0.937
15	15	0.30	0.295	0.931	0.273	0.927	0.296	0.922	0.302	0.913	0.320	0.935
15	15	0.50	0.349	0.928	0.311	0.930	0.362	0.903	0.358	0.916	0.385	0.941
15	15	0.70	0.299	0.922	0.263	0.924	0.318	0.871	0.311	0.899	0.323	0.945
15	15	0.90	0.136	0.910	0.121	0.926	0.149	0.827	0.146	0.871	0.141	0.945

6 DISCUSSION OF RESULTS AND CONCLUSIONS

As anticipated, the length of the intervals is shorter for extreme values of R. Larger record sequences result in shorter intervals too. The two bootstrap-t intervals appear to be anti-conservative with the percentile interval being slightly better. The performance of the Bayesian interval appears to be similar to that of the percentile interval. It is better than the other intervals in the case of unequal sample sizes. Interval based on the MLE have satisfactory performance only for large samples with equal sample sizes on both the stress and strength variables. The percentile interval is shorter than the Bayesian intervals for middle values of R and small sample sizes. For larger sample sizes, the percentile is shorter in all cases.

In conclusion, we recommend the percentile interval when the sample size is small, the percentile or Bayesian interval for larger sample sizes and the Bayesian interval for unequal sample sizes.

This research investigates the interesting stress-strength problem for the one parameter Pareto distribution case. Likelihood, bootstrap and Bayesian inference procedures are derived and their performance is investigated. We found that the bootstrap can provide viable solutions through the percentile interval when the sample size is relatively small. It provides satisfactory solution for larger samples through the bootstrap-t interval in addition to Bayesian intervals. On the other hand, the likelihood based intervals are not satisfactory and should be avoided unless very large samples are available.

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