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Comparative Analysis of Taylor Series and Runge-Kutta Fehlberg Methods in Solving the Lotka-Volterra Competitive Model

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ABSTRACT

This paper primarily scrutinizes the comparative efficacy of numerical methods, namely the Taylor Series and the Runge-Kutta Fehlberg methods, against the exact solution in solving mathematical models. These methods exhibit the capacity to handle the non-linearity of the Lotka-Volterra competitive model with a high degree of accuracy and reliability. The comparison of results obtained from both methods and the exact solution shows that the Runge-Kutta Fehlberg method provides a more precise approximation for the model than the Taylor Series method. This conclusion is supported by computations carried out using the Mathematica 13.2 software. The research data involves two species, Paramecium Caudatum and Stylonychia Pustulata, derived from Gause's experiment. Both species demonstrate an intraspecific interaction, with their populations rising steadily until reaching a constant level. For Paramecium Caudatum, the population peaks at 202 cells on the 16th day, while for Stylonychia Pustulata, it reaches a maximum of 41 cells on the 8th day. Equilibrium and stability analysis offer vital insights into the long-term behavior of the system and its reaction to perturbations. In mixed populations, when the carrying capacity of both species is less than the carrying capacity of another species divided by the competition coefficient, the species coexist in a stable equilibrium. Conversely, if the carrying capacity of one species is less than the carrying capacity of the other divided by the competition coefficient, one species may outcompete the other, leading to an unstable equilibrium or even extinction of the weaker species. The research thus provides valuable insights into the dynamics of competition and survival, with profound implications for the fields of ecology, conservation, and environmental management.

Keywords: Lotka-Volterra competitive model, RKF, Taylor Series, stability.

1 INTRODUCTION

The Lotka-Volterra competitive model represents interspecific and intraspecific competition among species in the same environment, competing over limited resources. The model is a mathematical expression of ecological dynamics, reflecting the reality that species' interactions affect population growth and resource allocation. In this context, two numerical methods, the Taylor Series Method and the Runge-Kutta-Fehlberg (RKF) Method, are applied to solve the model. The Taylor Series Method uses an infinite sum of function derivatives to approximate solutions, allowing for the direct evaluation of solution's accuracy through approximation and application of initial or boundary conditions. The RKF Method adapts step sizes based on calculation truncation errors, achieving similar accuracy to the Taylor Series without the need for higher derivative calculations. Both methods offer solutions where analytical ones may not be available or too challenging to derive.

In the existing literature, various numerical methods have been employed to solve the Lotka-Volterra competitive model, which provides a fundamental representation of population growth and decline [1]. [2] used the Differential Transformation Method (DTM) to solve the single species Lotka-Volterra equation. The DTM, compared to other methods like the variational iteration and Adomian decomposition method, is powerful for nonlinear equations. [3] employed the modified Lotka-Volterra model to stimulate value creation between upstream and downstream energy firms. This study broadened the application of the model beyond traditional ecological interactions. The study by [4] developed perturbation-iteration algorithms for first-order differential equation systems, providing approximations of Lotka-Volterra system solutions without the need for a small parameter assumption.

The accuracy of these solutions improved as the number of terms in the Taylor series expansion increased. [5] utilized numerical methods such as the Euler Method, Taylor Series Method, and Runge-Kutta Method to understand the effect of interspecific competition, with the Runge-Kutta method providing the most accurate approximation of the orbit's behavior. [6] compared the solutions obtained using the RKF and Laplace Adomian Decomposition Method (LADM) on the Lotka-Volterra model. The RKF method was found to be more accurate and reliable for solving differential equation models in population dynamics. [7] applied a moving mesh finite difference technique to a PDE system of the Lotka-Volterra competition model, proving the method's robustness and stability over a wide range of parameter values. [8] proposed an improved Taylor collocation method to solve the nonlinear delay differential equations of the Lotka-Volterra prey-predator model. This method demonstrated significant accuracy and reliability. [9] introduced a high-order method combining a fourth-order compact finite difference method with an implicit-explicit Runge Kutta scheme to solve the one-dimensional Lotka-Volterra-diffusion problem. The results confirmed the validity and effectiveness of the proposed method. [10] argued that the RKF method is more reliable and efficient in solving the Lotka-Volterra predator-prey model than the LADM method. [11] compared the RKF and 4-stage Runge-Kutta method for the Predator-Prey-Scavenger Model. They found the RKF45 method provided a better approximative solution. [12] developed a unique finite-difference technique to achieve periodic numerical solutions, proving to be dynamically consistent with the differential equations. Finally, [13] used two straightforward, reliable methods to study the effects of a predator-prey model on animal populations during the mating period, demonstrating the broad applicability and effectiveness of numerical methods in modeling ecological interactions.

This research aims to compare these methods' efficacy in solving the Lotka-Volterra competitive model and discern which one yields the best results. The study centers on employing numerical methods such as the Taylor Series Method and RKF to unravel mathematical models like the Lotka-Volterra competitive model. The researchers will develop a Lotka-Volterra competitive model to observe intraspecific interactions and establish an exact solution using sample data. The focus will be on determining the stability and equilibrium, investigating whether the species will achieve a stable equilibrium or undergo competitive exclusion. This research will utilize sample data from Gause's experiment involving *Paramecium Caudatum* and *Stylonychia Pustulata* [14].

2 METHODOLOGY

The data used in this research is from Gause's experiment on two species which are *Paramecium Caudatum* (*P. Caudatum*) and *Stylonychia Pustulata* (*S. Pustulata*). The logistic equation of both species is used to find the exact solution and will be compared to numerical methods, such as Taylor Series and Runge-Kutta Fehlberg methods. The Taylor series is a mathematical tool for representing a function as an infinite sum of terms, with each term derived by differentiating the function at a specified point. It provides a way to approximate a function using its derivatives at a given point. The RKF method is a numerical method used for solving ordinary differential equations (ODEs). A more accurate numerical approximation of the ODE solution can be obtained by iteratively applying the RKF method with smaller step sizes.

2.1 Exact Solution

The logistic equation from the Lotka-Volterra model is used to model the growth of an isolated population:

$$\frac{dy}{dx} = ry\left(1 - \frac{y}{K}\right) \qquad (1)$$

where $y(x)$ is the mean density (in individuals per $0.5cm^3$) at the time x (in days), r is the instantaneous rate of increase (births/deaths), and K is the carrying capacity per $0.5cm^3$. Assume constant K and r linear density dependence, no time lags, migration, age structure, or limited resources.

By solving the equation (1), the solution for the initial condition can be obtained, which gives:

$$y = \frac{K}{1 + \frac{1}{2}e^{-rx}(K-2)} \qquad (2)$$

To be able use the equation (2), the good fit value of r and K can be obtained by using curve fitting. The fit is reasonable for:

P. Caudatum in isolation where $r = 0.66$ and $K = 202.6$:

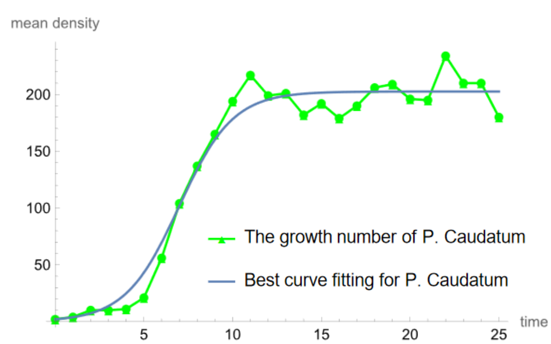

Figure 1: Curve fitting for growth of *P. Caudatum* in isolation.

S. Pustulata in isolation where $r = 0.89$ and $K = 41.7$:

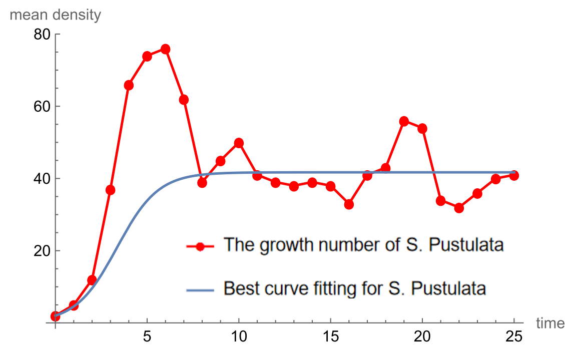

Figure 2: Curve fitting for growth of *S. Pustulata* in isolation.

Based on Figures 1 and 2, *P. Caudatum* produces a population of 202.6 in an isolated case while *S. Pustulata* only produces a population of 41.7. *S. Pustulata* consumes $1/41.7 = 0.02398$ food, while *P. Caudatum* $1/202.6 = 0.00494$ food in a single unit. In other words, *S. Pustulata* consumes 4.85 times as much food per unit as *P. Caudatum* $[0.02398/0.00494 = 4.85]$ and *P. Caudatum* consumes just $1/4.85 = 0.206$ times as much food per unit as *S. Pustulata*. These factors can make it recalculate the volume of one species into its equal in terms of the volume consumed by another species of food.

2.2 Taylor Series Method

In order to use the Taylor Series method, one must first determine the derivatives of $y(x)$. The solution $y(x)$ is a function of x differentiating the formula $y'(x) = f\big(x, y(x)\big)$ with respect to x to obtain $y''(x)$.

$$d_1 = y'(x) = ry\left(1 - \frac{y}{K}\right)$$
$$d_2 = y''(x) = \frac{d}{dy}\left[ry\left(1 - \frac{y}{K}\right)\right] \tag{3}$$

Next, the value of d_1 and d_2 can be obtained by substituting the initial value of y into the equation (3). Then, the value of the next y can be obtained by substituting the value of d_1 and d_2 into the equation (4) below:

$$y_{k+1} = y_k + d_1 h + \frac{d_2 h^2}{2!} \tag{4}$$

The step will be repeated to obtain a better approximation.

2.3 Runge-Kutta Fehlberg (RKF) Method

The fourth-order Runge-Kutta (RK4) technique is one of the most well-known constant-step procedures. The Runge-Kutta technique can approximate a Taylor Series approximation with reasonable accuracy without the requirement for higher derivative computations. To some extent, this technique may be seen as the basis upon which other techniques are built [10].

Consider the initial value problem:

$$y'(x) = f\big(x, y(x)\big)$$
$$y(x_0) = y_0 \tag{5}$$

The RKF is one way to try to resolve this problem. The problem is solving the initial value problem in the above equation using the Runge-Kutta methods of order 4 and order 5.

First, the definitions are:

$$k_1 = hf(x_i, y_i)$$
$$k_2 = hf\left(x_i + \frac{1}{4}h, y_i + \frac{1}{4}k_1\right)$$
$$k_3 = hf\left(x_i + \frac{3}{8}h, y_i + \frac{3}{32}k_1 + \frac{9}{32}k_2\right)$$
$$k_4 = hf\left(x_i + \frac{12}{13}h, y_i + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right) \tag{6}$$
$$k_5 = hf\left(x_i + h, y_i + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4\right)$$
$$k_6 = hf\left(x_i + h, y_i - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5\right)$$

A better value for the solution is determined using a Runge Kutta method of order 5:

$$y_{i+1} = y_i + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6 \tag{7}$$

Repeat the method and step to obtain a better approximation.

2.4 Numerical Methods for Competitive Equation

Given is the Lotka-Volterra competition model, or is also known as the coupled equation:

$$\frac{dN_1}{dt} = r_1 N_1(t)\left[\frac{K_1 - N_1(t) - \beta_{12}N_2(t)}{K_1}\right]$$
$$\frac{dN_2}{dt} = r_2 N_2(t)\left[\frac{K_2 - N_2(t) - \beta_{21}N_1(t)}{K_2}\right] \tag{8}$$

where N_1 represents the population density (in individual per $0.5cm^3$) of *P. Caudatum* and N_2 represents the population density (in individual per $0.5cm^3$) of *S. Pustulata*. The term r_1 represents the instantaneous rate of increase of *P. Caudatum*, meanwhile r_2 represents the instantaneous rate of increase of *S. Pustulata*, and K_1 represents the carrying capacity of *P. Caudatum*, meanwhile K_2

represents the carrying capacity of *S. Pustulata*. The parameter β_{12} represents the per capita effect of *S. Pustulata* on the population growth of *P. Caudatum*, and β_{21} represents the per capita effect of *P. Caudatum* on the population growth of *S. Pustulata*.

Since the coupled equation in this case consumes a lot of time if it is solved by using an exact analytical solution, the *Mathematica 13.2* software is used to solve it numerically.

3 RESULTS AND DISCUSSION

This research aims to make a comparison of the numerical methods, such as the Taylor Series method and the RKF method with the exact solution. Table 1and 2 show comparison among the Taylor Series, RKF and the exact solution for the single species of *P. Caudatum* and *S. Pustulata*.

Table 1: Comparison between numerical methods and the exact solution for *P. Caudatum* in isolation.

Day	Exact Solution	Taylor Series	RKF	Absolute Error	
				Taylor Series	RKF
0	2	2	2	0.00E+00	0.00E+00
1	3.834202799	3.729754000	3.834048532	1.04E-01	1.54E-04
2	7.289429365	6.914101076	7.288877169	3.75E-01	5.52E-04
3	13.664618760	12.676901340	13.643218960	9.88E-01	2.14E-02
4	24.835932870	22.784506310	24.833070650	2.05E+00	2.86E-03
5	43.112054600	39.544869290	43.107329690	3.57E+00	4.72E-03
6	69.573450380	64.775957600	69.567234680	4.80E+00	6.22E-03
7	101.899400600	97.319636080	101.892682100	4.58E+00	6.72E-03
8	134.103729600	131.129816600	134.097534500	2.97E+00	6.20E-03
9	160.285751800	158.693316700	160.281150700	1.59E+00	4.60E-03
10	178.275249000	177.147793600	178.273067500	1.13E+00	2.18E-03
11	189.253513700	188.206677000	189.253300400	1.05E+00	2.13E-04
12	195.475079300	194.533011600	195.475753000	9.42E-01	6.74E-04
13	198.853823600	198.092542800	198.854647100	7.61E-01	8.23E-04
14	200.646330300	200.084106600	200.647011400	5.62E-01	6.81E-04
15	201.585516700	201.196259500	201.586000900	3.89E-01	4.84E-04
16	202.074391400	201.816900000	202.074709400	2.57E-01	3.18E-04
17	202.327997500	202.163161700	202.328196700	1.65E-01	1.99E-04

18	202.459323900	202.356324500	202.459444700	1.03E-01	1.21E-04
19	202.527267000	202.464075700	202.527338600	6.32E-02	7.16E-05
20	202.562401300	202.524180800	202.562443100	3.82E-02	4.18E-05
21	202.580565300	202.557707900	202.580589400	2.29E-02	2.41E-05
22	202.589954700	202.576409400	202.589968400	1.35E-02	1.37E-05
23	202.594807900	202.586841100	202.594815700	7.97E-03	7.80E-06
24	202.597316400	202.592660000	202.597320800	4.66E-03	4.40E-06
25	202.598613000	202.595905700	202.598615400	2.71E-03	2.40E-06

Table 2: Comparison between numerical methods and the exact solution for *S. Pustulata* in isolation.

Day	**Exact Solution**	**Taylor Series**	**RKF**	**Absolute Error**	
				Taylor Series	**RKF**
0	2	2	2	0.00E+00	0.00E+00
1	4.556621749	4.376401238	4.556086590	1.80E-01	5.35E-04
2	9.591791693	9.088355933	9.590551115	5.03E-01	1.24E-03
3	17.560433260	17.002044700	17.559118830	5.58E-01	1.31E-03
4	26.653701240	26.700312850	26.652373320	4.66E-02	1.33E-03
5	33.852360800	34.180778820	33.851398920	3.28E-01	9.62E-04
6	38.075305760	38.105488950	38.076203190	3.02E-02	8.97E-04
7	40.131128890	39.952222080	40.132912200	1.79E-01	1.78E-03
8	41.041124960	40.834945900	41.042552470	2.06E-01	1.43E-03
9	41.426885980	41.267294890	41.427749810	1.60E-01	8.64E-04
10	41.587409570	41.482331370	41.587872840	1.05E-01	4.63E-04
11	41.653690400	41.590182850	41.653923810	6.35E-02	2.33E-04
12	41.680970240	41.644512660	41.681083510	3.65E-02	1.13E-04
13	41.692183220	41.671942570	41.692236830	2.02E-02	5.36E-05
14	41.696789640	41.685807150	41.696814550	1.10E-02	2.49E-05
15	41.698681500	41.692819150	41.681083510	5.86E-03	1.76E-02
16	41.699458580	41.696366500	41.699463750	3.09E-03	5.17E-06

17	41.699777660	41.698161360	41.699779980	1.62E-03	2.32E-06
18	41.699908690	41.699069580	41.699909730	8.39E-04	1.04E-06
19	41.699962500	41.699529170	41.699962960	4.33E-04	4.60E-07
20	41.699984600	41.699761740	41.699984800	2.23E-04	2.00E-07
21	41.699993680	41.699879430	41.699993770	1.14E-04	9.00E-08
22	41.699997400	41.699938980	41.699997440	5.84E-05	4.00E-08
23	41.699998930	41.699969120	41.699998950	2.98E-05	2.00E-08
24	41.699999560	41.699984380	41.699999570	1.52E-05	1.00E-08
25	41.699999820	41.699992090	41.699999820	7.73E-06	0.00E+00

The graphical representations of this model reveal that across both organisms, the Taylor Series exhibited considerably larger errors than the RKF method. The difference was more pronounced for *P. Caudatum*, with its maximum absolute error for Taylor Series reaching 4.80, whereas it's only 0.0214 for RKF. Similarly, for *S. Pustulata*, the Taylor Series had a maximum error of 0.558, in contrast to RKF's mere 0.0176. In biological modeling, accuracy is paramount. Large errors, like those exhibited by the Taylor Series, could lead to misinterpretations, potentially impacting decision-making, or interventions. It can be accepted that the RKF method is the most reliable approximation method than the Taylor Series method for solving the Lotka-Volterra competitive model. This can be applied for both species. The table demonstrates that populations of both species rise until they achieve a stable equilibrium, where their population levels remain constant for intraspecific interaction. *P. Caudatum* reaches its maximum population size of 202 cells on day 16^{th}, while *S. Pustulata* reaches its maximum population size of 41 cells on day 8^{th}.

In this case, the couple equation (8) does not have an exact analytical solution, thus it is solved numerically by using *Mathematica 13.2* programming. Those results are presented in Table 3.

Table 3: Numerical solution for *P. Caudatum* and *S. Pustulata*.

Day	**Numerical Solution**	
	(*P. Caudatum* in mixed)	(*S. Pustulata* in mixed)
0	2	2
1	3.751451974	4.504472679
2	6.808330731	9.304627060
3	11.683631120	16.591242660
4	18.612572770	24.421613580
5	27.522162810	30.003024910

6	38.313686050	32.453811910
7	50.911572770	32.557535140
8	65.061172600	31.279097530
9	80.206496750	29.258908940
10	95.554528050	26.882724210
11	110.267244900	24.399495860
12	123.667222000	21.977749000
13	135.357183700	19.725021040
14	145.223933000	17.698338470
15	153.364466400	15.916144810
16	159.991369700	14.371408780
17	165.355793800	13.043100870
18	169.699790500	11.904482790
19	173.233788400	10.928200410
20	176.130265700	10.088946640
21	178.525818200	9.364558700
22	180.526588800	8.736231420
23	182.214318400	8.188295469
24	183.651802800	7.707823312
25	184.887408900	7.284194294

Table 3 demonstrates that *P. Caudatum* populations in mixed populations are continuously growing, while *S. Pustulata* populations in mixed populations are also continuously growing but start to decrease after day 8^{th}. From day 1^{st} until day 5^{th}, *S. Pustulata* is outnumbered *P. Caudatum*. Starting from day 6^{th}, *S. Pustulata* started being driven out by *P. Caudatum*.

There are four different cases in stability of competition that are being studied, which are species 1 (*P. Caudatum*) win, species 2 (*S. Pustulata*) win, unstable equilibrium, and coexistence of both species.

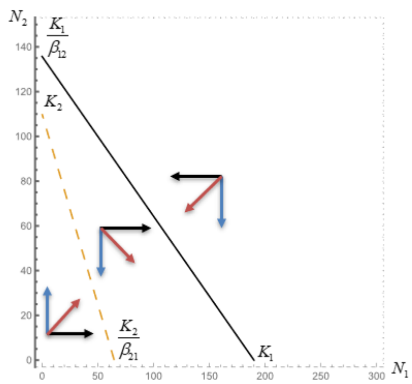

Figure 5: Case I (Species 1 win)

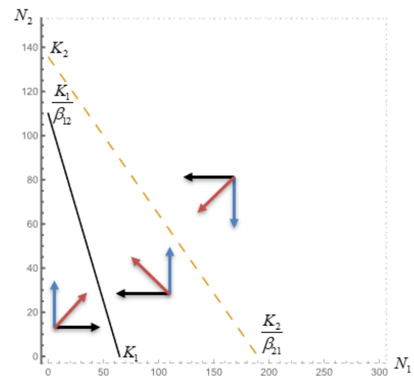

Figure 6: Case II (Species 2 win)

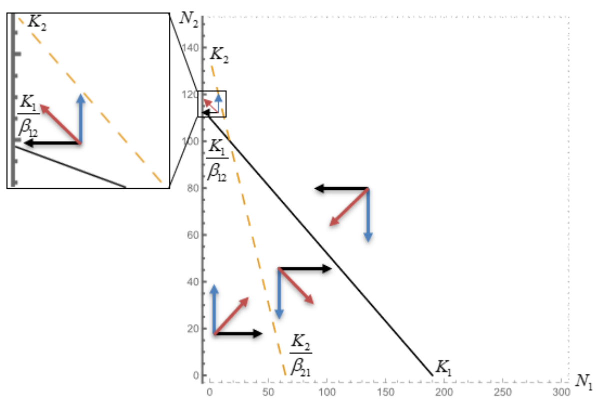

Figure 7: Case III (Unstable equilibrium)

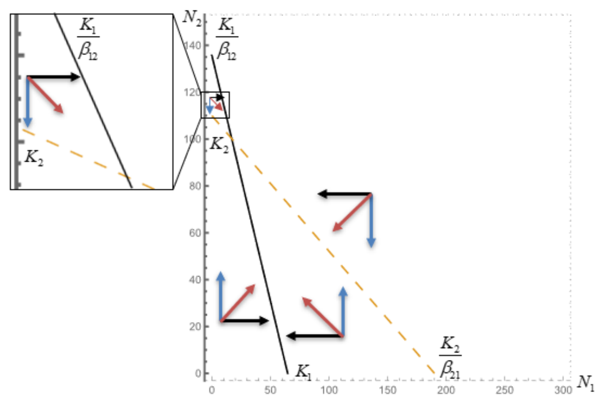

Figure 8: Case IV (Coexistence of both species)

Figure 5 demonstrates a graph where there is a situation in which species 1 will win in a competition for case I. *P. Caudatum* and *S. Pustulata* will increase where both isoclines are below them, whereas they will decrease if both isoclines are above them. *P. Caudatum* will increase and *S. Pustulata* will decrease at the point below *P. Caudatum*'s isocline and above *S. Pustulata*'s isocline. This trajectory will continue until *P. Caudatum* can be stable at its own carrying capacity, while *S. Pustulata* is driven to extinction. Figure 6 shows a graph where there is a situation which species 2 will win in a competition for case II. The point which at above and below both isoclines are the same as case I. However, *S. Pustulata* will increase and *P. Caudatum* will decrease at the point below *S. Pustulata*'s isocline and above *P. Caudatum*'s isocline. In the meantime, *P. Caudatum* will be driven to extinction, and this trajectory will last until *S. Pustulata* can stable at its own carrying capacity.

Figure 7 shows that there is also a situation called competitive exclusion or an unstable equilibrium in a competition for case III. The point which at above and below both isoclines are the same as the other cases. The competitive exclusion of *S. Pustulata* by *P. Caudatum* happens at the point that is below *P. Caudatum*'s isocline and above *S. Pustulata*'s isocline. However, the competitive exclusion of

P. Caudatum by *S. Pustulata* happens at the point below *S. Pustulata*'s isocline and above *P. Caudatum*'s isocline. It demonstrates that there is an unstable equilibrium, which one species outcompetes the other, resulting the extinction of the weaker species at the point where the two isoclines intersect, and it depends on the initial numbers of *P. Caudatum* and *S. Pustulata*. Figure 8 shows that there are stable equilibrium points, which represent a balanced coexistence, where the populations of both species remain constant over time for case IV. In this case, all the population trajectories will leave both species' populations at the intersection of the isoclines. The point is on both isoclines which neither population will grow any further and they will stabilize at this equilibrium. This is the only case in which competitive exclusion does not occur. There is a stable equilibrium point at the intersection of both isoclines where the two species can coexist without extinction when intraspecific competition is larger than interspecific competition.

4 CONCLUSION AND RECOMMENDATIONS

In summary, this research offers crucial insights into the use of numerical approximation methods, such as the Taylor Series and RKF method in addressing the Lotka-Volterra competitive model. Using data from Gause's experiment involving *P. Caudatum* and *S. Pustulata*, across both *P. Caudatum* and *S. Pustulata* models, the RKF method proved to be markedly more accurate than the Taylor Series. When prioritizing precision in biological modeling, RKF emerges as the more reliable choice between the two. In sum, while the Taylor Series may offer simplicity, its compromised accuracy, especially when compared to the RKF method, makes it less suitable for precise biological modeling of the organisms in question. Observations on the species' interactions, both intra- and interspecific, reveal dynamic population growths that stabilize over time, with *P. Caudatum* persistently outnumbering *S. Pustulata* in mixed populations due to competitive exclusion. The study underscores the critical role of carrying capacities in these outcomes. Furthermore, the commercial application of understanding these species' interactions could provide vital insights into freshwater ecology, serve as educational tools, and possibly aid in the development of bioindicators for assessing water quality. In essence, this study enriches our understanding of numerical approximation methods and their utility in ecological research, emphasizing the necessity for precise modeling and prediction of species interactions, population behavior, stability, equilibrium, and competitive behavior for effective ecological management and conservation. this research suggests a preference for the RKF method over the Taylor series method in solving the Lotka-Volterra competitive model due to its higher accuracy and adaptability. To enhance realism, models should be calibrated and validated using specific data of the populations being studied. Sensitivity analyses are recommended to ascertain the effect of parameter changes on numerical solutions, and environmental parameters should be included in the model to improve accuracy. The research also proposes comparing numerical solutions with real-world data to evaluate the model's predictions. Finally, the study encourages the application of the discussed numerical methods in studying interactions beyond the examined species, demonstrating their flexibility and applicability in analyzing ecological systems.

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REFERENCES

[1] J. C. Wildenberg, J. A. Vano and J. C. Sprott, “Complex spatiotemporal dynamics in Lotka–Volterra ring systems,” *Ecological Complexity*, vol. 3, no. 2, pp. 140–147, 2006.

[2] B. Batiha, “Comparison of numerical methods for solving one species Lotka Volterra equation,” *Analele Universităţii din Oradea. Fascicola Matematică*, vol. 19, pp. 243–253, 2012.

[3] L. Zhou, T. Wang, X. Lyu and J. Yu, “A modified Lotka–Volterra model for the evolution of coordinate symbiosis in energy enterprise,” *IOP Conference Series Earth and Environmental Science*, vol. 113, no. 1, 2018, http://doi.org/10.1088/1755-1315/113/1/012094.

[4] Y. Aksoy, U. Goktas, M. Pakdemirli and I. T. Dolapci, “Application of perturbation-iteration method to Lotka–Volterra equations,” *Alexandria Engineering Journal*, vol. 55, no. 2, 1661–1666, 2016, http://doi.org/10.1016/j.aej.2016.02.015.

[5] N. S. A. B. Razali and F. A. Abdullah,” Numerical methods for competitive hunters model,” *AIP Conference Proceedings*, vol. 140, no. 2013, 2013, 1522. https://doi.org/10.1063/1.4801116.

[6] S. Paul, S. P. Mondal, P. Bhattacharya and K. Chaudhuri, “Some comparison of solutions by different numerical techniques on mathematical biology problem,” *International Journal of Differential Equations*, vol. 2016, pp. 1–14, 2016, https://doi.org/10.1155/2016/8921710.

[7] M. J. Baines and K. Christou, “A moving-mesh Finite-Difference Method for segregated two-phase competition-diffusion,” *Mathematics*, vol. 9, no. 4, pp. 386, 2021, https://doi.org/10.3390/math9040386.

[8] E. Gokmen, O. R. Isik and M. Sezer, “Taylor collocation approach for delayed Lotka–Volterra predator–prey system,” *Applied Mathematics and Computation*, vol. 268, no. 2, pp. 671–684, 2015, http://dx.doi.org/10.1016/j.amc.2015.06.110.

[9] Y. A. Sabawi, M. A. Pirdawood and M. I. Sadeeq, “A compact fourth-order Implicit-Explicit Runge-Kutta Type method for solving diffusive Lotka Volterra system,” *Journal of Physics: Conference Series*, vol. 1999, no. 1, 2021, https://doi.org/10.1088/1742-6596/1999/1/012103.

[10] S. Paul, S. P. Mondal and P. Bhattacharya, “Numerical solution of Lotka Volterra prey predator model by using Runge–Kutta–Fehlberg Method and Laplace Adomian Decomposition Method,” *Alexandria Engineering Journal*, vol. 55, no. 1, pp. 613–617, 2016 https://doi.org/10.1016/j.aej.2015.12.026.

[11] A. A. Adeniji, S. E. Fadugba and M. Y. Shatalov, “Comparative analysis of Lotka-Volterra type

models with numerical methods using residuals in Mathematic,”. *Communication Mathematical Biology and Neuroscience*, vol. 2022, 2022, https://doi.org/10.28919/cmbn/7346.

[12] R. E. Mickens, “A nonstandard finite-difference scheme for the Lotka Volterra system,” *Applied Numerical Mathematics*, vol. 45, no. 2-3, pp. 309–314, 2003, https://doi.org/10.1016/s0168-9274(02)00223-4.

[13] N. H. Aljahdaly and M. A. Alqudah, “Analytical solutions of a modified Predator-Prey model through a new ecological interaction,” *Computational and Mathematical Methods in Medicine*, vol. 2019, pp. 1–7, 2019, https://doi.org/10.1155/2019/4849393.

[14] G. F. Gause, *Competition For Common Food in Protozoa* in Gause, G. F. (Ed), The Struggle for Existence, pp. 90–113, Baltimore, Williams and Wilkins Company, 1934.