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Solving Fuzzy Volterra Integral Equations via Fuzzy Sumudu Transform

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ABSTRACT

Fuzzy integral equations (FIEs) topic is an important branch in fuzzy mathematics. However, the methods proposed for handling FIEs are still very limited and often involve complex calculations. This paper introduced fuzzy Sumudu transform (FST) for solving FIEs, specifically fuzzy Volterra integral equations (FVIEs). For this purpose, a step-by-step procedure will be constructed for solving FVIEs. In order to illustrate the practicability of the method, two numerical examples will be given.

Keywords: fuzzy integral equations; Sumudu transform; Volterra integral equations.

1. INTRODUCTION

Fuzzy integral equations (FIEs) have drawn attention among researchers since the rising of the fuzzy control topic in recent years. The concept of integration of fuzzy functions was initially coined out by Dubois and Prade [1]. Later, it was discussed and explored by several authors [2, 3]. One of the earliest contributions on the application of FIEs was done by Wu and Ma [4], where they proved the existence of a unique solution for fuzzy Fredholm integral equations. It was then followed by a monograph discussing on the existence of solutions of FIEs in Banach spaces [5]. In [6], the authors solved linear fuzzy Fredholm integral equations of the second kind. This was done by considering the fuzzy number parametrically, and from this, the linear fuzzy Fredholm integral equation is separated into two crisp Fredholm integral equations. Recent works have been introducing methods for handling FIEs such as in [7] where the authors solved linear and nonlinear Abel FIEs using homotopy analysis method and in [8], the authors introduced Quadrature formula for solving FIEs. While in [9], fuzzy Laplace transform has been used to find the solution of FVIEs of the second kind.

One of the most recent methods in handling problems modelled under fuzzy environment is fuzzy Sumudu transform (FST). FST has been used for solving various kinds of fuzzy differential equations with fuzzy initial values. This includes handling solutions of fuzzy differential equations of integer order [10, 11], fuzzy fractional differential equations [12] and fuzzy partial differential equations [13]. By using FST, the problem is reduced to algebraic problem which is much simpler to be solved. This fall under the topic of operational calculus and under this topic, FST is considered as one of a powerful method alongside with fuzzy Laplace transform. One of the advantage of FST over fuzzy Laplace transform is that the variable u in the fuzzy Sumudu transformed function $G(u)$ is treated as the replica of the variable t in the original function $f(t)$. Whereas in fuzzy Laplace transform, the variable s in the fuzzy Laplace transformed function $F(s)$ is only treated as a dummy [14].

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This paper focused on utilizing FST for solving FIEs, particularly fuzzy Volterra integral equations (FVIEs) of the second kind. Therefore, the FVIEs will be converted into two crisp Volterra integral equations by using the concept of lower and upper bound of parametric fuzzy number. This will lead to the lower and upper function solutions of FVIEs. Theorems and properties of FST in [11] will be used in this paper, especially the theorem of FST for convolution terms.

This paper is organized as follows. In the next section, some preliminaries on fuzzy numbers and fuzzy functions will be revisited. Section 3 provides the previously proposed FST and its convolution theorem. Later in Section 4, a detailed procedure for solving FVIEs using FST is constructed. While in Section 5, two numerical examples are provided to demonstrate the proposed method. Finally in Section 6, conclusion is drawn.

2. PRELIMINARIES

In this section, several important definitions and properties shall be recall to provide better understanding of this paper. The notations $\mathbb{R}$ and $\mathcal{F}(\mathbb{R})$ denote the real number and fuzzy real number, respectively, throughout this paper.

2.1 Fuzzy Numbers

Definition 1

[15] A fuzzy number is a mapping $\widetilde{u}:\mathbb{R}\to[0,1]$ that satisfies the following conditions:

i. For every $\widetilde{u}\in\mathcal{F}(\mathbb{R})$, $\widetilde{u}$ is upper semi continuous,
ii. for every $\widetilde{u}\in\mathcal{F}(\mathbb{R})$, $\widetilde{u}$ is fuzzy convex, i.e., $\widetilde{u}(\gamma x+(1-\gamma)y)\geq\min\{\widetilde{u}(x),\widetilde{u}(y)\}$ for all $x,y\in\mathbb{R}$, and $\gamma\in[0,1]$,
iii. for every $\widetilde{u}\in\mathcal{F}(\mathbb{R})$, $\widetilde{u}$ is normal, i.e., $\exists x_0\in\mathbb{R}$ for which $\widetilde{u}(x_0)=1$,
iv. $\text{supp }\widetilde{u}=\{x\in\mathbb{R}|\widetilde{u}(x)>0\}$ is the support of $\widetilde{u}$, and it has compact closure $\text{cl}(\text{supp }\widetilde{u})$.

Definition 2

[16] Let $\widetilde{u}\in\mathcal{F}(\mathbb{R})$ and $\alpha\in[0,1]$. The α-level set of $\widetilde{u}$ is the crisp set $\widetilde{u}^\alpha$ that contains all the elements with membership degree greater than or equal to α, i.e.

$$\widetilde{u}^\alpha=\{x\in\mathbb{R}|\widetilde{u}(x)\geq\alpha\},$$

where $\widetilde{u}^\alpha$ denotes α-level set of fuzzy number $\widetilde{u}$.

Hence, it is clear that the α-level of any fuzzy number is bounded and closed. It will be denoted by $[\underline{u}^\alpha,\overline{u}^\alpha]$, for both lower and upper bound of $\widetilde{u}^\alpha$, respectively, in this paper.

Definition 3

[17, 18] A parametric form of an arbitrary fuzzy number $\widetilde{u}$ is an ordered pair $[\underline{u}^\alpha,\overline{u}^\alpha]$ of functions $\underline{u}^\alpha$ and $\overline{u}^\alpha$, for any $\alpha\in[0,1]$ that fulfil the following conditions.

i. $\underline{u}^\alpha$ is a bounded left continuous monotonic increasing function in $[0,1]$,
ii. $\overline{u}^\alpha$ is a bounded left continuous monotonic decreasing function in $[0,1]$,
iii. $\underline{u}^\alpha\leq\overline{u}^\alpha$.

Certain membership functions can be used to classify fuzzy numbers, some of which are the triangular, trapezoidal, Gaussian and generalized bell membership function. For example, triangular fuzzy number can be represented by the triplets (a_1, a_2, a_3) and the α-level is computed as follows [19].

$$\widetilde{u}^{\alpha} = [a_1 + (a_2 - a_3)\alpha, a_3 - (a_3 - a_2)\alpha], \quad \alpha \in [0,1].$$

For operations between fuzzy numbers, please see [17].

Theorem 1

[20] Let the fuzzy function $\widetilde{f} : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ represented by $[\underline{f}^{\alpha}(x), \overline{f}^{\alpha}(x)]$. For any $\alpha \in [0,1]$, assume that $\underline{f}^{\alpha}(x)$ and $\overline{f}^{\alpha}(x)$ are both Riemann-integrable on $[a,b]$ and assume that there are two positive $\underline{M}^{\alpha}$ and $\overline{M}^{\alpha}$ where $\int_a^b |\underline{f}^{\alpha}(x)|dx \leq \underline{M}^{\alpha}$ and $\int_a^b |\overline{f}^{\alpha}(x)|dx \leq \overline{M}^{\alpha}$, for every $b \geq a$. Then, $\widetilde{f}(x)$ is improper fuzzy Riemann-integrable on $[a, \infty[$ and the improper fuzzy Riemann-integrable is a fuzzy number. Furthermore, we have

$$\int_a^{\infty} \widetilde{f}(x)dx = \left[\int_a^{\infty} \underline{f}^{\alpha}(x)dx, \int_a^{\infty} \overline{f}^{\alpha}(x)dx\right].$$

Definition 4

[21] A fuzzy function $\widetilde{f} : [a,b] \to \mathcal{F}(\mathbb{R})$ is said to be continuous at $x_0 \in [a,b]$ if for each $\epsilon > 0$, there is $\delta > 0$ such that $D(\widetilde{f}(x), \widetilde{f}(x_0)) < \epsilon$, whenever $x \in [a,b]$ and $|x - x_0| < \delta$. We say that $\widetilde{f}$ is continuous on $[a,b]$ if $\widetilde{f}$ is continuous at each $x_0 \in [a,b]$.

Next, we recall the fuzzy Fredholm integral equation, which is given by

$$\text{x(t)=f(t)+}\lambda \int_a^b k(t-\tau)x(\tau)d\tau,$$

where $\lambda > 0,\ k(t-\tau)$ is an arbitrary kernel function defined over the square $a \leq t,\ u \leq b$ and $f(t)$ is a given fuzzy function of $t \in [a,b]$. If the kernel function satisfies $k(t-\tau) = 0,\ \tau > t$, then the FVIE has the following general form [22].

$$\text{x(t)=f(t)+}\lambda \int_0^t k(t-\tau)x(\tau)d\tau, \qquad t \in [0,T], T < \infty. \tag{1}$$

The parametric form of Eq. (1) is as follows.

$$\begin{cases} \underline{x}^{\alpha}(t) = \underline{f}^{\alpha}(t) + \lambda \displaystyle\int_0^t \underline{k^{\alpha}(t-\tau)x^{\alpha}(\tau)}d\tau, \\ \overline{x}^{\alpha}(t) = \overline{f}^{\alpha}(t) + \lambda \displaystyle\int_0^t \overline{k^{\alpha}(t-\tau)x^{\alpha}(\tau)}d\tau, \end{cases}$$

where

$$\underline{k^{\alpha}(t-\tau)x^{\alpha}(\tau)} = \begin{cases} k^{\alpha}(t-\tau)\underline{x}^{\alpha}(\tau) & k^{\alpha}(t-\tau) > 0, \\ k^{\alpha}(t-\tau)\overline{x}^{\alpha}(\tau) & k^{\alpha}(t-\tau) < 0, \end{cases}$$

and

$$\overline{k^{\alpha}(t-\tau)x^{\alpha}(\tau)} \begin{cases} k(t-\tau)^{\alpha}\overline{x}^{\alpha}(\tau) & k^{\alpha}(t-\tau) > 0, \\ k(t-\tau)^{\alpha}\underline{x}^{\alpha}(\tau) & k^{\alpha}(t-\tau) < 0, \end{cases}$$

for $\alpha \in [0,1]$. It is assumed that zero does not exists in support [9]. Further discussions on this matter can be seen in [17].

The next section provides the previously constructed FST. In addition, the convolution theorem of FST is also presented.

3. FUZZY SUMUDU TRANSFORM

In order to construct the solution for solving FVIEs, the definition and convolution theorem of FST is adopted [11].

Definition 5

[11] Let $\widetilde{f} : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ be a continuous fuzzy function. Suppose that $\widetilde{f}(ux) \odot e^{-x}$ is improper fuzzy Riemann-integrable on $[0, \infty[$, then $\int_0^\infty \widetilde{f}(ux) \odot e^{-x} dx$ is called fuzzy Sumudu transform and it is denoted by

$$\mathrm{G(u)=S}[\widetilde{f}(x)](u) = \int_0^\infty \widetilde{f}(ux) \odot e^{-x} dx, \quad u \in [-\tau_1, \tau_2],$$

where the variable u is used to factor the variable x in the argument of the fuzzy function and $\tau_1, \tau_2 > 0$.

FST also can be presented parametrically as follows

$$\mathrm{S}[\widetilde{f}(x)](u) = [s[\underline{f}^\alpha(x)](u), s[\overline{f}^\alpha(x)](u)].$$

Theorem 2

[11] Let $f, g : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ be two continuous fuzzy-valued functions. Let $M(u)$ and $N(u)$ be fuzzy Sumudu transforms for f and g respectively. Then, the fuzzy Sumudu transform of the convolution of f and g,

$$\mathrm{(f*g)(x)}=\int_0^t f(\tau)g(t-\tau)d\tau,$$

is given by

$$\mathrm{S[(f*g)(x)]=uM(u)N(u)}.$$

To explore other theorems of FST, please see in [10, 11].

4. FUZZY SUMUDU TRANSFORM FOR FUZZY VOLTERRA INTEGRAL EQUATIONS

This section presents the attempt to find the solution of FVIEs by using FST. The procedure is described in details as follows.

First, consider the Eq. (1). By using FST on both sides of the equation, the following equation was obtained

$$\mathcal{S}[x(t)] = \mathcal{S}[f(t)] + \lambda \mathcal{S}[\int_0^t k(t-\tau)x(\tau)d\tau], \qquad t \in [0,T], T < \infty,$$

where it can also be represented parametrically as follows.

$$s[\underline{x}^{\alpha}(t)] = s[\underline{f}^{\alpha}(t)] + \lambda u\underline{s\,[k^{\alpha}(t)]s[x^{\alpha}(t)]}, \tag{2}$$

and

$$s[\overline{x}^{\alpha}(t)] = s[\overline{f}^{\alpha}(t)] + \lambda u\overline{s\,[k^{\alpha}(t)]s[x^{\alpha}(t)]}. \tag{3}$$

Then, we may discuss the cases involved when handling $\underline{s\,[k^{\alpha}(t)]s[x^{\alpha}(t)]}$ and $\overline{s\,[k^{\alpha}(t)]s[x^{\alpha}(t)]}$. From Eq. (2) and (3), we have two cases as the following.

Case 1:

If $k^{\alpha}(t)$ is positive, then

$$s[\underline{x}^{\alpha}(t)] = s[\underline{f}^{\alpha}(t)] + \lambda us\,[k^{\alpha}(t)]s[\underline{x}^{\alpha}(t)]\,,$$

and

$$s[\overline{x}^{\alpha}(t)] = s[\overline{f}^{\alpha}(t)] + \lambda us\,[k^{\alpha}(t)]s[\overline{x}^{\alpha}(t)].$$

Then, we have

$$\mathrm{s}[\underline{\mathrm{x}}^{\alpha}(t)](1 - \lambda us[k^{\alpha}(t)]) = s[\underline{f}^{\alpha}(t)],$$

and

$$\mathrm{s}[\overline{x}^{\alpha}(t)](1 - \lambda us[k^{\alpha}(t)]) = s[\overline{f}^{\alpha}(t)].$$

From that we obtain $s[\underline{x}^{\alpha}(t)]$ and $s[\underline{x}^{\alpha}(t)]$ as follows:

$$\mathrm{s}[\underline{\mathrm{x}}^{\alpha}(t)] = \frac{s[\underline{f}^{\alpha}(t)]}{(1 - \lambda us[k^{\alpha}(t)])},$$

and

$$\mathrm{s}[\overline{x}^{\alpha}(t)] = \frac{s[\overline{f}^{\alpha}(t)]}{(1 - \lambda us[k^{\alpha}(t)])}.$$

By using the inverse of FST,

$$\underline{\mathrm{x}}^{\alpha}(t) = s^{-1}\left(\frac{s[\underline{f}^{\alpha}(t)]}{(1 - \lambda us[k^{\alpha}(t)])}\right),$$

and

$$\overline{x}^{\alpha}(t) = s^{-1}\left(\frac{s[\overline{f}^{\alpha}(t)]}{(1 - \lambda us[k^{\alpha}(t)])}\right).$$

Case 2: If $k^{\alpha}(t)$ is negative, then

$$\mathrm{s}[\underline{\mathrm{x}}^{\alpha}(t)] = s[\underline{f}^{\alpha}(t)] + \lambda us\,[k^{\alpha}(t)]s[\overline{x}^{\alpha}(t)]\,,$$

and

$$\mathrm{s}[\overline{x}^{\alpha}(t)] = s[\overline{f}^{\alpha}(t)] + \lambda us\,[k^{\alpha}(t)]s[\underline{x}^{\alpha}(t)].$$

By using similar steps as in Case 1, we have $s[\underline{x}^{\alpha}(t)]$ and $s[\overline{x}^{\alpha}(t)]$ as follows:

$$\mathrm{s}[\underline{\mathrm{x}}^{\alpha}(t)] = \frac{s[\underline{f}^{\alpha}(t)] + \lambda us[k^{\alpha}(t)]s[\overline{f}^{\alpha}(t)]}{(1-\lambda^2u^2s[k^{\alpha}(t)]s[k^{\alpha}(t)])},$$

and

$$\mathrm{s}[\underline{\mathrm{x}}^{\alpha}(t)] = \frac{s[\underline{f}^{\alpha}(t)] + \lambda us[k^{\alpha}(t)]s[\overline{f}^{\alpha}(t)]}{(1-\lambda^2u^2s[k^{\alpha}(t)]s[k^{\alpha}(t)])}.$$

By using the inverse of FST,

$$\underline{\mathrm{x}}^{\alpha}(t) = s^{-1}\left(\frac{s[\underline{f}^{\alpha}(t)] + \lambda us[k^{\alpha}(t)]s[\overline{f}^{\alpha}(t)]}{(1-\lambda^2u^2s[k^{\alpha}(t)]s[k^{\alpha}(t)])}\right),$$

and

$$\overline{x}^{\alpha}(t) = s^{-1}\left(\frac{s[\overline{f}^{\alpha}(t)] + \lambda us[k^{\alpha}(t)]s[\underline{f}^{\alpha}(t)]}{(1-\lambda^2u^2s[k^{\alpha}(t)]s[k^{\alpha}(t)])}\right).$$

5. EXAMPLE

Example 1

First, we consider the following integral equation.

$$\mathrm{x(t)}=[\alpha+1, 3-\alpha] + \lambda \int_0^t (\tau - t)x(\tau)d\tau, \qquad t \in T = [a,b]. \tag{4}$$

By using FST on both sides of Eq. (4),

$$\mathrm{S[x(t)]}=\mathrm{S}[[\alpha+1, 3-\alpha]] + \lambda u\mathcal{S}[(-t)]\mathcal{S}[x(t)]. \tag{5}$$

By using Theorem 2, we have Eq. (5) parametrically,

$$\mathrm{s}[\underline{\mathrm{x}}(\mathrm{t})]=\mathrm{s}[\alpha+1] + \lambda u\underline{s[(-t)]s[x(t)]},$$

and

$$\mathrm{s}[\overline{x}(t)] = s[3-\alpha] + \lambda u\overline{s[(-t)]s[x(t)]}.$$

Since $k(t-s)=-t$ is less than zero, we only consider Case 2, as in the previous section. So,

$$\mathrm{s}[\underline{\mathrm{x}}(\mathrm{t})]=\mathrm{s}[\alpha+1]+\lambda us[(-t)]s[\overline{x}(t)], \tag{6}$$

and

$$\mathrm{s}[\overline{x}(t)]=s[3-\alpha]+\lambda us[(-t)]s[\underline{x}(t)]. \tag{7}$$

Solving Eq. (6) and (7), we have $s[\underline{x}^{\alpha}(t)]$ and $s[\overline{x}^{\alpha}(t)]$ as follows:

$$\mathrm{s}[\underline{\mathrm{x}}^{\alpha}(t)]=(\alpha+1)\left[\frac{1}{2(1+\lambda u^2)}+\frac{1}{2(1-\lambda u^2)}\right]-(3-\alpha)\lambda\left[\frac{1}{2\lambda(1-\lambda u^2)}-\frac{1}{2\lambda(1+\lambda u^2)}\right],$$

and

$$\mathrm{s}[\underline{\mathrm{x}}^{\alpha}(t)]=(3-\alpha)\left[\frac{1}{2(1+\lambda u^2)}+\frac{1}{2(1-\lambda u^2)}\right]-(\alpha+1)\lambda\left[\frac{1}{2\lambda(1-\lambda u^2)}-\frac{1}{2\lambda(1+\lambda u^2)}\right].$$

By using the inverse of FST,

$$\underline{\mathrm{x}}^{\alpha}(t)=(\alpha+1)\left[\frac{\cos(\sqrt{\lambda}t)}{2}+\frac{\cosh(\sqrt{\lambda}t)}{2}\right]-(3-\alpha)\lambda\left[\frac{\cosh(\sqrt{\lambda}t)}{2\lambda}-\frac{\cos(\sqrt{\lambda}t)}{2\lambda}\right],$$

and

$$\overline{x}^{\alpha}(t)=(3-\alpha)\left[\frac{\cos(\sqrt{\lambda}t)}{2}+\frac{\cosh(\sqrt{\lambda}t)}{2}\right]-(\alpha+1)\lambda\left[\frac{\cosh(\sqrt{\lambda}t)}{2\lambda}-\frac{\cos(\sqrt{\lambda}t)}{2\lambda}\right].$$

The results obtained are then illustrated in Figure 1. For the illustration purpose, the value of λ is set to 1.

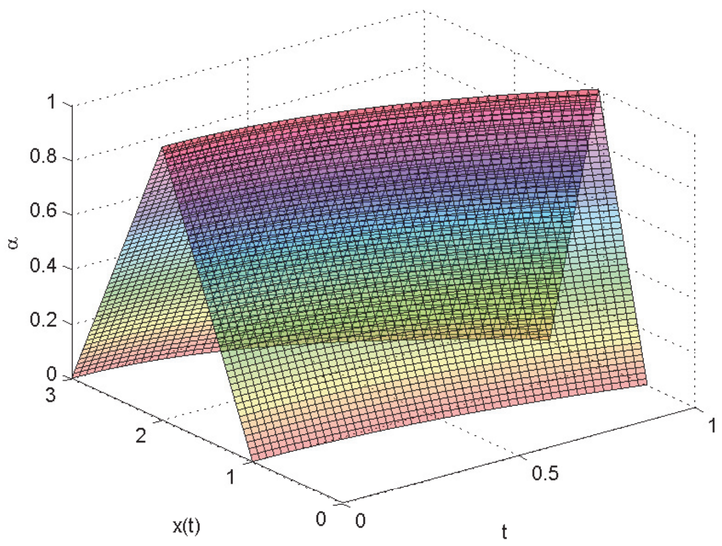

Figure 1. The solution of $x(t)$ for Equation (4).

Example 2

In this example, we consider the following integral equation.

$$\mathrm{x(t)}=[2+\alpha, 4-\alpha]t + \lambda \int_0^t (t-\tau)x(\tau)d\tau, \qquad t \in T = [a,b]. \tag{8}$$

By using FST on both sides of Eq. (8),

$$\mathrm{S[x(t)]=S}[(2+\alpha, 4-\alpha)t] + \lambda u \mathcal{S}[(t)]\mathcal{S}[x(t)],$$

Since $k(t-s) = t$ is more than zero, we only consider Case 1, as in the previous section. So, we obtain $s[\underline{x}^\alpha(t)]$ and $s[\overline{x}^\alpha(t)]$ as follows.

$$\mathrm{s}[\underline{\mathrm{x}}^\alpha(t)] = \frac{[2+\alpha]u}{(1-\lambda u^2)},$$

and

$$\mathrm{s}[\overline{x}^\alpha(t)] = \frac{[4-\alpha]u}{(1-\lambda u^2)}.$$

By using the inverse of FST,

$$\underline{\mathrm{x}}^\alpha(t) = [2+\alpha]\frac{1}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}t),$$

and

$$\overline{x}^\alpha(t) = [4-\alpha]\frac{1}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}t).$$

The results obtained are then illustrated in Figure 2. Similar to Figure 1, the value of λ is set to 1.

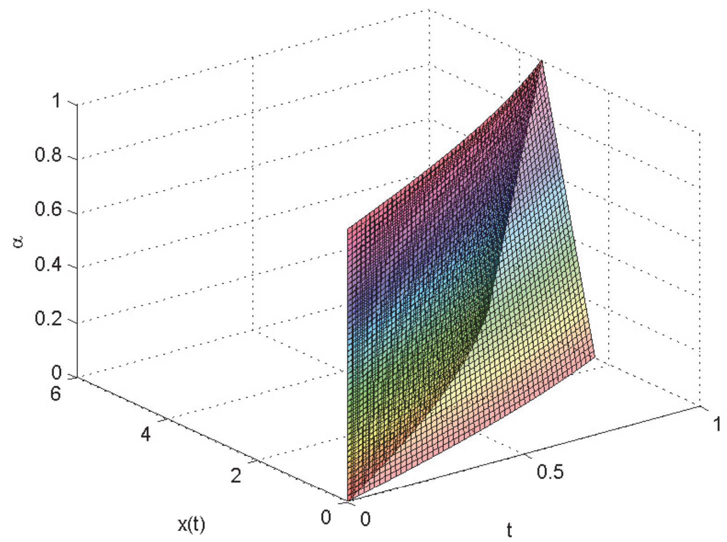

Figure 2. The solution of $x(t)$ for Equation (8).

CONCLUSION

In this paper, FST have been proposed for solving FVIEs of the second kind. For this purpose, a detailed procedure for handling FVIEs has been constructed. To demonstrate the functionality of the method proposed, two numerical examples have been solved.

REFERENCES

[1] D. Dubois and H. Prade, Fuzzy Sets and Systems **8**, 1-17, 106-111, 225-233 (1982).
[2] J. Mordeson and W. Newman, Information Sciences **87**(4), 215-229 (1995).
[3] J. J. Buckley, E. Eslami, and T. Feuring, "Fuzzy Integral Equations," in *Fuzzy mathematics in Economics and Engineering*, (Springer, 2002), pp. 229-241.
[4] W. Congxin and M. Ming, Journal of Harbin Institute of Technology **21**, 11-19 (1990).
[5] J. Y. park, Y. C. Kwun, and J. U. Jeong, Fuzzy Sets and Systems **72**(3), 373-378 (1995).
[6] S. Abbasbandy, E. Babolian, and M. Alavi, Chaos, Solitons & Fractals **31**(1), 138-146 (2007).
[7] F. Mirzaee, M. K. Yari, and M. Paripour, Journal of Taibah University for Science **9**(1), 104-115 (2015).
[8] A. Taherpour, A Hosseinpour, A. Hosseinpour, and L. Abasabadi, Indian Journal of Science and Technology **9**(28), 1-6 (2016).
[9] S. Salahshour, M. Khazerloo, S. Hajighasemi, and M. Khorasany, International Journal of Industrial Mathematics **4**(1), 21-29 (2012).
[10] M. Z. Ahmad and N. A. Abdul Rahman, International Journal of Applied Physics and Mathematics **5**(2), 86-93 (2015).
[11] N. A. Abdul Rahman and M. Z. Ahmad, Entropy **17**(7), 4582-4601 (2015).
[12] N. A. Abdul Rahman and M. Z. Ahmad, Journal of Nonlinear Sciences and Applications **10**, 2620-2632 (2017).
[13] N. A. Abdul Rahman and M. Z. Ahmad, Journal of Nonlinear Science and Applications **9**(5), 3226-3239 (2016).
[14] G. K. Watugala, International Journal of Mathematical Education in Science and Technology **24**(1), 35-43 (1993).
[15] L. A. Zadeh, Information and Control **8**(3), 338-353 (1965).
[16] O. Kaleva, Nonlinear Analysis: Theory, methods & Applications **64**(5), 895-900 (2006).
[17] M. Friedman, M. Ma, and A. Kandel, Fuzzy Sets and Systems **106**(1), 35-48 (1999).
[18] M. Ma, M. Friedman, and A. Kandel, Fuzzy Sets and Systems **105**(1), 133-138 (1999).
[19] A. Kaufmann and M. M. Gupta, *Introduction to Fuzzy Arithmetic: Theory and Applications* (Van Nostrand Reinhold, New York, 1985).
[20] H. C. Wu, Information Sciences **111**(1), 109-137 (1998).
[21] Z. Guang-Quan, Fuzzy Sets and Systems **43**(2), 159-171 (1991).
[22] M. Ghanbari, International Journal of Industrial Mathematics **2**(2), 73-87 (2010).

APPENDIX

If any, the appendix should appear directly after the references without numbering, and on a new page.