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Asymptotic Behavior of Nonlinear Operator

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ABSTRACT

A quadratic stochastic operator (QSO) describes the time evolution of different species in biology. QSOs are the simplest non-linear operators however the main problem with a non-linear operator is its behavior. The behavior of non-linear operator has not been studied in depth; even QSOs, which are the simplest non-linear operators, have not been studied thoroughly. This paper investigates the global behavior of an operator V_a *taken from* $\xi^{(s)}$ *-QSO when the parameter* $a \in \{0, 0.5, 1\}$.

Keywords: quadratic stochastic operator; non-linear operators; asymptotic behavior.

1. INTRODUCTION

The first appearance of quadratic stochastic operator (QSO) was in Bernstein's work [1]. Many different fields have been using the properties of dynamical, QSO, as feed up of analysis. For examples, physics [24,32], economics, mathematics [15,17,20,33] and biology [1,14,15,18,19,20,30,35].

The QSO is generally used to present time evolution of species in biology. In [1], the system of QSO related to genetic population has been investigated. The system grows as follows. Suppose a population that contains of m species (traits) $1, 2, \cdots, m$. We denote a set of all species by $I = \{1, 2, \cdots, m\}$, the probability distribution of species at an initial state denoted by $x^{(0)} = \left(x_1^{(0)}, \cdots, x_m^{(0)}\right)$. The coefficient $p_{ij,k}$ means the probability that individual in i^{th} and j^{th} species hybridize to produce an individual from k^{th} species. Thus, the probability distribution of the species in the first generation, namely $x^{(1)} = \left(x_1^{(1)}, \cdots, x_m^{(1)}\right)$ can be calculated as a total probability i.e.,

$$x_k^{(1)} = \sum_{i,j=1}^{m} P_{ij,k} x_i^{(0)} x_j^{(0)}, \quad k = \overline{1, m}.$$

Consequently, this formula means that the relation $x^{(0)} \to x^{(1)}$ find a mapping V which known as evolution operator. The population is develop by starting from an arbitrary state $x^{(0)}$ then passing to the state $x^{(1)} = V(x^{(0)})$ (known as first generation), after that to the state

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$x^{(2)} = V(x^{(1)}) = V(V(x^{(0)})) = V^{(2)}\left(x^{(0)}\right)$ (called second generation) and so on. Hence, the discrete dynamical system discussed the population system evolution states by the following.

$$x^{(0)}, \quad x^{(1)} = V\left(x^{(0)}\right), \quad x^{(2)} = V^{(2)}\left(x^{(0)}\right), \quad \cdots.$$

In other meaning, if the distribution of the current generation is given, then the QSO can characterize the distribution probability of the next generation. In [20], the most interesting application of QSO to population genetics were provided. The open problem and recent achievement in the theory of QSO was provided in [11]. Researcher usually tries to study the behavior of non-linear operators which is considered as the main problem in non-linear operator but this problem has not been fully studied. The reason is that the problem depends on the given cubic matrix $(P_{ijk})_{i,j,k=1}^{m}$. An asymptotic behavior of QSO even in small dimensional is simplex [6, 30, 31, 33, 34].

Many researchers devoted their study to introduce a special class of QSO and investigated its behavior such as F-QSO [28], Volterra-QSO [7, 8, 9, 16, 33], permutated Volterra-QSO [12, 13], ℓ-Volterra-QSO [26,27], Quasi-Volterra-QSO [5], non-Volterra-QSO [6, 31], strictly non-Volterra-QSO [29] and non-Volterra operators which produced by measurements [3, 4, 25]. However, all these classes together would not cover a system of QSOs. Yet, there are many classes of QSO need to study. Recently, [21, 23, 36] introduced $\xi^{(as)}$-QSO which is a new class of QSO that depend on a partition of the coupled index set (which have couple traits) $\mathbf{P}_m = \{(i,j) : i < j\} \subset I \times I$. In case of 2D simplex $(m = 3)$, $\mathbf{P}_3$ have five possible partitions. Studies by [23, 21, 36, 37, 38] investigated the $\xi^{(as)}$-QSO that correspond to the point partition and the studies also examined the dynamics.

In [36], the $\xi^{(s)}$-QSO related to $|\xi| = 2$ was investigated. Moreover, 36 operators were described and the operators were classified into 20 non-conjugate classes. Despite that, the dynamics of the classes are not fully studied. Therefore, this paper will described the dynamics of V_a where $a \in \{0, 0.5, 1\}$.

2. PRELIMINARIES

In this section, some basic concepts are recalled.

Definition 1.

QSO is a mapping of the simplex

$$S^{m-1} = \left\{ x = (x_1, \cdots, x_m) \in \mathbb{R}^m : \sum_{i=1}^{m} x_i = 1,\ x_i \geq 0,\ i = \overline{1,m} \right\} \tag{1}$$

Into itself of the form

$$x'_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \quad k = \overline{1,m}, \tag{2}$$

where $V(x) = x' = (x'_1, \cdots, x'_m)$ and $P_{ij,k}$ is a coefficient of heredity which satisfies the following conditions

$$P_{ij,k} \geq 0, \quad P_{ij,k} = P_{ji,k}, \quad \sum_{k=1}^{m} P_{ij,k} = 1. \tag{3}$$

From the above definition, it can be concluded that each QSO $V : S^{m-1} \to S^{m-1}$ can be uniquely defined by a cubic matrix $\mathrm{P} = \left(P_{ijk}\right)_{i,j,k=1}^{m}$ with condition (3). For $V : S^{m-1} \to S^{m-1}$, the set of fixed points by $Fix(V)$ is denoted. Moreover, for $x^{(0)} \in S^{m-1}$, the set of limiting point by $\omega_V(x^{(0)})$ is denoted.

Recall that a Volterra-QSO is defined by (2), (3) and the additional assumption

$$P_{ij,k} = 0 \quad \text{if} \quad k \notin \{i, j\}. \tag{4}$$

The biological treatment of condition (4) is clear: the offspring repeats the genotype (trait) of one of its parents. One can see that a Volterra-QSO has the following form:

$$x'_k = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right), \; k \in I, \tag{5}$$

where

$$a_{ki} = 2P_{ik,k} - 1 \;\text{ for } i \neq k \text{ and } a_{ii} = 0, i \in I. \tag{6}$$

Moreover,

$$a_{ki} = -a_{ik} \;\text{ and }\; |a_{ki}| \leq 1.$$

In [2, 7, 8, 9, 16, 33], this type of Volterra-QSO was intensively studied. The concept of ℓ-Volterra-QSO was introduced in [26]. This concept is recalled as follows. Let $\ell \in I$ be fixed, and suppose that the heredity coefficient $\left\{P_{ij,k}\right\}$ satisfy

$$P_{ij,k} = 0 \;\text{ if }\; k \notin \{i, j\} \;\text{ for any }\; k \in \{1, \ldots, \ell\}, \; i, j \in I, \tag{7}$$

$$P_{i_0 j_0,k} > 0 \;\text{ for some } (i_0, j_0), i_0 \neq k, \, j_0 \neq k, \; k \in \{\ell + 1, \ldots, m\}. \tag{8}$$

Thus, the QSO defined by (2), (3), (7) and (8) is called ℓ-Volterra-QSO.

Remark 1.

An ℓ-Volterra-QSO is a Volterra-QSO if and only if $\ell = m$.

No periodic trajectory exists for Volterra-QSO [7]. However, such trajectories exist for ℓ-Volterra-QSO [26]. By following [36], each element $x \in S^{m-1}$ is a probability distribution of the set $I = \{1, \ldots, m\}$. Let $x = (x_1, \cdots, x_m)$ and $y = (y_1, \cdots, y_m)$ be vectors taken from S^{m-1}. x is equivalent to y, if $x_k = 0 \Leftrightarrow y_k = 0$. The relation is denoted by $x \sim y$. Let $supp(x) = \{i : x_i \neq 0\}$ be a support of $x \in S^{m-1}$. We say that x is singular to y and denoted by $x \perp y$, if

$supp(x) \cap supp(y) = \varnothing$. Note that if $x, y \in S^{m-1}$, then $x \perp y$ if and only if $(x, y) = 0$, where $(\cdot,\cdot)$ stands for a standard inner product in $\mathbb{R}^m$.

Sets of coupled indexes were denoted by

$$\mathbf{P}_m = \{(i,j) : i < j\} \subset I \times I, \quad \Delta_m = \{(i,i) : i \in I\} \subset I \times I. \qquad (\#)$$

For a given pair $(i,j) \in \mathbf{P}_m \cup \Delta_m$, a vector $\mathrm{P}_{ij} = \left(P_{ij,1}, \cdots, P_{ij,m}\right)$ is set. Clearly, because of condition (3), $\mathrm{P}_{ij} \in S^{m-1}$. Let $\xi_1 = \{A_i\}_{i=1}^N$ and $\xi_2 = \{B_i\}_{i=1}^M$ be some fixed partitions of $\mathbf{P}_m$ and Δ_m respectively i.e.

$$A_i \bigcap A_j = \varnothing\ ,\ B_i \bigcap B_j = \varnothing, \text{ and } \bigcup_{i=1}^{N} A_i = \mathbf{P}_m\ ,\ \bigcup_{i=1}^{M} B_i = \Delta_m, \text{ where } N, M \le m. \qquad (\#)$$

Definition 2.

QSO $V : S^{m-1} \to S^{m-1}$ given by [2] and [3] is called a $\xi^{(as)}$-QSO w.r.t. the partitions ξ_1, ξ_2 (where the "as" stands for absolutely continuous-singular), if the following conditions are satisfied:

(i) For each $k \in \{1, \ldots, N\}$ and any $(i,j),\ (u,v) \in A_k$, one has $\mathrm{P}_{ij} \sim \mathrm{P}_{uv}$;

(ii) For any $k \ne \ell,\ k, \ell \in \{1, \ldots, N\}$ and any $(i,j) \in A_k$ and $(u,v) \in A_\ell$ one has $\mathrm{P}_{ij} \perp \mathrm{P}_{uv}$;

(iii) For each $d \in \{1, \ldots, M\}$ and any (i,i), $(j,j) \in B_d$, one has $\mathrm{P}_{ii} \sim \mathrm{P}_{jj}$;

(iv) For any $s \ne h$, $s, h \in \{1, \ldots, M\}$ and any $(u,u) \in B_s$ and $(v,v) \in B_h$, one has that $\mathrm{P}_{uu} \perp \mathrm{P}_{vv}$.

In [36], 36 operators of the $\xi^{(s)}$-QSO was investigated, where the operators were then classified into 20 non-conjugacy classes. In this paper, the following operators are studied:

$$V_a = \begin{cases} x' = x^2 + 2ayz \\ y' = y^2 + 2(1-a)yz \\ z' = z^2 + 2x(1-x) \end{cases} \qquad (9)$$

3. FIXED POINT OF $V_{0.5}$

In this section, the fixed point of V_a where $a = 0.5$ will be discovered. Thus, the operator given by (9) can be rewrite as follows.

$$V_{0.5} = \begin{cases} x' = x^2 + yz \\ y' = y^2 + yz \\ z' = z^2 + 2x(1-x) \end{cases} \qquad (10)$$

Theorem 1.

Consider $V_{0.5}: S^2 \to S^2$ as a quadratic stochastic operator given by (10). One has that $Fix(V_{19,0.5}) = \{e_1, e_2, e_3\}$.

Proof

To find the fixed point of $V_{0.5}$, the following system need to be solved:

$$\begin{cases} x = x^2 + yz \\ y = y^2 + yz \\ z = z^2 + 2x(1-x) \end{cases} \tag{11}$$

Now, by subtracting second equation from first equation in system (11), we obtain that

$$x - y = x^2 - y^2 \tag{12}$$

In Eq. (12), two different cases are discussed.

Case 1. If $x \neq y$, then the $x + y = 1$ is obtained. Thus, $z = 0$. Moreover, substitute $z = 0$ in third equation of system (11) this yield

$$0 = 2x(1-x) \tag{13}$$

After solving Eq. (13), the $x \in \{0,1\}$ is obtained. It is easy to see the fixed point of $V_{19,0.5}$ when $x \neq y$ are $\{e_1, e_2\}$ because $x + y + z = 1$.

Case 2. If $x = y$, then $x + y + z = 1$ can be written as $z = 1 - 2x$. Now, after substitute $z = 1 - 2x$ in third equation of system (11), the following equation is obtained:

$$1 - 2x = (1-2x)^2 + 2x(1-x) \tag{14}$$

By solving Eq. (14), $x = 0$ is obtained. Thus, $y = 0$ and $z = 1$. Therefore, the fixed point of $V_{0.5}$ when $x = y$ is $\{e_3\}$. This process completes the proof.

4. DYNAMIC OF $V_{0.5}$

In this section, the dynamics of V_a when $a = 0.5$ given by (10) is studied by finding the set of limiting point.

Let us introduce the following lines:

$$\ell_1 := \{(x,y,z) \in S^2 : x = y\}\} \quad (\#)$$

$$\ell_2 := \{(x,y,z) \in S^2 : x = z, x \neq y\} \quad (\#)$$

Proposition 1.

Suppose that $V_{0.5} : S^2 \to S^2$ be a QSO. The line ℓ_1 is an invariant line under $V_{0.5}$.

Proof.

Let $(x^{(0)}, y^{(0)}, z^{(0)}) \in \ell_1$ be an initial point in S^2. Since $(x^{(0)}, y^{(0)}, z^{(0)}) \in \ell_1$ it can be easily observed that $x^{(0)} = y^{(0)}$. Thus, the first iteration of $V_{0.5}$ contain $x' = (x^{(0)})^2 + 2y^{(0)}z^{(0)}$ and $y' = (x^{(0)})^2 + 2y^{(0)}z^{(0)}$. It is easily to see that $x' = y'$. Therefore, $V_{0.5}(x^{(0)}, y^{(0)}, z^{(0)}) \in \ell_1$. Hence, ℓ_1 is an invariant line under $V_{0.5}$. This process completes the proof.

Theorem 2.

Suppose that $V_{19,0.5} : S^2 \to S^2$ given by (10) be a QSO and let $x^{(0)} = (x,y,z) \notin Fix(V_{0.5})$ be any initial point. Then, the following statement holds true.

(1) If $(x^{(0)}, y^{(0)}, z^{(0)}) \in \ell_1$, then $\omega_{V_{0.5}}(x^{(0)}, y^{(0)}, z^{(0)}) = e_3$.

(2) If $(x^{(0)}, y^{(0)}, z^{(0)}) \in S^2 \setminus \ell_1$, then $\omega_{V_{0.5}}(x^{(0)}, y^{(0)}, z^{(0)}) = e_3$.

Proof.

(1) Let $(x^{(0)}, y^{(0)}, z^{(0)}) \in \ell_1$ be an initial point in S^2. Thus, $x^{(0)} = y^{(0)}$ and $z^{(0)} = 1 - 2x^{(0)}$. Moreover, $x' = (x^{(0)})^2 + 2y^{(0)}z^{(0)} = (x^{(0)})^2 + 2x^{(0)}(1 - 2x^{(0)})$. It is easily to see that $x' = x^{(0)} - (x^{(0)})^2$. Suppose that $f(x) = x - x^2$, it can be easily checked that $f(x)$ increasing on $(0, 0.5)$ and decreasing on $(0.5, 1)$. In order to study the dynamic of $f(x)$, firstly a fixed point of $f(x)$ need to be found by solving the equation $x = x - x^2$. $f(x)$ has a fixed point at $x = 0$. Moreover, $f(x) - x < 0$ for all $x \in [0,1]$. Furthermore, $f^{(n+1)}(x) < f^{(n)}(x)$ indicate that the sequence $\{f^{(n)}(x)\}_{n=1}^{\infty}$ is decreasing and bounded. Thus, the sequence $f^{(n)}(x)$ converge to fixed point x^* which is $x^* = 0$. Therefore, $x^{(n)} \to 0$. In the same manner, $y^{(n)} \to 0$. Since $x^{(n)}$ and $y^{(n)}$ converge to zero. Thus, $z^{(n)} \to 1$. Therefore, the limiting point is $e_3 = (0,0,1)$.

(2) To prove (2) the following claim is needed.

Claim 1.

Suppose that $(x^{(0)}, y^{(0)}, z^{(0)}) \in S^2 \setminus \ell_1$, then the $n_k \in \mathbb{N}$ exist such that $V_{(n_k)}(x^{(0)}, y^{(0)}, z^{(0)}) \in \ell_1$.

Proof.

By contrast, let $(x^{(0)}, y^{(0)}, z^{(0)}) \in S^2 \setminus \ell_1$ and let $V^{(n_k)}(x^{(0)}, y^{(0)}, z^{(0)}) \notin \ell_1$ for all $n_k \in \mathbb{N}$. Suppose that $(x^{(0)}, y^{(0)}, z^{(0)}) \in \ell_2$, thus $x^{(0)} = z^{(0)}$ and $x^{(0)} \neq y^{(0)}$. Furthermore, $x' = (x^{(0)})^2 + y^{(0)}x^{(0)}$ and $y' = (y^{(0)})^2 + y^{(0)}x^{(0)}$. On the other hand, $x' + y' = (x^{(0)} + y^{(0)})^2$. Since $0 < x^{(0)} + y^{(0)} < 1$, it can be observed that $(x^{(0)} + y^{(0)})^2 < x^{(0)} + y^{(0)}$. Thus, the new sequence $\left\{x^{(n)} + y^{(n)}\right\}$ converges to zero. Moreover, the sequences $x^{(n)}$ and $y^{(n)}$ are positive sequences. Hence, $x^{(n)}$ converges to zero and $y^{(n)}$ converges to zero. So $\exists n_k \in \mathbb{N}$ such that $x^{(n_k)} = y^{(n_k)} = 0$, where $\left(x^{(n_k)}, y^{(n_k)}, z^{(n_k)}\right) \in \ell_1$ which is contradiction.

Due to Claim (1), the set of limiting point is $\omega_{V_{0.5}}(x^{(0)}, y^{(0)}, z^{(0)}) = e_3$ when $(x^{(0)}, y^{(0)}, z^{(0)}) \in S^2 \setminus \ell_1$.

5. DYNAMIC OF V_a WHERE a ϵ {0,1}

In this section, the behavior of $V_{19,a}$ when the parameter $a = 0$ and $a = 1$ is studied by finding the set of fixed points and the set of limiting point to each operators.

Proposition 2.

Consider $f : [0,1] \to [0,1]$ be a function given by $f(x) = x^2$. Then, the following statement hold true

(i) $Fix(f(x)) = \{0,1\}$.
(ii) $\omega_f\left(x^{(0)}\right) = 0$ for any $x^{(0)} \in (0,1)$.

Proof.

(i) The following Eq. (15) must be solved in order to find the fixed points of $f(x)$. From Eq. (15) it is easily to see that $x \in \{0,1\}$.

$$x = x^2 \tag{15}$$

(ii) Let $x^{(0)} \in (0,1)$, so $f(x^{(0)}) - x^{(0)} < 0$. Since $f(x)$ is increasing on $(0,1)$, therefore $f^{(n+1)}(x^{(0)}) < f^{(n)}(x^{(0)})$ for any $n \in \mathbb{N}$. This means that the sequence $\left\{f^{(n)}(x^{(0)})\right\}_{n=1}^{\infty}$ is decreasing and bounded. Consequently, it converge to some point $\tilde{x}$ and $\tilde{x}$ should be a fixed point; that is $\tilde{x} = 0$. This means that $\omega_f\left(x^{(0)}\right) = 0$.

If $a = 0$, then the operator in (9) can be rewrite as follows

$$V_0 = \begin{cases} x' = x^2 \\ y' = y^2 + 2yz \\ z' = z^2 + 2x(1-x) \end{cases} \tag{16}$$

Theorem 3.

Consider $V_0 : S^2 \to S^2$ be a QSO given by (16) and let $\left(x^{(0)}, y^{(0)}, z^{(0)}\right) \notin Fix\left(V_0\right)$ be an initial point in S^2. One has that:

i. $Fix\left(V_0\right) = \{e_1, e_2, e_3\}$,
ii. $\omega_{V_0}\left(x^{(0)}, y^{(0)}, z^{(0)}\right) = e_2$

Proof.

(i) To find fixed point of V_0, we are going to solve the following system

$$\begin{cases} x = x^2 \\ y = y^2 + 2yz \\ z = z^2 + 2x(1-x) \end{cases} \tag{17}$$

From first equation of system (17), we obtain that $x = x^2$. Thus, $x = 0$ or $x = 1$. If $x = 1$, then $y = z = 0$, because $x + y + z = 1$. On the other hand, after substituting $x = 0$ in third equation of system (17) we obtain that $z = 0$ or $z = 1$. If $z = 0$, then $y = 1$ and otherwise. Thus, the fixed points are $\{e_1, e_2, e_3\}$.

(ii) Suppose that $\left(x^{(0)}, y^{(0)}, z^{(0)}\right) \notin Fix\left(V_0\right)$ be an initial point in S^2 since $x' = x^2$. Due to proposition (2), the sequence $\{x^{(n)}\}_{n=1}^{\infty}$ converges to zero. On the other hand, let $z' = g(z) + h(x)$, where $g(z) = z^2$ and $h(x) = 2x(1-x)$. Therefore, $z^{(n+1)}\left(x^{(0)}, y^{(0)}, z^{(0)}\right) = g^{(n)}\left(z^{(0)}\right) + h^{(n)}\left(x^{(0)}\right)$. It can be easily see that $h^{(n)}$ depend only on $x^{(n)}$ and $x^{(n)}$ converges to zero, then $h^{(n)}\left(x^{(0)}\right)$ converges to zero. Furthermore, due to proposition (2), the $g^{(n)}\left(z^{(0)}\right)$ converges to zero. Therefore, the sequence $\{z^{(n)}\}_{n=1}^{\infty}$ converges to zero. Furthermore, $\{y^{(n)}\}_{n=1}^{\infty}$ converges to one since $x^{(n)} + y^{(n)} + z^{(n)} = 1$.

Now, consider $a = 1$, then the operator in (9) can be rewritten as follows:

$$V_1 = \begin{cases} x' = x^2 + 2yz \\ y' = y^2 \\ z' = z^2 + 2x(1-x) \end{cases} \tag{18}$$

Theorem 4.

Let $V_1 : S^2 \to S^2$ be a QSO given by (18) and let $\left(x^{(0)}, y^{(0)}, z^{(0)}\right) \notin Fix\left(V_0\right)$ be an initial point in S^2. One has that:

i. $Fix\left(V_1\right) = \left\{e_1, e_2, e_3\right\}$,

ii. $\omega_{V_1}\left(x^{(0)}, y^{(0)}, z^{(0)}\right) = e_3$

Proof.

(i) To find fixed point of V_1, the following system have to be solved

$$\begin{cases} x = x^2 + 2yz \\ y = y^2 \\ z = z^2 + 2x(1-x) \end{cases} \tag{19}$$

From the second equation in system (19), we obtain that $y = 0$ or $y = 1$. Now, substitute $y = 0$ in first equation of system (19) which will yield $x = 0$ or $x = 1$. If $x = 0$, then $z = 1$ and otherwise. Moreover, $x = z = 0$. Thus, the fixed points are $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$.

(ii) Let $\left(x^{(0)}, y^{(0)}, z^{(0)}\right) \notin Fix\left(V_1\right)$ be an initial point in S^2, since $y^{'} = y^2$. Due to proposition (2), the sequence $\left\{y^{(n)}\right\}_{n=1}^{\infty}$ converges to zero. On the other hand, consider that $M(x,y,z) = x^2 + 2yz = h(x) + f(y,z)$. Thus, $f(y,z) = 2yz$ depend only on y and z. Now, consider $f(y,z) = m(y)k(z)$, where $m(y) = y$ and $k(z) = 2z$. Consequently, $f^{(n)}\left(y^{(0)}, z^{(0)}\right) = m^{(n)}\left(y^{(0)}\right)k^{(n)}\left(z^{(0)}\right)$ but $\left\{m^{(n)}\left(y^{(0)}\right)\right\}_{n=1}^{\infty}$ converges to zero. Therefore, $\left\{f^{(n)}\left(y^{(0)}, z^{(0)}\right)\right\}_{n=1}^{\infty}$ converges to zero. Due to proposition (2), the sequence is $\left\{h^{(n)}\left(x^{(0)}\right)\right\}_{n=1}^{\infty}$. Therefore, the sequence $\left\{M^{(n)}\left(x^{(0)}, y^{(0)}, z^{(0)}\right)\right\}_{n=1}^{\infty}$ converges to zero. Thus, $x^{(n)}$ converges to zero. Since $x^{(n)} + y^{(n)} + z^{(n)} = 1$. Thus, $\left\{z^{(n)}\right\}_{n=1}^{\infty}$ converges to 1.

6. CONCLUSION

In this paper, we investigated the dynamics behavior of V_a when $a \in \left\{0, 0.5, 1\right\}$. Moreover, we conclude that the global behavior of V_a goes to e_3 when $a = \left\{0.5, 1\right\}$. Further, the global behavior of V_a goes to e_2 when $a = 0$.

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APPENDIX

If any, the appendix should appear directly after the references without numbering, and on a new page.