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Symbolic Solution of Short Beam on Elastic Foundation Using the Computer Algebra System

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ABSTRACT

In this paper, the aim is to utilize Computer Algebra Systems to derive a closed-form solution for foundation-structure response. Typically, the subgrade reactive pressure beneath a loaded foundation region is modelled using a series of independent linear spring systems, but this approach has limitations and is not practical due to the lack of continuity between the springs and the unloaded foundation medium. To address this issue, a short beam structure has been introduced on an elastic foundation, which can restrict the vertical displacement on both the loaded and unloaded regions of the foundation. To obtain a closed-form solution for displacement and the bending moment of the foundation based on the imaginary beam inclusion, a MAPLE code has been developed to symbolically solve the governing differential equation using Computer Algebra Systems. Additionally, a comparison between the present solution and the analytical solution is provided for two prominent practical problems in the civil engineering applications.

Keywords: beam on elastic foundation, Computer Algebra System, linear springs, short beam

1 INTRODUCTION

In the realm of undergraduate engineering education, the conventional approach for resolving intricate differential equations pertaining to short beams on elastic foundations (SBOF) has predominantly rested on algebraic and analytic techniques. This approach has been characterized by systematic algorithms that concentrate on specific types of differential equations, as outlined by Lozada et al. [1]. Nonetheless, the educational sphere has undergone a significant transformation with the advent of Computer Algebra Systems (CAS), which have proven to be invaluable tools for enhancing the teaching and learning processes. CAS have found a particularly pertinent role in the domain of mathematics, where the ability to tackle complex differential equations holds paramount importance.

The potential offered by CAS has garnered the attention of esteemed researchers who have extensively investigated the merits of incorporating distinct CAS components within the classroom setting, as evidenced by the works of Attia et al. [2], Kurniadi et al. [3], and Vargas et al. [4]. Notably, there exists a considerable gap in the research concerning the application of CAS technology in the instructional approach for undergraduate civil engineering—a relatively novel area of exploration. Furthermore, the engineering industry itself acknowledges the imperative for engineers to make well-informed decisions and adeptly address industrial challenges. Hence, the introduction of alternative solutions, like CAS, to engineering undergraduates during their educational journey is advocated, as emphasized by [5].

While one might contemplate alternative solutions such as the Finite Element Method (FEM) to address the current issue, exemplified by the work of [6] where the soil medium is discretized into linear springs, it is crucial to acknowledge that employing FEM demands a profound grasp of intricate mathematical principles, substantial input data, and time-intensive processes to solve the pertinent problem. This intricate nature renders FEM unsuitable for undergraduate civil engineering students. Consequently, this poses challenges for both educators and students, necessitating a mastery of extensive knowledge and skills to complete assignments within stipulated timeframes for solving the differential equations inherent to the SBOF problem. In contrast, CAS offers a more streamlined approach for undergraduate civil engineering students, alleviating the burdensome and manual mathematical manipulations typically involved in solving differential equations within SBOF problems. CAS serves to expedite the SBOF solution process, enhance the efficiency of solution derivation, and elevate the quality of short beam designs—qualities of great relevance within the engineering field.

The analysis of the SBOF problem stands as a well-established and fundamental challenge within the realm of civil engineering. This predicament revolves around comprehending the behavior and response of a beam that finds support upon an elastic foundation, as illustrated in Figure 1. Such structures find applications in diverse civil engineering scenarios, including building foundations anchored in soil, vertically embedded pile structures within soil strata, rail structures and tracks (Lamprea-Pineda et al. [7]), and pavement structures resting atop concrete foundations. The descriptor "short" denotes the beam's proportions, indicating that its width significantly surpasses its length. This beam is subject to various loads and initial conditions, with the presence of an elastic foundation introducing additional intricacies to its behavior. Unlike a beam supported by rigid foundations, an elastic foundation facilitates the dispersion of forces and deformations through the underlying soil or supporting materials. This interplay between the beam and elastic foundation assumes a pivotal role in determining the overall response and performance of the structure.

Figure 1 illustrates a beam that is subjected to various arbitrary loadings, including a vertical point load F(x), uniformly distributed loads q(x), and a point bending moment M(x) that varies along the beam span variation parameters, x. As a consequence of these loadings, the soil supporting the beam exerts a subgrade reactive pressure, R(x), in a direction perpendicular to the beam, in accordance with Newton's Third Law (Capecchi) [8]. The soil beneath a beam structure presents a complex and noticeably non-uniform nature, posing challenges when attempting to accurately represent it. However, the primary focus of civil engineers lies in comprehending the behavior that occurs at the interface between the beam and the soil. Consequently, the problem can be simplified by striving to develop a mathematically descriptive model as given in (1) that is both reasonably accurate and capable of representing the contact interaction at the interface.

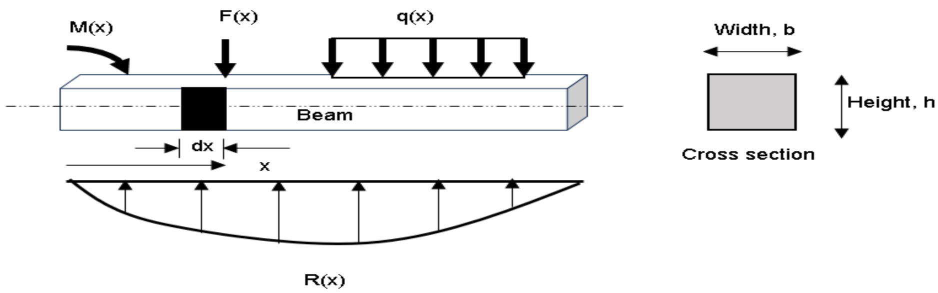

Figure 1: Beam structure under various loadings and the elastic foundation response

$$EI\frac{d^4w}{dx^4} + kw = Q(x) \tag{1}$$

where EI is the flexural rigidity of the beam, k is the modulus subgrade reaction of soil, Q(x) is the acting load on the beam and w is the vertical displacement of the beam. To analytically solve (1), four initial conditions must be considered for the short beam problem, which depend on the restrictions imposed on the degrees of freedom or the force conditions at the beam's ends.

A firm grasp of calculus, especially the concepts of differentiation and integration, proves indispensable in deriving the analytical expression for the vertical displacement, denoted as "w," of the beam. Nevertheless, it has come to notice that a considerable proportion of undergraduate students face hurdles when applying differential and integral calculus. This challenge has manifested as a notable rate of academic non-performance and diminished enthusiasm among undergraduate engineering students, particularly in the mechanical resolution of this specific problem, as documented by Bigotte et al. [9]. Furthermore, [10] identified persistent errors in integration techniques among engineering students enrolled in MAT183 and MAT235 courses at UiTM Penang. Addressing these issues promptly is of utmost significance, especially given the exposure of our new generation of undergraduate engineering students to computer technology and artificial intelligence, which have the potential to enhance student interest and motivation.

In response to this pressing challenge, we present a pioneering and captivating approach that harnesses Computer Algebra Systems (CAS) through MAPLE code. This approach aims to tackle two noteworthy practical real-world problems within the domain of Short Beams on Elastic Foundations (SBOF). Specifically, our objective is to attain symbolic solutions for soil responses in terms of displacement and the resulting bending moment developed along the beam. To accomplish this, we employ a MAPLE code and subsequently juxtapose the outcomes with analytically derived solutions.

2 BACKGROUND ON THE DIFFERENTIAL EQUATION OF SBOF

Given the context provided in Figures 1 and 2, the governing equation represented in equation (2) can be deduced by examining the equilibrium of induced shear force and moment forces—V and M, respectively—stemming from the incremental load qdx. This equilibrium accounts for shear force increments, denoted as dV, on either side of the differential element dx.

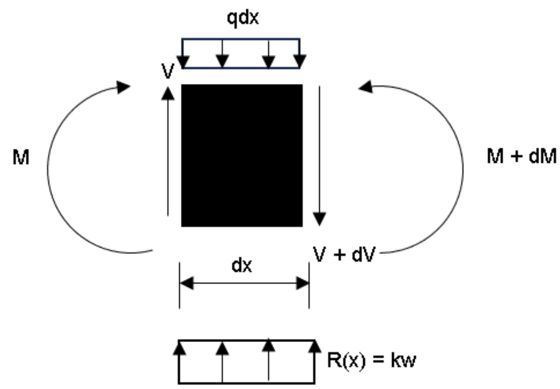

Figure 2: A differential element, of the short beam and k is the modulus of the foundation

$$V - (V + dV) + kwdx - qdx = 0 \tag{2}$$

By using structural mechanics knowledge and using V=dM/dx and simplifying (2) yields (3). It is worth to note that β is defined as $\sqrt[4]{}$(k/EI).

$$\frac{d^4w}{dx^4} + 4\beta^4 w = \frac{q}{EI} \tag{3}$$

The classical closed-form solution of the above equation can be expressed as shown in (4). It's important to mention that (3) holds true exclusively under the condition of no existent of shear deformation in the beam (Polizzotto) [11]. the resulting foundation pressure demonstrates a linear relationship with the applied load on the beam. This assumption also requires that both the material properties of the foundation and the beam adhere to Hooke's Law.

$$w = e^{\beta x}\left(C_1 sin(\beta x) + C_c cos(\beta x)\right) + e^{-\beta x}\left(C_3 sin(\beta x) + cos(\beta x)\right)\frac{q}{4EI\alpha^4} \tag{4}$$

where C_1, C_2, C_3 and C_4 are the integration constants determined from the initial conditions of the beam. Notably, it is worth mentioning that significant findings from [12] indicate that students still face challenges and encounter difficulties in analytically solving (3). In conjunction with this issue, demonstration of alternative CAS approach will be explored on prominent SBOF problems in civil engineering applications.

In a general sense, the solution outlined in equation (3) is achieved through symbolic manipulation using Maple software. The Maple code is developed by initially formulating the differential equation for the SBOF and its associated loading, q, within the Maple worksheet. The 'dsolve' command, which is Maple's differential equation solver, is then employed to derive the overall symbolic solution of the SBOF differential equation. This solution is expressed in terms of the problem's independent and dependent variables.

Furthermore, by incorporating the 'numeric' option command, it becomes possible to acquire numerical values for the given problem. Executing each line of the Maple command involves pressing the spacebar on the computer keyboard, prompting automatic computation of the output for each command. Notably, the Maple software provides a set of plotting tools that enable the generation of graphical representations of the solution in an automated manner.

3 RESULTS AND DISCUSSIONS

3.1 Classical beam problem on elastic foundation

Figure 3 illustrates a SBOF problem consist of a prismatic rectangular timber beam with free ends, supported on an elastic foundation. The geometrical section of the beam, the applied loads (P and M), and its mechanical engineering properties can be found in Table 1. The elastic foundation is composed of a homogeneous material with a subgrade modulus of 13789.51 kN/m^2. The width of the foundation is 0.254 m. It is important to note that this example is sourced from a famous geotechnical engineer, Hetenyi's [13], which employs an Imperial unit of measurement to illustrate the approach.

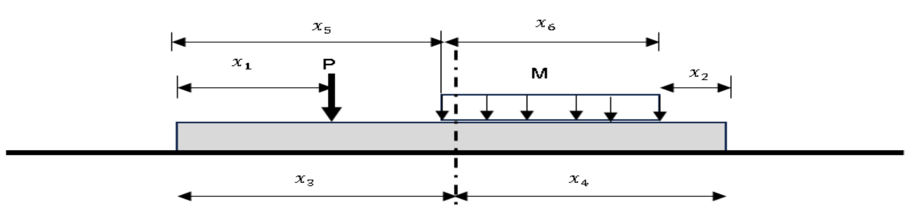

Figure 3: Timber beam resting on elastic foundation

In accordance to (1), with the substitutions of the parameters given in Table 1, yields a differential equation as given in (5) below;

$$1836.681382\frac{d^4w}{dx^4} + 13789.51w = q \quad (5)$$

Table 1: Parameters of the Problem 1

Parameter	Values or Formulas	Units
Beam width	0.254	m
Beam height	0.2032	m
P	22.24	kN
M	17.513	kN/m
E	10342135.94	kN/m²
I	$\frac{\text{Beam } width \times Beam\ height^3}{12}$	m^4
x_1	0.762	m
x_2	0.508	m
x_3	1.524	m
x_4	1.524	m
x_5	1.3208	m
x_6	1.2192	m

From (5), the load acting on the imaginary beam, q can be expressed as combination of contribution of the single point load P and two singularity function to represent the net effect of the uniformly distributed load, M, as shown in (6). δ(x-distance) and H (x- distance) are the Dirac Delta function and the Heaviside function implemented to represent the various loading, q, that acting on the beam.

$$\mathrm{q} = \mathrm{P}\delta(\mathrm{x} - x_1) + MH(x - x_5) - MH\big(x - (x_5 + x_6)\big) \quad (6)$$

The numerical solution for (5) and (6) were obtained by using CAS MAPLE software by employing a straightforward command, i.e. 'evalf(dsolve{DE, IC},w(x))'. This command facilitates the automated determination of the beam's vertical displacement, w(x), without necessitating the intricate calculus of integration and differentiation. To effectively address the free ends and resolve equation (2) symbolically, the requisite initial conditions (IC) must be specified.

Applying the foundational principles of structural mechanics, the portrayal of forces—specifically, displacement w and bending moment M—is achieved using equation (7). These graphical representations, depicted in Figure 4, furnish a visual aid that enhances engineering students' grasp of the solution to the SBOF problem. The horizontal axis corresponds to the distance along the beam from the left end, while the vertical axis portrays the magnitudes of displacement w and bending moment M.

$$M = EI\frac{d^2w}{dx^2} \tag{7}$$

The necessary expression for bending moment, as analytically presented in equation (7), can be automatically computed by students employing the provided MAPLE command 'evalf(EIdiff(w, x$2)'. A comparison between outcomes obtained via this CAS approach and the theoretical solution put forth by [13] is outlined in Table 2. Notably, exact solutions for pivotal points along the beam,x, are demonstrated, offering precise values for both beam displacement and bending moment. This assurance empowers students and practicing engineers to confidently adopt this approach.

Table 2: Comparison of CAS results with the reference problem

Parameters	**Present CAS solution**	**Hetenyi's analytical solution**
Displacement at left end, wA (m)	0.0077114	0.0077114 (0.03036 inch)
Displacement at mid-point, wc (m)	0.001319022	0.001319022 (0.05193 inch)
Displacement at right end, wB (m)	0.000159512	0.000159512 (0.00628 inch)
Bending moment at mid-point, MC (kNm)	1.087309543	1.087309543 (9623.5-inch lbs)

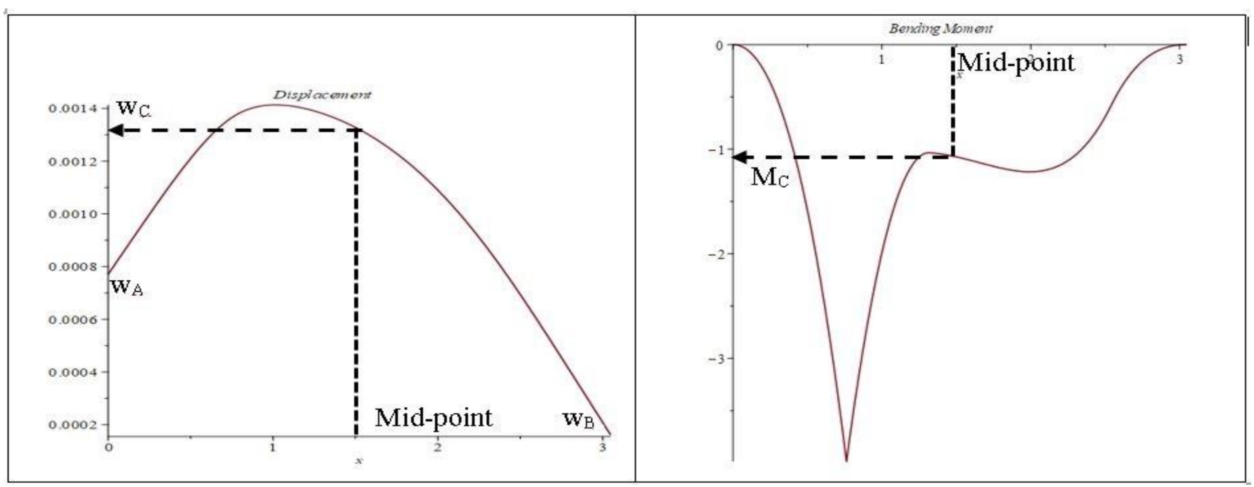

Figure 4: The plotted diagram of the beam displacement, w, (left) and the bending moment (right) with CAS

```
> restart:
> E := 10342135.94 : Iner := 0.254·0.2032^3/12 :
> de := E·Iner·diff(w(x), x$4) + 13789.51·w(x) = q;
```

$$de := 1836.681382 \frac{d^4}{dx^4} w(x) + 13789.51\, w(x) = q \tag{1}$$

```
> q := 22.24·Dirac(x − 0.762) + 17.513·Heaviside(x − 1.3208) − 17.513·Heaviside(x − 2.54042);
```

$$q := 22.24\, \mathrm{Dirac}(x - 0.762) + 17.513\, \mathrm{Heaviside}(x - 1.3208) - 17.513\, \mathrm{Heaviside}(x - 2.54042) \tag{2}$$

```
> de;
```

$$1836.681382 \frac{d^4}{dx^4} w(x) + 13789.51\, w(x) = 22.24\, \mathrm{Dirac}(x - 0.762) + 17.513\, \mathrm{Heaviside}(x - 1.3208) - 17.513\, \mathrm{Heaviside}(x - 2.54042) \tag{3}$$

```
> IC := w''(0) = 0, w''(3.048) = 0, w'''(3.048) = 0, w'''(0) = 0;
```

$$IC := D^{(2)}(w)(0) = 0,\ D^{(2)}(w)(3.048) = 0,\ D^{(3)}(w)(3.048) = 0,\ D^{(3)}(w)(0) = 0 \tag{4}$$

```
> solu := evalf(dsolve({de, IC}, w(x)));
```

$$\begin{aligned} solu := w(x) = {} & \left(\int_{0.}^{x} 4.244093322 \times 10^{-8}\, e^{1.170479346\,\_z1} \left(\cos(1.170479346\,\_z1)^3 \right.\right. \\ & - 1.\, \sin(1.170479346\,\_z1) \cos(1.170479346\,\_z1)^2 + \cos(1.170479346\,\_z1) \sin(1.170479346\,\_z1)^2 \\ & \left. - 1.\, \sin(1.170479346\,\_z1)^3 \right) (22240.\, \mathrm{Dirac}(\_z1 - 0.7620000000) + 17513.\, \mathrm{Heaviside}(\_z1 - 1.320800000) \\ & \left. - 17513.\, \mathrm{Heaviside}(\_z1 - 2.540420000))\, d\_z1 \right) e^{-1.170479346\,x} \cos(1.170479346\,x) + \left(\int_{0.}^{x} 4.244093322 \right. \\ & \times 10^{-8}\, e^{1.170479346\,\_z1} \left(\cos(1.170479346\,\_z1)^3 + \sin(1.170479346\,\_z1) \cos(1.170479346\,\_z1)^2 \right. \\ & \left. + \cos(1.170479346\,\_z1) \sin(1.170479346\,\_z1)^2 + \sin(1.170479346\,\_z1)^3 \right) (22240.\, \mathrm{Dirac}(\_z1 - 0.7620000000) \\ & \left. + 17513.\, \mathrm{Heaviside}(\_z1 - 1.320800000) - 17513.\, \mathrm{Heaviside}(\_z1 - 2.540420000))\, d\_z1 \right) \\ & e^{-1.170479346\,x} \sin(1.170479346\,x) + \left(\int_{0.}^{x} -4.244093322 \times 10^{-8}\, e^{-1.170479346\,\_z1} \left(\cos(1.170479346\,\_z1)^3 \right.\right. \\ & + \sin(1.170479346\,\_z1) \cos(1.170479346\,\_z1)^2 + \cos(1.170479346\,\_z1) \sin(1.170479346\,\_z1)^2 \end{aligned} \tag{5}$$

Figure 5: The developed MAPLE code in worksheet to solve section 3.1 problem

3.2 Steel pile structure embedded in soil and under a lateral load

Another prominent real-world practical civil engineering problem as provided by [14] is used in this paper. In this problem, a vertical pile structure made from steel is subjected to a lateral load, P, at its capping beam location as shown in Figure 6 (left). The pile structure is considered acting as a vertical beam structure that is supported by the elastic soil's stratum on its both sides and modelled by beam on elastic foundation as shown in the right side of Figure 4, with z-axis is pointed downwards. The length of the pile, L, is 19 meters. The steel section of the pile is designed with HP 360 x 174, where the pile width is 0.378 m, and moment of inertia, I, is 0.5080×10^{-3} m^4.

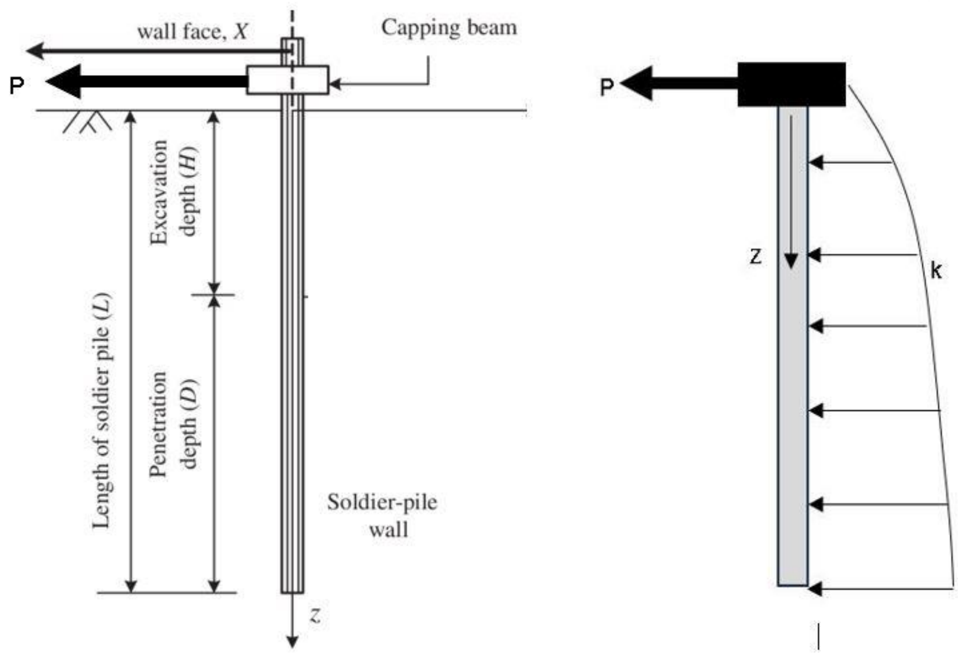

Figure 6: Vertical steel pile structure embedded in soil stratum (left) and its SBOF model (right)

The whole process of solving symbolically in section 3.1 can be obtained by executing several lines of developed MAPLE code as shown in Figure 5.

As the pile structure is made from steel material, the Young Modulus, E, is taken as 210 GPa. The lateral load, P, that acting on the capping beam is computed as 50.74 kN. The top end of the pile was fairly assumed to be fixed at its position, does not allowing any lateral displacement and rotational movement. According to [14], the most general form for the lateral modulus of subgrade reaction, k, distribution can be approximated by using (8) as follows,

$$k = A_s + B_s z^n \tag{8}$$

where A_s = constant for horizontal members

B_s = coefficient for depth variation

z = depth of interest below the ground

n = exponent to give k the best model fit (by load test)

Using the suggested values given by the problem, the lateral modulus of subgrade reaction, k, is given as shown in (9).

$$k = 200 + 50z^{\frac{1}{2}} \quad (9)$$

By substitute the corresponding parameters in (1) and using proper initial conditions, the following differential equation of the SBOF of the imaginary beam is given by (10).

$$101600\frac{d^4w}{dz^4} + \left(75.6 + 18.90z^{\frac{1}{2}}\right)w = 0 \quad (10)$$

Table 3: Comparison of CAS results and Bowles displacement numerical values of Problem 2

Depth, z(m)	Beam displacement (m) -CAS	Beam displacement (m) – Bowles (FEM)	Percentage of difference
0	0.06223363563696	0.06206	0.28
1.0	0.06129043199190	0.06112	0.28
2.0	0.05876736630641	0.05859	0.30
3.0	0.05508131492566	0.05489	0.35
4.5	0.04813901511214	0.04794	0.42
8.0	0.02975035434996	0.02959	0.54
10.0	0.01980885229967	0.01973	0.40
13.0	0.00680400544472	0.00697	2.40
16.0	-0.004336314313413	-0.00378	14.71
19.0	-0.014730820085470	-0.01361	8.24

Table 4: Comparison of CAS results and Bowles bending moment numerical values of Problem 2

Depth, z(m)	Bending Moment (kNm) -CAS	Bending Moment (kNm) – Bowles	Percentage of difference
0	208.1543808	208.4830	0.16
1.0	160.0327014	159.9880	0.03
2.0	117.6589389	117.1610	0.43
3.0	81.27689873	80.3200	1.19
4.5	37.79899538	36.2810	4.18
8.0	-17.65397106	-20.2180	12.68
10.0	-26.78026532	-29.8760	10.36
13.0	-21.78399237	-24.2090	10.02
16.0	-7.92666562	-9.5350	16.87
19.0	0.00000000	0.0000	0.00

Expression (10) was solved symbolically using the MAPLE software's differential equation solver, dsolve, with the following code*: dsolve({de,IC},w(x), numeric).* The results obtained from this symbolic solution in the present study were compared with the suggested solution by Bowles [14] using the FEM. The maximum difference between the displacement values was found to be 14.71 percent, while the maximum difference for the bending moment values was 12.68 percent. It should be noted that the accuracy of FEM results by Bowles were depends greatly on the number of divisions and the interpolation order used to model the problem. By employing a finer mesh or a higher order interpolation function, the accuracy of Bowles' results can be made comparable to the results obtained using the Computer Algebra System (CAS) approach. However, according to [15], they mentioned that FEM is an approximation method, and achieving accurate results relies on the engineer's experience and the correct application of techniques. For the sake of providing a comprehensive depiction of the approach, Figure 7 presents a screenshot of the devised MAPLE code utilized to symbolically resolve Problem 2. This code encompasses a sequence of straightforward MAPLE commands, well within the grasp of undergraduate civil engineering students.

```
> restart :
> de := 101600·diff(w(x), x$4) + (75.6 + 18.9·x^0.5)·w(x) = 0;
```

$$de := 101600 \frac{d^4}{dx^4} w(x) + \left(75.6 + 18.9\sqrt{x}\right) w(x) = 0 \qquad (1)$$

```
>
> 
> IC := w'(0) = 0, w'''(0) = 50.78/101600, w''(19) = 0, w'''(19) = 0;
```

$$IC := D(w)(0) = 0, D^{(3)}(w)(0) = 0.0004998031496, D^{(2)}(w)(19) = 0, D^{(3)}(w)(19) = 0 \qquad (2)$$

```
> c := dsolve({de, IC}, w(x), numeric)
```

$$c := \mathbf{proc}(x\_bvp)\ \ldots\ \mathbf{end\ proc} \qquad (3)$$

```
> c(0)
```

$$\left[x = 0., w(x) = 0.0622336356369596, \frac{d}{dx} w(x) = 0., \frac{d^2}{dx^2} w(x) = -0.00204876359042215, \frac{d^3}{dx^3} w(x) = 0.000499803149600000\right] \qquad (4)$$

```
> c(19)
```

$$\left[x = 19., w(x) = -0.0147308200854702, \frac{d}{dx} w(x) = -0.00344284623593818, \frac{d^2}{dx^2} w(x) = 0., \frac{d^3}{dx^3} w(x) = 0.\right] \qquad (5)$$

Figure 7: The developed MAPLE code in Maple worksheet to solve section 3.2 problem symbolically

4 CONCLUSION

The MAPLE Computer Algebra System (CAS) software was employed to demonstrate the solution for two prominent real-world civil engineering problems related to SBOF. The results revealed a strong agreement between the analytical solutions provided by 13] and [14], which the solutions obtained using CAS. This demonstration introduces an innovative educational strategy, offering an alternative avenue to empower undergraduate civil engineering students in their approach to differential equations. This approach is particularly conducive to active learning, serving as an effective pedagogical tool for contexts like non-examinable assessments and individual mini-projects. The application of this approach necessitates the execution of straightforward MATLAB codes, placing minimal emphasis on intricate theories. This design imparts an engaging mechanism that resonates with students. While demonstrating remarkable precision, it's worth acknowledging that this method might not be suitable for beams exhibiting substantial shear deformation or displaying non-linear material characteristics. Future endeavours will focus on expanding this work to encompass shear-deformable beam structures and the incorporation of non-linear material behaviours. This progressive refinement will contribute to a more comprehensive approach, accommodating a broader spectrum of real-world scenarios.

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