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Curvature Comparison of Bézier Curve, Ball Curve and Trigonometric Curve in Preserving the Positivity of Real Data

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ABSTRACT

The curvature of a curve is important in designing roads, construction of smooth surfaces, or grinding workpieces. Curvature is the tool to measure the smoothness of curves and surfaces. Therefore, in this paper, the curvature profile of three functions will be compared where these three functions also preserved the positivity of the data. These three functions are a rational cubic Bézier curve, a rational cubic Ball curve, and a cubic trigonometric Bézier curve. Conditions were imposed to preserve the positivity of the data and the results are presented. Then, the curvature profile of the curves are compared and analysed. It was found that the interpolated curve by cubic Bézier curve is the best among those three types of curves based on the lowest amplitude value of curvature. However, for all three curves, the curvatures are not continuous since the interpolated curves represent C^1 continuity only.

Keywords: Ball, Bézier, Curvature, Positivity, Trigonometric.

1 INTRODUCTION

Curve interpolation plays an important role particularly in automotive design [1], designing roads [2], construction of smooth surfaces [3] and molding workpieces [4]. It is important to preserve the properties (i.e. positivity) of the real data and be able to construct a visually pleasing curve. Positivity of a data refers to physical quantity that cannot be negative such as stability of radioactive substance, population statistics and rainfall measurements. The positive data can be analyzed and forecasted further via data interpolation. There are many techniques or methods to interpolate data and one of them is spline interpolation. Spline interpolation schemes are common methods used in interpolation of real data. Various splines are used to construct visually pleasing and fair curves such as radial basis function, piecewise Hermite, Bézier, Ball and trigonometric splines. [5] proposed the use of a radial basis function (RBF) to preserve the positivity of 2D and 3D real data. The derivatives method is used in the analysis to provide the evidence that the method managed to preserve properties of a positive curve and surface.

[6] used classical piecewise rational cubic Hermite spline to preserve positivity of the data. [7] also used piecewise cubic Bézier interpolation in the application of estimating solar radiation missing value in Penang, Malaysia. [8] proposed a new scheme that used rational spline to preserve the

positivity of positive data. [9] proposed a rational cubic spline with three shape parameters that ensures positivity everywhere in the curve. He then came out with a rational quartic spline to preserve positivity of the data in [10].

Curve interpolation using Ball basis function has also been developed. [11] used cubic Ball with four shape parameters to preserve the positivity of the data. Then, [12] proposed a rational cubic Ball with two parameters and later applied in image interpolation [13]. The latest development would be in [14] where the author proposed a piecewise rational quartic Said-Ball functions where the function used linear denominator to preserve positivity of the data. There is also interpolation scheme that used a rational cubic Bézier function expressed in terms of Ball control points and weights with four shape parameters to preserve positivity of the data [15].

For trigonometric curves, some researchers used rational while some used non-rational curves in the interpolation. Variety of trigonometric curves are used in respective schemes. [16] proposed a non-rational quadratic trigonometric spline with three shape parameters to preserve positivity. [17] presented schemes using non-rational quadratic trigonometric spline that managed to preserve positivity, monotonicity and convexity of the data. This trigonometric spline with three shape parameters proposed utilized two of the shape parameters to maintain the shape of the data and one of the parameters was used to modify the shape of the interpolant. Further, [18] used cubic trigonometric spline with two parameters to preserve positivity of the data. For rational trigonometric curves, [19] used rational quadratic trigonometric with four local parameters to interpolate positive data.

The schemes proposed by all these splines give visually pleasing results but it is not easy to decide which one is better than the other. Hence, in this paper, we proposed the use of curvature as a tool to analyse the fairness of curve. Recently, there are lots of research that use the curvature to determine the smoothness of curves and surfaces such as to resolve the manoeuvring speed estimation of a lane-change system [20]. Another research relies on curvature to find the best shortest path based on the calculated nodes [21]. In this research, the positive data will be interpolated using three different types of curves. The interpolated curves later will be compared using the curvature plot.

In this paper, curve interpolation schemes using three curves which are rational cubic Bézier, rational cubic Ball and non-rational cubic trigonometric splines will be proposed. Then, the curvature results for all three curves will be presented and discussed. This paper is organized as follows. Section 2 presents the overview of all three curves which include the sufficient conditions imposed to preserve positivity of the data. Section 3 describes the curvature formulation while Section 4 presents the graphical results which includes the comparison of the curvature for all three splines. Section 5 concludes the paper.

2 OVERVIEW

This section provides the construction of three curves and provides the sufficient conditions imposed on each curve to ensure the positivity of real data. We have chosen a positive data taken from Sarfraz (2007) for all curves.

2.1 Rational Cubic Bézier Curve

A rational cubic/cubic (referring to a cubic numerator/cubic denominator) Bézier curve, *Be(x)* For $x \in [x_i, x_{i+1}]$ where $i = 1, 2, ..., n - 1$, is defined as:

$$Be(x) = Be(x_i + h_i\theta) \equiv \frac{Pe_i}{Qe_i} \quad (1)$$

where

$$h_i = x_{i+1} - x_i,\ \theta = \frac{(x - x_i)}{h_i},$$

$$Pe_i(\theta) = \alpha_i L_i(1-\theta)^3 + 3c_i M_i \theta(1-\theta)^2 + 3g_i N_i \theta^2(1-\theta) + \beta_i R_i \theta^3 \text{ and} \quad (2)$$

$$Qe_i(\theta) = \alpha_i(1-\theta)^3 + 3c_i\theta(1-\theta)^2 + 3g_i\theta^2(1-\theta) + \beta_i\theta^3. \quad (3)$$

α_i, β_i, c_i and g_i are the weights. L_i, M_i, N_i and R_i are the control points. The following interpolating properties are used to ensure *Be(x)* achieve C^1 continuity where the first derivative of the endpoint of the segments are the same:

$$Be(x_i) = f_i, \qquad Be(x_{i+1}) = f_{i+1}, \qquad Be'(x_i) = d_i, \qquad Be'(x_{i+1}) = d_{i+1} \quad (4)$$

where d_i are the estimated derivative values at given knots.

Using the properties given in (4), the following shape parameter values are derived as to ensure the Bézier curve, *Be(x)* in (1) preserves the positivity of the data.

$$\alpha_i, \beta_i > 0, \qquad c_i > max\left\{0, -\frac{\alpha_i h_i d_i}{3f_i}\right\}, \qquad g_i > max\left\{0, -\frac{\beta_i h_i d_{i+1}}{3f_{i+1}}\right\}. \quad (5)$$

2.2 Rational Cubic Ball Curve

A piecewise rational cubic Ball function *B(x)* in cubic/cubic form which is defined as:

$$B(x) = B(x_i + h_i\theta) \equiv \frac{p_i(\theta)}{q_i(\theta)} \quad (6)$$

where $p_i(\theta) = u_i U_i(1-\theta)^2 + 2vV_i\theta(1-\theta)^2 + 2w_i W_i\theta^2(1-\theta) + z_i Z_i\theta^2$ and

$$q_i(\theta) = u_i(1-\theta)^2 + 2a_i\theta(1-\theta)^2 + 2b_i\theta^2(1-\theta) + z_i\theta^2.$$

u_i, v_i, w_i and z_i are the shape parameters while U_i, V_i, W_i and Z_i are the control points. Using the properties given in (4), the Ball curve, *B(x)* will preserve the positivity of the data when the following shape parameters were imposed:

$$u_i, z_i > 0, \qquad a_i > max\left\{0, -\frac{u_i}{2} - \frac{u_i h_i d_i}{2f_i}\right\}, \qquad b_i > max\left\{0, -\frac{z_i}{2} + \frac{z_i h_i d_{i+1}}{2f_{i+1}}\right\}. \quad (7)$$

2.3 Cubic Trigonometric Bézier Curve

[22] proposed a cubic trigonometric Bézier function for every subinterval $I_i = [x_i , x_{i+1}]$ for all i = 0, 1, 2, ..., n − 1 as:

$$S_i(x) = \sum_{i=0}^{3} \omega_i P_i,$$

$$\omega_0 = \left[1 - sin\left(\frac{\pi\theta}{2}\right)\right]^2 \left[1 - \gamma sin\left(\frac{\pi\theta}{2}\right)\right], \quad \omega_1 = sin\left(\frac{\pi\theta}{2}\right)\left[1 - sin\left(\frac{\pi\theta}{2}\right)\right]\left[2 + \gamma - \gamma sin\left(\frac{\pi\theta}{2}\right)\right]$$
$$\omega_2 = cos\left(\frac{\pi\theta}{2}\right)\left[1 - cos\left(\frac{\pi\theta}{2}\right)\right]\left[2 + \sigma - \sigma sin\left(\frac{\pi\theta}{2}\right)\right], \quad \omega_3 = \left[1 - cos\left(\frac{\pi\theta}{2}\right)\right]^2 \left[1 - \sigma cos\left(\frac{\pi\theta}{2}\right)\right] \tag{8}$$

and

$$P_0 = f_i, \quad P_1 = \frac{\pi f_i(2+\gamma_i) + 2h_i d_i}{\pi(2+\gamma_i)}, \quad P_2 = \frac{\pi f_{i+1}(2+\sigma_i) - 2h_i d_{i+1}}{\pi(2+\sigma_i)}, \quad P_3 = f_{i+1}$$

where ω_0, ω_1, ω_2 and ω_3 are the trigonometric cubic basis functions, γ and σ are the shape parameters and P_0, P_1, P_2 and P_3 are the control points. Based on [22], for a positive data set $\{(t_i , f_i) : i = 0, 1, 2, ..., n\}$, the author presented a piecewise cubic trigonometric spline that preserves the positivity of the interpolant that satisfied the following conditions:

$$\gamma_i > max\left\{0, \frac{-2h_i d_i}{\pi f_i}\right\}, \quad \sigma_i > max\left\{0, \frac{-2h_i d_{i+1}}{\pi f_{i+1}}\right\}. \tag{9}$$

The weight or shape parameter will provide control of the curve. Different value of shape parameter will result in different shape of curve interpolated. This property will help the designer to modify the interpolated curve compared to the classical Bézier curve. In this paper, an arbitrary value of 0.1 is chosen as the shape parameter for all curves to provide fair comparison for analysis purposes.

3 CURVATURE

Let $z(x) = (z_x(t), z_f(t))$ be two-dimensional parametric curve. $z'(x)$ and $z''(x)$ is the first and second order derivative of $z(x)$ respectively. The curvature equation of a curve is defined as follows:

$$\kappa(x) = \frac{|z'(x) \times z''(x)|}{|z'(x)|^3} \tag{10}$$

Generally, the ideal desired curve should have monotone curvature segments [23]. Curvature plot is one of the ways to help determine the smooth shape [24]. Based on the curvature plot, the inflection point can be found by detecting the value of curvature starts to change its sign as reported in [25]. In this paper, the chosen positive data will be interpolated using several types of curves namely rational cubic/cubic Bézier, rational cubic/cubic Ball curve, and cubic trigonometric Bézier curve. The curvature value for each interpolated curve will be plotted and the amplitude of the curvature will be used as an indicator to determine the smooth curve.

4 RESULTS AND DISCUSSIONS

Table 1 shows a positive W-shaped data from [26] where the data consists of conductance values taken at a specific time. This data is an example of a positive data where the conductance value can never be negative.

Table 1: A positive data set of conductance values from [26]

i	1	2	3	4	5	6	7
x_i	2	3	7	8	9	13	14
f_i	10	2	3	7	2	3	10

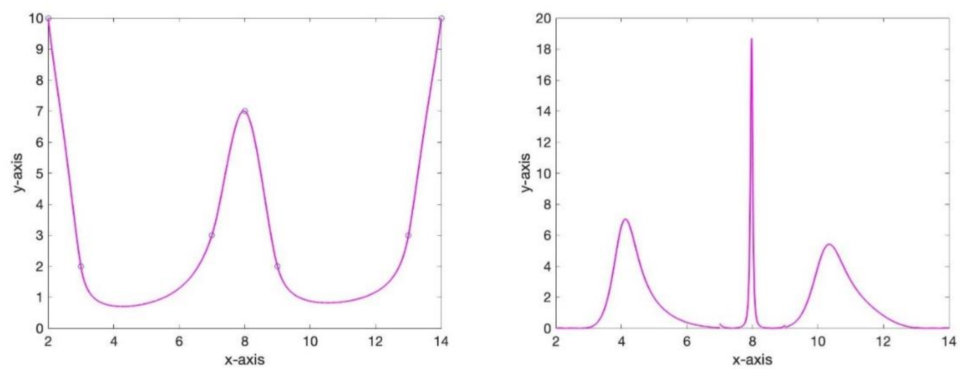

Figure 1: Rational cubic Bézier curve with shape parameters $c_i = g_i = 0.1$ (left) and its curvature profile (right)

Figure 1 (left) shows a shape preserving curve from data in Table 1 by using formulation of rational cubic Bézier curve as in Equation (1) and imposing parameter values as in Equation (5). c_i and g_i were set to be 0.1 act as its shape parameters and the smoothness of the curve can be seen in Figure 1 (right). The value of shape parameter 0.1 is arbitrarily chosen and the same value of shape parameters are fixed for three types of curves in this research. The curviness of the interpolated curve at each point can be seen via curvature profile for each curve as in Figure 1 (right), Figure 2 (right) and Figure 3 (right). Equation (6) was used to interpolate the data in Table 1 to generate rational cubic Ball with the value of shape parameters $a_i = b_i = 0.1$ as in Figure 2 (left) and the curvature plot for rational cubic Ball using the same shape parameters is provided in Figure 2 (right). The interpolated curves in both Figure 1 (left) and Figure 2 (left) produced smooth interpolants at the connected points.

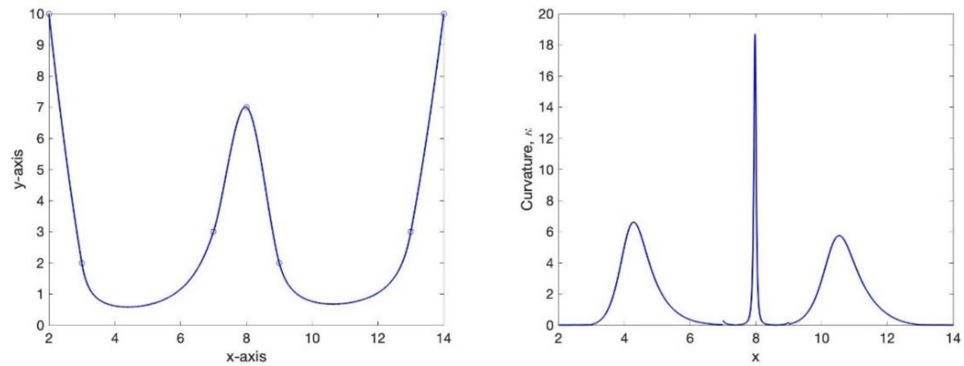

Figure 2: Rational cubic Ball curve with shape parameters $a_i = b_i = 0.1$ (left) and its curvature profile (right)

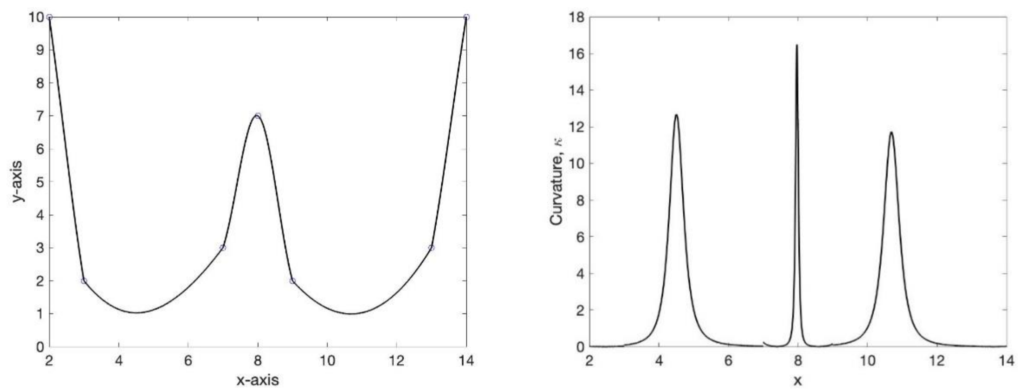

Figure 3: Cubic trigonometric Bézier curve with shape parameters $\gamma = \sigma = 0.1$ (left) and its curvature profile (right)

The third type of curve that was compared with is cubic trigonometric Bézier curve as in Equation (8). Since the curve already have shape parameters of γ and σ, so the rational form of the curve were not needed. By using the same value of shape parameters, the shape preserving curve of cubic trigonometric Bézier curve in Figure 3 (left) are sharp-edged at the connected points. This statement is supported by the curvature plot in Figure 3 (right) where the first and third amplitude of the curvature profile is higher than previous two curves. The curves and curvatures comparison can be observed in Figure 4.

From Figure 1 (right), Figure 2 (right), and 3 (right), the curvatures are not connected smoothly. These show that the connected joint at data points $x = 7$ and $x = 9$ are not curvature continuous. The curvature is discontinuous due to the interpolated curve possessed C^1 continuity or first-order derivative of parametric continuity. This research can be further extended by imposing another condition by integrating the curvature continuity or C^2 so that the curvature profile of the interpolated curve is not disconnected.

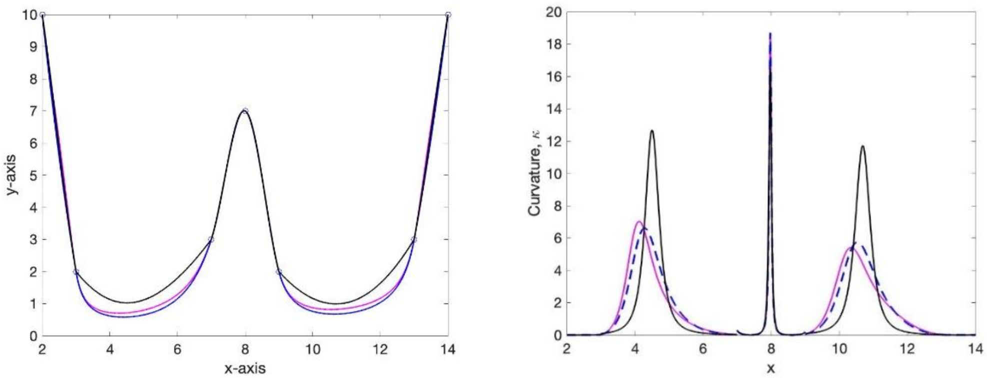

Figure 4: Curves comparison where Bézier curve (pink), Ball curve (blue) and trigonometric curve (black) (left) and their curvatures profile (right).

5 CONCLUSIONS

All three curves managed to preserve the positivity of the data and resulted in smooth interpolation and achieved C^1 continuity. By implementing curvature formulation on the shape-preserving curves, the curvature values for all three different functions namely, cubic Bézier curve, cubic Ball curve, and cubic trigonometric Bézier curve are successfully computed. Since the interpolated curve itself is not enough to validate that our data points are connected smoothly, therefore the curvature information for each type of curves are crucial to determine the smoothness of shape preserving scheme amongst them all.

The interpolated curve by cubic Bézier curve is the best based on the lowest amplitude value of curvature ($x = 10$) among those three types of curves. The cubic Ball curve recorded similar curvature profile as Bézier curve. However, the cubic trigonometric Bézier curve shows higher curvature amplitude than the other two. High curvature indicates that the interpolated curve is a bit tight since the curve have sharp turn at connected point. Moreover, it was found that the curvatures of all three functions are discontinuous. This is expected since curvature is a function of first and second derivatives.

The result is based on one value of shape parameter which is 0.1 for all types of curves and it might give different indicator when different value of shape parameters for each curve is used in the future. For future work, C^2 continuity or curvature continuity conditions can be applied to meet curvature continuity and different value of shape parameters can be used to find the variation of the best fit curve.

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REFERENCES

[1] F. Li, G. Hu, M. Abbas, and K. T. Miura, "The generalized h-bézier model: geometric continuity conditions and applications to curve and surface modeling," *Mathematics*, vol. 8, no. 6, p. 924, 2020.

[2] M. Y. Misro, A. Ramli, and L. K. Hoe, "Determining degree of road elevation using spatial bézier curve," in *AIP Conference Proceedings*. AIP Publishing LLC, 2019, p. 060043.

[3] N. H. M. Ismail and M. Y. Misro, "Surface construction using continuous trigonometric bézier curve," in *AIP Conference Proceedings*. AIP Publishing LLC, 2020, p. 040012.

[4] S. B. Z. Adnan, A. A. M. Ariffin, and M. Y. Misro, "Curve fitting using quintic trigonometric bézier curve," in *AIP Conference Proceedings*. AIP Publishing LLC, 2020, p. 040009.

[5] A. Ahmad, W. E. Ong, and A. R. Mansor, "Preserving the positivity of curves and surfaces using radial basis function," in *2019 4th International Conference and Workshops on Recent Advances and Innovations in Engineering (ICRAIE).* IEEE, 2019, pp. 1–6.

[6] X. Han, "Shape-preserving piecewise rational interpolant with quartic numerator and quadratic denominator," *Applied Mathematics and Computation*, vol. 251, pp. 258–274, 2015.

[7] A. Saaban, M. L. Zainudin, and M. N. A. Bakar, "Piecewise positivity preserving cubic bezier interpolation for estimating solar radiation missing value in penang, malaysia," *Journal of Mathematics and Statistics*, vol. 12, no. 4, pp. 302–307, Dec. 2016.

[8] Y. Zhu, "C2 positivity-preserving rational interpolation splines in one and two dimensions," *Applied Mathematics and Computation*, vol. 316, pp. 186–204, 2018.

[9] S. A. A. Karim, "Construction new rational cubic spline with application in shape preservations," *Cogent Engineering*, vol. 5, no. 1, pp. 1–19, 2018.

[10] N. A. Harim, S. A. A. Karim, Karim, M. Othman, A. Saaban, A. Ghaffar, K. S. Nisar, and D. Baleanu, "Positivity preserving interpolation by using rational quartic spline," *AIMS Mathematics*, vol. 5, no. 4, p. 3762, 2020.

[11] A. Tahat, A. Mt Piah, and Z. Yahya, "Positivity preserving curves using rational cubic ball interpolant," vol. 1682, 2015.

[12] S. A. A. Karim, "Positivity preserving interpolation by using rational cubic ball spline," *Jurnal Teknologi*, vol. 78, no. 11, pp. 141–148, 2016.

[13] S. A. A. Karim and A. Saaban, "Shape preserving interpolation using rational cubic ball function and its application in image interpolation," *Mathematical Problems in Engineering*, vol. 2017, 2017.

[14] S. A. A. Karim and V. T. Nguyen, "Positivity-Preserving Interpolation Using Rational Quartic Spline Functions," in *Theoretical, Modelling and Numerical Simulations Toward Industry 4.0. Studies in Systems, Decision and Control*, vol. 319, Singapore: Springer, 2021.

[15] A. Ahmad, A. Ashaari, and N. B. M. Isa, "Using bézier-ball function to interpolate positive real data and it's application," in *2020 5th IEEE International Conference on Recent Advances and Innovations in Engineering (ICRAIE)*. IEEE, 2020, pp. 1–6.

[16] M. Dube and P. Rana, "Positivity preserving interpolation of positive data by rational quadratic trigonometric spline," *IOSR Journal of Mathematics*, vol. 10, no. 2, pp. 42–47, 2014.

[17] M. Sarfraz, M. Z. Hussain, and F. Hussain, "Shape preserving curves using quadratic trigonometric splines," *Applied Mathematics and Computation*, vol. 265, pp. 1126–1144, 2015.

[18] N. A. A. A. Munir, F. Yahya, and N. A. Hadi, "Cubic trigonometric spline for preserving positive data," *ASM Science Journal*, vol. 12, no. Special Issue 6, pp. 67–73, 2019.

[19] S. Liu, Z. Chen, and Y. Zhu, "C1 rational quadratic trigonometric interpolation spline for data visualization," *Mathematical Problems in Engineering*, vol. 2015, p. 20, 03 2015.

[20] N. A. Othman, U. Reif, A. Ramli, and M. Y. Misro, "Manoeuvring speed estimation of a lane-change system using geometric hermite interpolation," *Ain Shams Engineering Journal*, vol. 12, no. 4, pp. 4015-4021, 2021.

[21] M. Safaruddin and M. Misro, "Multi-objectives path planning using bézier curve," *Malaysian Journal of Mathematical Sciences*, vol. 15, no. 1, pp. 45–59, 2021.

[22] M. Abbas, A. A. Majid, and J. M. Ali, "Positivity preserving interpolation of positive data by cubic trigonometric spline," vol. 27, 2011.

[23] G. Farin and N. Sapidis, "Curvature and the fairness of curves and surfaces," *IEEE Computer Graphics and Applications*, vol. 9, no. 2, pp. 52–57, 1989.

[24] J. Miller, "Shape curve analysis using curvature," Ph.D. dissertation, University of Glasgow, 2009.

[25] M. Y. Misro, A. Ramli, and J. M. Ali, "Extended analysis of dynamic parameters on cubic trigonometric bézier transition curves," in *2019 23rd International Conference in Information Visualization–Part II*. IEEE, 2019, pp. 141–146.

[26] M. Sarfraz, *Interactive Curve Modeling: With Applications to Computer Graphics, Vision and Image Processing*. Springer London, 2007.