Applied Mathematics and Computational Intelligence
Volume 15, No. 2, 2026 [33 − 39]

On The Exponential Diophantine Equation $p^{2m} + (6r+1)^n = z^2$

Nuralia Amira Mohammad Ikhram[1], Siti Hasana Sapar[2*], Kai Siong Yow[3]

[1,2,3]Department of Mathematics and Statistics, Faculty of Science, Universiti Putra Malaysia 43400 UPM, Serdang, Selangor, Malaysia.

* Corresponding author: sitihas@upm.edu.my

Received: 19 September 2025
Revised: 22 September 2025
Accepted: 7 February 2026

ABSTRACT

A polynomial equation with two or more unknowns for which the integer solutions are sought out is called a Diophantine equation. When exponents are introduced into the equation, a simple linear Diophantine equation transforms into a more complex exponential Diophantine equation. This paper concentrates on finding the solution to the exponential Diophantine equation $p^{2m} + (6r+1)^n = z^2$ *where* $p, m, r, n, z \in \mathbb{Z}^+$*. There is no integral solution to the equation when* p *is an odd prime,* $m \leq 5$*,* $r \leq 25$ *and* $n \leq 10$*.*

Keywords: Exponential Diophantine equation, integral solutions, prime number.

1 INTRODUCTION

The equation in which only integer solutions are sought is called a Diophantine equation. The Diophantine equation where the unknown variables as exponents is called an exponential Diophantine equation. Due to the presence of numerous exponents in an equation, solving exponential Diophantine equations is even more challenging than solving polynomial Diophantine equations. Moreover, no algorithm can effectively solve all exponential Diophantine equations. Numerous branches of mathematics and computer science, such as cryptography, coding theory, and algorithm design, use exponential Diophantine equations [1].

Diophantine equations have been studied by many authors in various types and forms. [2] was taken into consideration in the form of $x^2 + 4 \cdot 7^b = y^{2r}$, where x, y, and b are positive integers and $r > 1$. Positive integer solutions were found by giving explicit generators $x_{b,i} = 7^{i-1}(7^{b-2i+2} - 1)$ and $y^r_{b,i} = 7^{i-1}(7^{b-2i+2} + 1)$ for valid indices i, while excluding cases with negative x. [3] explored on the Diophantine equations in the form of $(p+1)^x - p^y = z^2$ and $p^y - (p+1)^x = z^2$ where p is a prime number for $(x+y) = 2,\ 3,$ or 4 and found that the solutions for these equations are $(p, x, y, z) = (p, 1, 1, 1)$ when prime $p \geq 2$ and $(p, x, y, z) = (2, 1, 2, 1)$ for $(p+1)^x - p^y = z^2$ and also discovered the unique solution for $p^y - (p+1)^x = z^2$ when $(p, x, y, z) = (2, 3, 1, 5)$.

[4] considered the Diophantine equation in the form of $a^x + (a+2)^y = z^2$ where $a \equiv 5 \pmod{42}$ and showed that it has no solution to the equation where x, y and z are non-negative integers.

Then, [5] examined on the Diophantine equations $379^x + 397^y = z^2$ and found that there is no unique integer solution when x, y and z are non-negative integers. [6] looked at the Diophantine equation in the form $x^2 + 8 \cdot 7^b = y^{2r}$, where b and r are positive integers. It proved that the equation has only a finite number of integer solutions for certain values of r. [6] categorize every potential solution in these situations using algebraic number theory and when available, give explicit forms of the answers.

[1] conducted a study to obtain the solutions using the proposed Diophantine equation $(13^{2m}) + (6r+1)^n = z^2$ by using the Calatan's Conjecture and found that it has no solution in the whole number. Thus, there are four cases that the author consider in order to solve this equation as follows:

Case 1 : $m = 0$ while n, r, z are any whole numbers.
Case 2 : $n = 0$ while m, r, z are any whole numbers.
Case 3 : m, n are positive integers while r, z are any whole numbers.
Case 4 : $m, n = 0$ while r, z are any whole numbers.

[7] investigated on the exponential Diophantine equation $(19^{2m}) + (12\gamma + 1)^n = \rho^2$ and found that there is no solution to the equation in whole numbers. [8] solved the exponential Diophantine equation in the form of $(19^{2m}) + (6\gamma + 1)^n = \rho^2$, where m, n, γ, ρ are whole numbers. They concluded that this equation has no solutions in whole numbers. [9] considered on the Diophantine equation $n^x + 13^y = z^2$ where $n \equiv 2\,(mod\,39)$ and $n+1$ is not a square number. They discovered that it has an unique solution which is $(n, x, y, z) = (2, 3, 0, 3)$ for x, y, and z are non-negative integers.

[10] studied the integer solutions to the Diophantine equation, in the form $x^2 + 16 \cdot 7^b = y^{2r}$, where b and r are positive integers. They established that there are only a finite number of solutions by using methods from algebraic number theory and examining particular situations for small values of r. Hence, they gave detailed explanations of those solutions when practical. [11] explored on the Diophantine equation $143^x + 85^y = z^2$, where x, y, and z are non-negative integers and found that $(x, y, z) = (1, 0, 12)$ is the only unique solution, using Catalan's Conjecture.

[12] have made a research on non-linear exponential Diophantine equation $\beta^x + (\beta + 18)^y = z^2$, where x, y, z are non-negative integers and β, $(\beta + 18)$ are prime numbers with β having the form $6n + 1$ of a natural number n. This equation has no solution in non-negative integers.

[13] considered on the Diophantine equation in the form of $11^x - 17^y = z^2$ and discovered that it has an unique solution which are $(x, y, z) = (2, 0, 10)$ when x, y and z are non-negative integers. By considering in four main cases including $x = 0$ and $y = 0$, $x = 0$ and $y > 0$, $x > 0$ and $y = 0$ and $x > 0$ and $y > 0$.

By the paper from [1], this paper concentrates on finding an integral solution to the Diophantine equation in the form of $p^{2m} + (6r+1)^n = z^2$, for p is an odd prime, $m \leq 5$, $r \leq 25$ and $n \leq 10$.

2 RESULTS

In this section, we will explore methods for determining the integral solutions of the Diophantine equation in the form of $p^{2m} + (6r+1)^n = z^2$, for the range of p as an odd prime, $m \leq 5$, $r \leq 25$ and $n \leq 10$. Now, we start with $n = 2$ and $m = 1$ as in the following theorem.

Theorem 2.1 *Let* p, r *and* z *be any positive integers. The exponential Diophantine equation* $p^2 + (6r+1)^2 = z^2$ *for* $p \equiv 1\,(mod\,4)$ *with any odd integer has no trivial positive integer solution.*

Proof. From the hypothesis, we have $p \equiv 1\,(mod\,4)$. Then, by squaring this equation, we obtain $p^2 \equiv 1\,(mod\,4)$. Next, from the equation

$$p^2 + (6r+1)^2 = z^2, \tag{1}$$

we consider for the expression $(6r+1)^2\,(mod\,4)$. Now, we have

$$r \equiv 0\,(mod\,4) \Rightarrow (6r+1) \equiv 1\,(mod\,4)$$
$$r \equiv 1\,(mod\,4) \Rightarrow (6r+1) \equiv 3\,(mod\,4)$$
$$r \equiv 2\,(mod\,4) \Rightarrow (6r+1) \equiv 1\,(mod\,4)$$
$$r \equiv 3\,(mod\,4) \Rightarrow (6r+1) \equiv 3\,(mod\,4).$$

Therefore,

$$(6r+1) \equiv 1 \,\text{or}\, 3\ (mod\,4). \tag{2}$$

Next, by squaring Eq. (2), we obtain $(6r+1)^2 \equiv 1\,(mod\,4)$ for all cases. Then, by substituting the congruence equations into Eq. (1), we have

$$p^2 + (6r+1)^2 \equiv 2\,(mod\,4).$$

That is, $z^2 \equiv 2\,(mod\,4)$ from Eq. (1).

Thus, this is contradiction since z^2 is a perfect square.
Hence, it can only be congruent to 0 or 1 $(mod\,4)$.
Therefore, there is no solution to the Eq. (1). ☐

In the following theorem, we will consider for the general case with $m = 1$ and n is an even integer.

Theorem 2.2 *Let* p, r, k *and* z *be any positive integers. The exponential Diophantine equation* $p^2 + (6r+1)^{2k} = z^2$ *for* $p \equiv 1\,(mod\,4)$ *with any odd integer has no trivial positive integer solution.*

Proof. From the hypothesis, we have $p \equiv 1\,(mod\,4)$. Then, by squaring this equation, we obtain $p^2 \equiv 1\,(mod\,4)$. Next, from the equation

$$p^2 + (6r+1)^{2k} = z^2, \tag{3}$$

we consider for the expression $(6r+1)^{2k}\,(mod\,4)$. Now, we have

$r\equiv 0\,(mod\,4)\Rightarrow(6r+1)\equiv 1\,(mod\,4)$
$r\equiv 1\,(mod\,4)\Rightarrow(6r+1)\equiv 3\,(mod\,4)$
$r\equiv 2\,(mod\,4)\Rightarrow(6r+1)\equiv 1\,(mod\,4)$
$r\equiv 3\,(mod\,4)\Rightarrow(6r+1)\equiv 3\,(mod\,4)$.

Therefore,

$$(6r+1)\equiv 1\,\text{or}\,3\ (mod\,4). \tag{4}$$

Next, by considering at expression $(6r+1)^{2k}$, we obtain $(6r+1)^{2k}\equiv 1\,(mod\,4)$ for all cases. Then, by substituting the congruence equations into Eq. (3), we have

$$p^2+(6r+1)^{2k}\equiv 2\,(mod\,4).$$

That is, $z^2\equiv 2\,(mod\,4)$ from Eq. (3).

Since z^2 is a perfect square, then it can only be congruent to 0 or 1 $(mod\,4)$.
However, this is a contradiction.
From this, we deduce that the problem has no solution to the Eq. (3). □

In the following theorem, we will consider for the Diophantine equation when $m=2$ and $n=2$.

Theorem 2.3 *Let p, r, k and z be any positive integers. The exponential Diophantine equation $p^4+(6r+1)^2=z^2$ for $p\equiv 1\,(mod\,4)$ with any odd integer has no trivial positive integer solution.*

Proof. From the hypothesis, we have $p\equiv 1\,(mod\,4)$. Then, by raising both sides to the power of 4, we obtain $p^4\equiv 1\,(mod\,4)$. Next, from the equation

$$p^4+(6r+1)^2=z^2, \tag{5}$$

we consider for the expression $(6r+1)^2\,(mod\,4)$. Now, we have

$r\equiv 0\,(mod\,4)\Rightarrow(6r+1)\equiv 1\,(mod\,4)$
$r\equiv 1\,(mod\,4)\Rightarrow(6r+1)\equiv 3\,(mod\,4)$
$r\equiv 2\,(mod\,4)\Rightarrow(6r+1)\equiv 1\,(mod\,4)$
$r\equiv 3\,(mod\,4)\Rightarrow(6r+1)\equiv 3\,(mod\,4)$.

Therefore,

$$(6r+1)\equiv 1\,\text{or}\,3\ (mod\,4). \tag{6}$$

Next, by squaring Eq. (6), we obtain $(6r+1)^2 \equiv 1\,(mod\,4)$ for all cases. Then, by substituting the congruence equations into Eq. (5), we have

$$p^4 + (6r+1)^2 \equiv 2\,(mod\,4).$$

That is, $z^2 \equiv 2\,(mod\,4)$ from Eq. (5).

Thus, this is contradiction since z^2 is a perfect square.
Hence, it can only be congruent to 0 or 1 $(mod\,4)$.
Thus, we can determine that no solution to the Eq. (5). $\square$

In the following theorem, we will consider for the general case with $m = 2$ and n is an even integer.

Theorem 2.4 *Let p, r, k and z be any positive integers. The exponential Diophantine equation $p^4 + (6r+1)^{2k} = z^2$ for $p \equiv 1\,(mod\,4)$ with any odd integer has no trivial positive integer solution.*

Proof. From the hypothesis, we have $p \equiv 1\,(mod\,4)$. Then, by raising both sides to the power of 4, we obtain $p^4 \equiv 1\,(mod\,4)$. Next, from the equation

$$p^4 + (6r+1)^{2k} = z^2, \tag{7}$$

we consider for the expression $(6r+1)^{2k}\,(mod\,4)$. Now, we have

$$r \equiv 0\,(mod\,4) \Rightarrow (6r+1) \equiv 1\,(mod\,4)$$
$$r \equiv 1\,(mod\,4) \Rightarrow (6r+1) \equiv 3\,(mod\,4)$$
$$r \equiv 2\,(mod\,4) \Rightarrow (6r+1) \equiv 1\,(mod\,4)$$
$$r \equiv 3\,(mod\,4) \Rightarrow (6r+1) \equiv 3\,(mod\,4).$$

Therefore,

$$(6r+1) \equiv 1 \,\text{or}\, 3\ (mod\,4). \tag{8}$$

Next, by considering at expression $(6r+1)^{2k}$, we obtain $(6r+1)^{2k} \equiv 1\,(mod\,4)$ for all cases. Then, by substituting the congruence equations into Eq. (7), we have

$$p^4 + (6r+1)^{2k} \equiv 2\,(mod\,4).$$

That is, $z^2 \equiv 2\,(mod\,4)$ from Eq. (7).

Since z^2 is a perfect square, then it can only be congruent to 0 or 1 $(mod\,4)$.
However, this is a contradiction since the congruity is differ from each other.
Therefore, it demonstrates that the equation has no solution to the Eq. (7). $\square$

3 CONCLUSION

This project studied the exponential Diophantine equation in the form of $p^{2m} + (6r + 1)^n = z^2$ by extending previous work of [1] under the constraints $m \leq 5$, $r \leq 25$, $n \leq 10$ and p is an odd prime. This study found no positive integer solutions to the Diophantine equation under the constraint that we consider. Future research could explore broader values of m, r and n to further analyze this equation.

ACKNOWLEDGEMENT

I would like to thank the reviewers for their valuable comments.

REFERENCES

[1] S. Aggarwal and S. Kumar, "On the exponential diophantine equation $(13^{2m}) + (6r + 1)^n = z^2$," *International Journal of Latest Technology in Engineering, Management & Applied Science (IJLTEMAS)*, vol. X, no. Issue III, pp. 1–5, 2021.

[2] K. S. Yow, S. H. Sapar, and K. A. Atan, "On the diophantine equation $x^2 + 4 \cdot 7^b = y^{2r}$," *Pertanika Journal of Science & Technology*, vol. 21, no. 2, pp. 443–458, 2013.

[3] N. Burshtein, "All the solutions of the diophantine equations $(p+1)^x - p^y = z^2$ and $p^y - (p+1)^x = z^2$ when p is prime and $x + y = 2, 3, 4$," *Annals of Pure and Applied Mathematics*, vol. 19, no. Issue III, pp. 53–56, 2019.

[4] R. Dokchann and A. Pakapongpun, "On the diophantine equation $a^x + (a + 2)^y = z^2$ where $a \equiv 5\,(\mathit{mod}\,42)$," *Tatra Mountains Mathematical Publications*, vol. 77, p. 39–42, 2020.

[5] S. Aggarwal and N. Sharma, "On the non-linear diophantine equation $379^x + 397^y = z^2$," *Open Journal of Mathematical Sciences*, vol. 4, pp. 397–399, 2020.

[6] K. S. Yow and S. H. Sapar, "A generalisation of the diophantine equation $x^2 + 8 \cdot 7^b = y^{2r}$," *Malaysian Journal of Science*, vol. 40, no. 2, pp. 25–39, 2021.

[7] S. Aggarwal and S. Kumar, "On the exponential diophantine equation $(19^{2m}) + (12\gamma + 1)^n = \rho^2$," *International Journal of Research and Innovation in Applied Science*, vol. 6, no. 3, pp. 14–16, 2021.

[8] ——, "On the exponential diophantine equation $(19^{2m}) + (6\gamma + 1)^n = \rho^2$," *International Journal of Research and Innovation in Applied Science (IJRIAS)*, vol. 6, no. 2, pp. 112–113, 2021.

[9] N. Viriyapong and C. Viriyapong, "On the diophantine equation $n^x + 13^y = z^2$ where $n \equiv 2\,(\mathit{mod}\ 39)$ and $n + 1$ is not a square number," *WSEAS Transactions on Mathematics*, vol. 20, pp. 442–445, 2021.

[10] K. S. Yow, S. H. Sapar, and C. Y. Low, "Solutions to the diophantine equation $x^2 + 16 \cdot 7^b = y^{2r}$," *Malaysian Journal of Fundamental and Applied Sciences*, vol. 18, no. 2, pp. 489–496,

2022.

[11] S. Aggarwal, C. Swarup, D. Gupta, and S. Kumar, “Solution of the diophantine equation $143^x + 85^y = z^2$,” *International Journal of Progressive Research in Science and Engineering*, vol. 4, no. 2, pp. 5–7, 2023.

[12] S. Aggarwal, A. T. Shahida, E. Pandey, and A. Vyas, “On the problem of solution of non-linear (exponential) diophantine equation $\beta^x + (\beta + 18)^y = z^2$,” *Mathematics and Statistics*, vol. 11, no. 5, 2023.

[13] S. Thongnak, T. Kaewong, and W. Chuayjan, “On the exponential diophantine equation $11^x - 17^y = z^2$,” *International Journal of Mathematics and Computer Science*, vol. 19, no. 1, pp. 181–184, 2024.

[14] K. H. Rosen, “Elementary number theory and its applications,” *AddisonWesley*, 1984.

[15] H. Rose, “Number theory, algebraic and analytic,” *In Elsevier eBooks*, p. 1–13, 2003.