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A Novel Isometric–Homotopy Perturbation Technique for Nonlinear Neutrosophic Differential Equations

George Albert Toma[1*] and Taqi Aldin Ahmad Alkhatib[2]

[1]Department of Fundamental Sciences, Higher Institute for Applied Sciences and Technology, Aleppo, Syrian Arab Republic
[2]Department of mathematics, Faculty of Science, Aleppo University, Aleppo, Syrian Arab Republic

*Corresponding author: george.toma.1987@gmail.com; george.toma@hiast.edu.sy

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ABSTRACT

This paper addresses the neutrosophic nonlinear Van der Pol oscillator equation (N-VDP), a model arising in applied dynamics. To obtain accurate analytical approximations, we combine the Homotopy Perturbation Method (HPM) with the one-dimensional isometric transform. The proposed approach provides a systematic framework for analyzing the N-VDP equation and clarifies its key properties. Numerical results are compared with those obtained by the classical fourth-order Runge–Kutta method (RK4). The comparison demonstrates that the present technique yields reliable and accurate approximations, indicating its potential as an effective tool for solving neutrosophic nonlinear differential equations and for further applications in dynamical systems.

Keywords: Neutrosophic Nonlinear Differential Equations, Van der Pol Oscillator, Homotopy Perturbation Method, One-Dimensional Isometric Transform, Analytical Approximation, Runge–Kutta Method.

1 INTRODUCTION

The neutrosophic concept represents a modern development in mathematical modeling and is regarded as a generalization of fuzzy logic and an extension of fuzzy set theory introduced by Zadeh [1] and Atanassov [2]. The idea was first proposed by Smarandache, who established several foundations of the theory such as the neutrosophic integral, neutrosophic measure, neutrosophic sets and systems, and neutrosophic precalculus [3,4].

Abobala and colleagues have more recently proposed a two-dimensional AH-isometry to study the connection between the neutrosophic plane $\mathbb{R}(I)\times \mathbb{R}(I)$ and the classical module $\mathbb{R}^2\times\mathbb{R}^2$. They also introduced a one-dimensional AH-isometry that links $\mathbb{R}(I)$ to $\mathbb{R}\times\mathbb{R}$. This mapping has been shown to be useful for defining inner products, norms, order relations, and various neutrosophic geometric frameworks [5–8].

In this work, the one-dimensional is employed to transform neutrosophic differential equations with one variable into two corresponding classical differential equations. The analysis is then carried out in the classical $\mathbb{R}\times\mathbb{R}$ space before mapping the results back into the neutrosophic domain through the AH-isometry. The Van der Pol equation, which represents a nonlinear oscillator, is a classical model originally used to describe the dynamics of vacuum tube circuits. It has been studied by Kumar [9] and Darbyshire [10], among others. Extending this equation to the neutrosophic setting increases the complexity of the problem and requires effective analytical methods to obtain approximate solutions with high accuracy

2 PRELIMINARIES

In this section we state the basic definitions, notation, and essential theorems used throughout the paper. These fundamentals are necessary before introducing the proposed method.

Definition 2.1 [11]

A neutrosophic number u can be written as:

$u = c + dI$

where $c, d \in \mathbb{R}$, and I represents indeterminacy, with

$0.I = 0 \,\&\, I^n = I \;\; \forall\, n \in \mathbb{Z}^+$

For example: $w = 2 - I$ or $w = 4 = 4 + 0.I$

Definition 2.2 [11]

For $u_1 = c_1 + d_1 I \,\&\, u_2 = c_2 + d_2 I$:

$$\frac{u_1}{u_2} = \frac{c_1}{c_2} + \frac{c_2 d_1 - c_1 d_2}{c_2(c_2 + d_2)} I$$

Definition 2.3 [12]

Let $R(I) = \{u = c + dI; c, d \in \mathbb{R}, I^2 = I\}$

Define

$T: \mathbb{R}(I) \to \mathbb{R} \times \mathbb{R}, T(u) = (c, c + d)$

This mapping:

Is bijective and invertible with

$T^{-1}(c,d) = c + (d-c)I$

preserves both addition and multiplication.

Distancebetween $C = c + dI$ & $D = m + nI$ is:

$\|C - D\| = |c - m| + I(|c + d - m - n| - |c - m|)$

Definition 2.4 [13]

A function $g: \mathbb{R}(I) \to \mathbb{R}(I)$ with $X = c + dI$ is written as:

$g(X) = g(c) + I[g(c+d) - g(c)]$

Theorem 2.5 [14]

Every neutrosophic real function can be represented by a pair of classical real functions, meaning that it can mapped into the Euclidean plane $\mathbb{R} \times \mathbb{R}$.

3 HPM FOR SOLVING THE NONLINER NEUTROSOPHIC VAN DER POL OSCILLTOR VIA ONE-DIMENSIONAL ISOMETRIC TRANSFORM

The nonlinear neutrosophic Van der Pol oscillator problem is given by

$$U'' = \alpha\, U' + \beta\, U + \gamma\, U^2\, U', \qquad \alpha = \alpha(X)\ \&\ U = U(X) \tag{1}$$

with the initial conditions,

$$\text{for } x = 0 \text{ then } U = a \text{ and } U' = b. \tag{2}$$

Hence, $X = x + yI$ and $U = u_1 + u_2 I \in \mathbb{R}(I)$. $\alpha, \beta, \gamma \in \mathbb{R}(I)$ are analytic and can be written as follows

$\alpha = \alpha_1 + \alpha_2 I$ & $\beta = \beta_1 + \beta_2 I$ and $\gamma = \gamma_1 + \gamma_2 I$

with initial values $a = a_1 + a_2 I, b = b_1 + b_2 I \in \mathbb{R}(I)$.

By applying the one-dimensional isometric transform to equation (1), we obtain

$$T[U''] = T[\alpha]T[U'] + T[\beta]T[U] + T[\gamma]T[U^2]T[U'] \tag{3}$$

which leads to

$$\begin{aligned} &[u''_1(x), (u_1 + u_2)''(x+y)] \\ &\qquad = [\alpha_1(x), \alpha_2(x+y)][u'_1(x), (u_1+u_2)'(x+y)] \\ &\qquad + [\beta_1(x), \beta_2(x+y)][u_1(x), (u_1+u_2)(x+y)] \\ &\qquad + [\gamma_1(x), \gamma_2(x+y)][u_1(x)^2, (u_1+u_2)(x+y)^2][u'_1(x), (u_1+u_2)'(x+y)] \end{aligned} \tag{4}$$

From (2) become

$u_1(0) = a_1 \quad ; \quad u_1(0)' = b_1$

$(u_1 + u_2)(0) = a_1 + a_2 \quad ; \quad (u_1 + u_2)'(0) = b_1 + b_2$

Thus, two classical problems are obtained:

$$\begin{cases} u''_1(x) = \alpha_1(x)u'_1(x) + \beta_1(x)u_1(x) + \gamma_1(x)\, u_1{}^2(x)u'_1(x) \\ \qquad u_1(0) = a_1 \quad ; \quad u_1{}'(0) = b_1 \end{cases} \tag{5}$$

$$\begin{cases} (u_1 + u_2)''(x + y) \ = (\alpha_1 + \alpha_2)(x + y)\,(u_1 + u_2)'(x + y) + \\ \qquad (\beta_1 + \beta_2)(x + y)(u_1 + u_2)(x + y) + \\ \qquad (\gamma_1 + \gamma_2)(x + y)(u_1 + u_2)^2(x + y)(u_1 + u_2)'(x + y) \\ (u_1 + u_2)(0) = a_1 + a_2 \quad ; \quad (u_1 + u_2)'(0) = b_1 + b_2 \end{cases} \tag{6}$$

By applying HPM [15-17] to (5), we construct the Homotopy:

$$H(u_1, P) = u_1{}'' - P(\alpha_1 u_1{}' + \beta_1 u_1 + \gamma_1 u_1{}^2 u_1{}') = \ 0 \tag{7}$$

where $P \in [0,1]$ is an embedding parameter

According to HPM, the solution of (7) can be expressed as

$$u_1 = \sum_{i=0}^{\infty} P^i (u_1)_i \tag{8}$$

Substituting (8) into (7), we obtain

$$\sum_{i=0}^{\infty} P^i\,(u_1)_i'' - P\left(\alpha_1 \sum_{i=0}^{\infty} P^i (u_1)_i' + \beta_1 \sum_{i=0}^{\infty} P^i (u_1)_i + \gamma_1 \left(\sum_{i=0}^{\infty} P^i (u_1)_i\right)^2 \left(\sum_{i=0}^{\infty} P^i (u_1)_i'\right)\right) = 0 \tag{9}$$

By equating (9) the coefficients of like powers of P, yields:

P^0: $(u_1)_0'' = 0$

P^1: $(u_1)_1'' = \alpha_1 (u_1)_0' + \beta_1 (u_1)_0 + \gamma_1 (u_1)_0^2 (u_1)_0'$

From the initial conditions:

$(u_1)_0 = a_1 + b_1 x$

$$(u_1)_1 = \iint\limits_{0\,0}^{x\,x} \left(a_1 \alpha_1(t) + (a_1 + b_1 t)\beta_1(t) + a_1(a_1 + b_1 t)^2 \gamma_1(t)\right) dt dt$$

Hence, the solution of problem (9) can be expressed as

$$u_1(x) = \sum_{i=0}^{\infty} (u_1)_i \tag{10}$$

For the second problem (6), we construct the Homotopy:

$$\big((u_1+u_2),P\big)=(u_1+u_2)''(x+y) - P\begin{pmatrix}(\alpha_1+\alpha_2)(x+y)(u_1+u_2)'(x+y)+\\(\beta_1+\beta_2)(x+y)(u_1+u_2)(x+y)+\\(\gamma_1+\gamma_2)(x+y)(u_1+u_2)^2(x+y)(u_1+u_2)'(x+y)\end{pmatrix} \tag{11}$$

where the solution of (11) is assumed in the form

$$(u_1+u_2)=\sum_{i=0}^{\infty}P^i(u_1+u_2)_i \tag{12}$$

By substituting (12) into (11) and comparing coefficients of powers of

$$\sum_{i=0}^{\infty}P^i\,(u_1+u_2)_i''-P\begin{pmatrix}(\alpha_1+\alpha_2)(x+y)\sum_{i=0}^{\infty}P^i(u_1+u_2)_i'+\\(\beta_1+\beta_2)(x+y)\sum_{i=0}^{\infty}P^i(u_1+u_2)_i+\\(\gamma_1+\gamma_2)(x+y)\left(\sum_{i=0}^{\infty}P^i(u_1+u_2)_i\right)^2\left(\sum_{i=0}^{\infty}P^i(u_1+u_2)_i'\right)\end{pmatrix}=0 \tag{13}$$

Then:

P^0: $(u_1+u_2)_0''=0$

P^1: $(u_1+u_2)_1''=(\alpha_1+\alpha_2)(u_1+u_2)_0'+(\beta_1+\beta_2)(u_1+u_2)_0+(\gamma_1+\gamma_2)(u_1+u_2)_0^2(u_1+u_2)_0'$

with the initial conditions:

$$(u_1+u_2)_0(x+y)=(a_1+a_2)+(b_1+b_2)(x+y)$$
$$=\iint_{00}^{xx}\begin{pmatrix}(a_1+a_2)\big(\alpha_1(t)+\alpha_2(t)\big)+\\\big((a_1+a_2)+(b_1+b_2)t\big)\big(\beta_1(t)+\beta_2(t)\big)+\\(a_1+a_2)\big((a_1+a_2)+(b_1+b_2)t\big)^2\big(\gamma_1(t)+\gamma_2(t)\big)\end{pmatrix}dtdt$$

Thus, the solution of (13) can be expressed by:

$$(u_1+u_2)_1(x+y)=\sum_{i=0}^{\infty}(u_1+u_2)_i \tag{14}$$

Finally, by applying the inverse isometric transform to relations (13) and (14), we obtain the solution of the neutrosophic (1):

$$U(X)=T^{-1}\big(u_1(x),(u_1+u_2)_1(x+y)\big)=\sum_{i=0}^{\infty}(u_1)_i+I\left(\sum_{i=0}^{\infty}(u_1+u_2)_i-\sum_{i=0}^{\infty}(u_1)_i\right)=\sum_{i=0}^{\infty}U_i$$

4 APPLICATION

Let us consider the following problem:

$$U'' = -2U' + 3U - U^2U' \tag{15}$$

with

$$\text{For } x = 0 \text{ then } U = 0 \ \text{ and } U' = 1 \tag{16}$$

Hence, $X = x + yI$ and $U = u_1 + u_2 I \in \mathbb{R}(I)$

By applying the isometric transform to equation (15), we obtain:

$$T[U''] = -2T[U'] + 3T[U] - T[U^2]T[U'] \tag{17}$$

which leads to:

$$\begin{aligned}&[u_1''(x), (u_1+u_2)''(x+y)] \\ &\qquad = -2[u_1', (u_1+u_2)'] + 3[u_1, (u_1+u_2)] - [u_1^2, (u_1+u_2)^2][u_1', (u_1+u_2)']\end{aligned} \tag{18}$$

Hence, we obtain two classical problems:

$$\begin{cases} u_1'' = -2u_1' + 3u_1 - u_1^2 u_1' \\ u_1(0) = 0 \qquad ; \qquad u_1'(0) = 1 \end{cases} \tag{19}$$

$$\begin{aligned}&(u_1+u_2)'' = -2(u_1+u_2)' + 3(u_1+u_2) - (u_1+u_2)^2(u_1+u_2)' \\ &(u_1+u_2)(0) = 0 \qquad ; \qquad (u_1+u_2)'(0) = 1\end{aligned} \tag{20}$$

By applying the Homotopy Perturbation Method (HPM) to problem (19), we construct:

$$u_1'' = P(-2u_1' + 3u_1 - u_1^2 u_1') \tag{21}$$

Substituting expansion (8) into (21), we obtain:

$$\sum_{i=0}^{\infty} p^i (u_1)_i'' = P\left(-2\sum_{i=0}^{\infty} p^i (u_1)_i' + 3\sum_{i=0}^{\infty} p^i (u_1)_i - \left(\sum_{i=0}^{\infty} p^i (u_1)_i\right)^2 \left(\sum_{i=0}^{\infty} p^i (u_1)_i'\right)\right) \tag{22}$$

By equating coefficients of like powers of$P$, yields:

$p^0 :\ (u_1)_0'' = 0$

$p^1 :\ (u_1)_1'' = 3(u_1)_0 - 2(u_1)_0' - (u_1)_0^{\,2}(u_1)_0'$

$p^2 :\ v_2''(x) = 3(u_1)_1 - 2(u_1)'_1 - (u_1)_0^{\,2}(u_1)'_1 - 2(u_1)_0(u_1)_1(u_1)_0'$

Thus, the perturbation series solution of equation (19) is:

$(u_1)_0 = x$

$$(u_1)_1 = -x^2 + \frac{x^3}{2} - \frac{x^4}{12}$$

$$(u_1)_2 = \frac{2x^3}{3} - \frac{x^4}{4} + \frac{23x^5}{120} - \frac{11x^6}{120} - \frac{x^7}{84}$$

$$(u_1)_3 = -\frac{x^4}{3} + \frac{3x^5}{40} - \frac{29x^6}{360} + \frac{451x^7}{5040} - \frac{79x^8}{1680} + \frac{19x^9}{1260} - \frac{7x^{10}}{4320}$$

$$\begin{aligned}(u_1)_4 = {} & \frac{2x^5}{12} - \frac{x^6}{48} + \frac{43x^7}{1008} - \frac{2473x^8}{40320} + \frac{7243x^9}{120960} - \frac{604x^{10}}{14175} + \frac{4091x^{11}}{184800} - \frac{48583x^{12}}{4989600} + \frac{3709x^{13}}{95356000} \\ & - \frac{4901x^{16}}{50803200} + \frac{57221x^{17}}{3454617600} - \frac{8333x^{18}}{3886444800} + \frac{x^{19}}{8864640}\end{aligned}$$

Similarly, the perturbation series solution of equation (20) is:

$$(u_1 + u_2)_0 = x + y$$

$$(u_1 + u_2)_1 = -(x+y)^2 + \frac{(x+y)^3}{2} - \frac{(x+y)^4}{12}$$

$$(u_1 + u_2)_2 = \frac{2(x+y)^3}{3} - \frac{(x+y)^4}{4} + \frac{23(x+y)^5}{120} - \frac{11(x+y)^6}{120} - \frac{(x+y)^7}{84}$$

$$(u_1 + u_2)_3 = -\frac{(x+y)^4}{3} + \frac{3(x+y)^5}{40} - \frac{29(x+y)^6}{360} + \cdots$$

$$(u_1 + u_2)_4 = \frac{2(x+y)^5}{12} - \frac{(x+y)^6}{48} + \frac{43(x+y)^7}{1008} - \frac{2473(x+y)^8}{40320} + \cdots$$

By applying the inverse isometric transform, the solution of the neutrosophic problem (15) takes the form:

$$U = T^{-1}[u_1, u_1 + u_2]$$

$$\begin{aligned}T^{-1}[&(u_1)_0 + (u_1)_1 + (u_1)_2 + (u_1)_3 + (u_1)_4 + \cdots, (u_1+u_2)_0 + (u_1+u_2)_1 + (u_1+u_2)_2 + (u_1+u_2)_3 \\ & + (u_1+u_2)_4 + \cdots] \\ & = (u_1)_0 + (u_1)_1 + (u_1)_2 + (u_1)_3 + (u_1)_4 + \cdots \\ & + I[(u_1+u_2)_0 + (u_1+u_2)_1 + (u_1+u_2)_2 + (u_1+u_2)_3 + (u_1+u_2)_4 + \cdots \\ & - ((u_1)_0(x) + (u_1)_1(x) + (u_1)_2(x) + (u_1)_3(x) + (u_1)_4(x) + \cdots)] = \sum_{i=0}^{\infty} U_i(X)\end{aligned}$$

Table 1: Compares HPM and RK4 solutions for the neutrosophic Van der Pol problem.

X	S_{RK4}	S_{HPM} $n=4$	S_{HPM} $n=5$
0	0	0	0
0.001	$9.990011657507490\,e-04$	$9.990011657506078\,e-04$	$9.990011657507410\,e-04$
0.003	$2.991031426033710\,e-03$	$2.991031425897445\,e-03$	$2.991031425929749\,e-03$
0.005	$4.975145262758745\,e-03$	$4.975145262309570\,e-03$	$4.975145262724187\,e-03$
0.007	$6.951397985652526\,e-03$	$6.951397975913045\,e-03$	$6.951397978138644\,e-03$
0.009	$8.919844605804774\,e-03$	$8.919844521395253\,e-03$	$8.919844529199635\,e-03$
0.01	$9.901157698531246\,e-03$	$9.901157560313845\,e-03$	$9.901157573518107\,e-03$
0.03	$2.913077502126183\,e-02$	$2.913077190773699\,e-02$	$2.913077505968901\,e-02$
0.05	$4.764033561432903\,e-02$	$4.764028632168047\,e-02$	$4.764032622605293\,e-02$
0.07	$6.547933048890503\,e-02$	$6.547912069076151\,e-02$	$6.547933242119513\,e-02$
0.09	$8.269440112585977\,e-02$	$8.269368482549287\,e-02$	$8.269442024692327\,e-02$
0.1	$9.108179307220726\,e-02$	$9.108058758622024\,e-02$	$9.108182687589945\,e-02$
0.2	$1.680671797415925\,e-01$	$1.680311813180952\,e-01$	$1.680699243605246\,e-01$
0.3	$2.354916069229221\,e-01$	$2.352264803134821\,e-01$	$2.355257445257613\,e-01$
0.4	$2.967879021285581\,e-01$	$2.956803647390476\,e-01$	$2.970009912846601\,e-01$
0.5	$3.545848967266488\,e-01$	$3.511884416852679\,e-01$	$3.554892029556845\,e-01$

5 CONCLUSION

In this study, the Homotopy Perturbation Method (HPM) combined with the one-dimensional geometric isometric transformation was employed to solve the nonlinear neutrosophic Van der Pol equation. The analytical and numerical results show that this hybrid approach is not only highly accurate but also significantly more efficient than the classical RK4 method, thereby confirming the capability of HPM in addressing complex neutrosophic differential equations. Moreover, integrating HPM with the isometric transformation provides a flexible and robust mathematical framework, enabling future extensions to a broad class of neutrosophic differential equations in mathematical modeling, dynamical systems analysis, and the study of physical and engineering phenomena characterized by uncertainty and indeterminacy. This approach opens new avenues for precise, scalable, and practically applicable solutions in advanced scientific research.

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