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# Some Cases On The Diophantine Equation $a^x + b^y = (a + 2m)^z$

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**ABSTRACT**

*A Diophantine equation is a polynomial equation in several unknowns, where the value of the unknowns are integers. Exponential Diophantine equation is a mathematical equation that involve at least one variable appears as an exponent in the equation. This study investigates specific cases of the exponential Diophantine equation $a^x + b^y = (a + 2m)^z$, which arises in number theory and has implications in fields such as crypthography and algebraic geometry. We consider the cases where $a$ is a positive integer and $x, y, z \leq 4$. The analysis and exploration of solution patterns will be performed using C-language programming implemented in Code::Blocks. This project found that the integral solutions to the equation $a^x+b^y = (a+2m)^z$ are: $(a, b, m, x, y, z) = (1, (1+2m)^z-1, m, 1, 1, z), (a, (a+4)^2-a, 2, 1, 1, 2)$ and for the case of $a \in \mathbb{Z}^+$ and $(y, z) = (3, 3)$, there are no positive integral solutions to the equation.*

**Keywords:** Diophantine equation, number theory, integer solutions.

## 1 INTRODUCTION

A Diophantine equation is a type of equation that encompasses sums, products, and powers, with all the coefficients being integers, and also the integer solutions are sought. The term "Diophantine" originates from the name of Diophantus of Alexandria, who investigate these types of equations during the 3rd century [1]. Hilbert's tenth problem poses a famous question about the existence of algorithms or formulas capable of determining solutions for any Diophantine equation [2]. Determining the solvability of a given diophantine equation is typically challenging.

Diophantine equation has been studied by many authors with different type of equations. [3] discussed an open problem given by [4] that the Diophantine equation $8^x + p^y = z^2$ where $x, y$ and $z$ are positive integers has only three solutions namely $(x, y, z) = (1, 1, 5), (2, 1, 9)$ and $(3, 1, 23)$ for $p = 17$. [5] established that the solution $(x, y, z) = (1, 0, 12)$ is the unique non-negative integer solution to the Diophantine equation $143^x + 145^y = z^2$.

[6] proved the Diophantine equation with three unknowns, $2^x + 47^y = z^2$ have two solutions including $(x, y, z) = (3, 0, 3)$ and $(1, 1, 7)$ where $x, y$ and $z$ are non-negative integer. The research by [7] explored the Diophantine equation $a^x + b^y = (a + 2)^z$, proposing further investigation into the

properties of the solutions when $a \equiv -1 \pmod{b}$. The objective of this problem was to determine all positive integer solutions $(x, y, z)$ for the equation when $a$ and $b$ are co-prime integers with specific congruence conditions. Their findings provide a basis for extending values of $a, b$ and $z$, particularly in special cases such as Pythagorean triples or polynomial-based forms, illustrating the complexity of solving such equations even with restricted variable ranges.

According to [8], the Diophantine equation $7^x + 10^y = z^2$, where $x, y$ and $z$ are positive integers, does not have any positive integer solutions. [9] examined on the simultaneous Diophantine equation of the form $17^x + 83^y = z^2$ and $29^x + 71^y = z^2$ and found that both have a unique solution which is $(x, y, z) = (1, 1, 10)$.

Inspired by [7] suggestion, this paper concentrates on finding an integral solution to the Diophantine equation in the form of $a^x + b^y = (a + 2m)^z$, for the range of $a \in \mathbb{Z}^+$ and $x, y, z \leq 4$.

## 2 RESULTS

We consider small bounded values for the variables and apply algebraic manipulation and number-theoretic reasoning to determine whether the equation has positive integer solutions. In particular, we fix certain variables to simplify the search and test for general patterns. We will discuss on finding the integral solutions to the Diophantine equation in the form of $a^x + b^y = (a + 2m)^z$, for the range of $a \in \mathbb{Z}^+$ and $x, y, z \leq 4$. The results are as follows:

**Theorem 2.1** *Let a, b, m, x, y, z be positive integers. Suppose $a = x = 1$. There only exists a solution $(a, b, m, x, y, z) = (1, (1+2m)^z-1, m, 1, 1, z)$ to the Diophantine equation $a^x+b^y = (a+2m)^z$.*

**Proof.** We consider two cases, based on the value of $y$.
**Case 1:** We begin by analyzing the base case where all variables are minimized (a = x = y = 1). This allows us to identify simple patterns and test for initial solvability, which will guide further case extensions. Then, we substitute this values into equation $a^x + b^y = (a + 2m)^z$, we have:

$$1 + b = (1 + 2m)^z.$$

That is,

$$b = (1 + 2m)^z - 1. \tag{1}$$

From Eq. (1), the general solution we obtain is of the form expressed as $(a, b, m, x, y, z) = (1, (1 + 2m)^z - 1, m, 1, 1, z)$ for $z, m \in \mathbb{Z}^+$. Now, we consider the next case.

**Case 2:** Now, we continue with $a = x = 1, y = 2$ and $z = 3$. Then, substitute into equation $a^x + b^y = (a + 2m)^z$, we obtain:

$$1 + b^2 = (1 + 2m)^3.$$

That is,

$$1 + b^2 = 1 + 6m + 12m^2 + 8m^3$$
$$b^2 = 8m^3 + 12m^2 + 6m + 1 - 1$$
$$b^2 = 8m^3 + 12m^2 + 6m.$$

Let $m = 2^\beta k$ ; $(2,k) = 1$, $\beta \geq 1$, we have:

$$b^2 = 8(2^\beta k)^3 + 12(2^\beta k)^2 + 6(2^\beta k)$$
$$b^2 = 8(2^{3\beta} k^3) + 12(2^{2\beta} k^2) + 6(2^\beta k)$$
$$b^2 = 2^\beta k(6 + 12 \cdot 2^\beta k + 8 \cdot 2^{2\beta} k^2).$$

Since $b^2$ is a perfect square and $2^\beta k(6 + 12 \cdot 2^\beta k + 8 \cdot 2^{2\beta} k^2)$ is not a perfect square. From both cases, the solution to the Diophantine equation for case of $a = x = 1$ is $(a,b,m,x,y,z) = (1,(1+2m)^z - 1, m, 1, 1, z)$. □

**Theorem 2.2** *Let a, b, m, x, y, z be positive integers. Suppose $x = y = 1$ and $z = m = 2$. There exists a solution $(a,b,m,x,y,z) = (a,(a+4)^2 - a, 2, 1, 1, 2)$ to the Diophantine equation $a^x + b^y = (a+2m)^z$.*

**Proof.** We consider two cases, based on the parity of $a$ and $b$.
**Case 1:** Suppose both a and b are even. Suppose $x = y = 1$ and $z = m = 2$. Then, substitute into equation $a^x + b^y = (a+2m)^z$, we have:

$$a + b = (a + 2(2))^2.$$

That is,

$$a + b = (a+4)^2$$
$$b = (a+4)^2 - a. \quad (2)$$

From Eq. (2), the solution to the equation is $(a,b,m,x,y,z) = (a,(a+4)^2 - a, 2, 1, 1, 2)$. Now, we consider the next case.

**Case 2:** Suppose $a$ is even and $b$ is odd. Let $a = 2^\alpha k$ and $b = 2^\beta r + 1$ ; $(2,k) = (2,r) = 1$, $\alpha, \beta \geq 1$. Then, substitute into equation $a^x + b^y = (a+2m)^z$, we have:

$$(2^\alpha k) + (2^\beta r + 1) = (2^\alpha k + 4)^2.$$

That is,

$$2^\alpha k + 2^\beta r + 1 = 2^{2\alpha} k^2 + 8(2^\alpha k) + 16$$
$$2^\alpha k + 2^\beta r - 2^{2\alpha} k^2 - 8(2^\alpha k) = 15$$
$$2^\alpha k + 2^\beta r - 2^{2\alpha} k^2 - 2^{\alpha+3} k = 15.$$

Hence, it is a contradiction since $\alpha, \beta \geq 1$. The left hand side always be even while the right hand side always be odd value. From both cases, the solution to the Diophantine equation for case of $x = y = 1$ and $z = m = 2$ is $(a, b, m, x, y, z) = (a, (a + 4)^2 - a, 2, 1, 1, 2)$. □

**Theorem 2.3** *Let a, b, m, x, y, z be positive integers. There is no integral solution to the Diophantine equation* $a^x + b^y = (a + 2m)^z$ *for* $x = 2$ *and* $y = z = 3$.

**Proof.** Suppose $x = 2$ and $y = z = 3$. Substitute all values into equation $a^x + b^y = (a + 2m)^z$, we have:

$$a^2 + b^3 = (a + 2m)^3.$$

That is,

$$\begin{aligned}
a^2 + b^3 &= a^3 + 3a^2(2m) + 3a(2m)^2 + (2m)^3 \\
a^2 + b^3 &= a^3 + 6a^2m + 12am^2 + 8m^3 \\
b^3 &= a^3 + 6a^2m + 12am^2 + 8m^3 - a^2 \\
b^3 - 8m^3 &= a^3 + 6a^2m + 12am^2 - a^2 \\
b^3 - (2m)^3 &= a^3 + 6a^2m - a^2 + 12am^2 \\
b^3 - (2m)^3 &= a^3 + a^2(6m - 1) + 12am^2 \\
b^3 - (2m)^3 &= a(a^2 + a(6m - 1) + 12m^2) \\
(b - 2m)(b^2 + 2bm + 4m^2) &= a(a^2 + a(6m - 1) + 12m^2). \qquad (3)
\end{aligned}$$

By comparing both sides from Eq. (3), we have:

$$b - 2m = a \qquad (4)$$

and

$$b^2 + 2bm + 4m^2 = a^2 + a(6m - 1) + 12m^2. \qquad (5)$$

Substitute Eq. (4) into Eq. (5), we have:

$$b^2 + 2bm + 4m^2 = (b - 2m)^2 + (b - 2m)(6m - 1) + 12m^2.$$

That is,

$$\begin{aligned}
b^2 + 2bm + 4m^2 &= b^2 - 4bm + 4m^2 + 6bm - b - 12m^2 + 2m + 12m^2 \\
b^2 + 2bm + 4m^2 &= b^2 - 4bm + 6bm + 4m^2 - 12m^2 + 12m^2 - b + 2m \\
b^2 + 2bm + 4m^2 &= b^2 + 2bm + 4m^2 - b + 2m \\
b &= 2m. \qquad (6)
\end{aligned}$$

Substitute Eq. (6) into Eq. (4), we obtain:

$$(2m) - 2m = a$$
$$a = 0.$$

However, $a$ must be positive integers, and $a = 0$ contradicts the requirement that a must be positive. Therefore, we found that there is no integral solution to the Diophantine equation for case of $x = 2$ and $y = z = 3$. □

**Theorem 2.4** *Let a, b, m, x, y, z be positive integers. There is no integral solution to the Diophantine equation $a^x + b^y = (a + 2m)^z$ for $x = y = z = 3$ and $m = 2$ .*

**Proof.** Suppose $x = y = z = 3$ and $m = 2$. Substitute all values into equation $a^x + b^y = (a + 2m)^z$, we have:

$$a^3 + b^3 = (a + 2(2))^3.$$

That is,

$$a^3 + b^3 = (a + 4)^3$$
$$a^3 + b^3 = a^3 + 12a^2 + 48a + 64$$
$$b^3 = 12a^2 + 48a + 64$$
$$b^3 = 4(3a^2 + 12a + 16).$$

To determine whether the RHS is a perfect cube, we analyze each term individually. Both terms must be perfect cubes for the RHS to be a perfect cube. Now, we start with the first term on RHS which is 4.

$$4 = 2^2.$$

Next, we continue with the second term on RHS by compute the discriminant of the quadratic $3a^2 + 12a + 16$ where $a = 3, b = 12$ and $c = 16$, to determine whether it factors into rational components, which would be necessary for it to represent a perfect cube. A negative discriminant indicates complex roots, meaning the expression cannot be simplified into a perfect cube.

$$\Delta = b^2 - 4ac$$
$$\Delta = (12)^2 - 4(3)(16)$$
$$\Delta = -48.$$

Since $\Delta < 0$, the quadratic has complex roots. It cannot be factored into simpler form that could make it a cube and is irreducible. LHS is a perfect cube and RHS cannot be a perfect cube. Thus, we found that there is no integral solution to the Diophantine equation for case of $x = y = z = 3$ and $m = 2$. □

## 3 CONCLUSION

This study contributes to the broader understanding of exponential Diophantine equations by identifying compact solution forms for bounded variable cases and proving the non-existence of solutions in others. These findings offer insight into the behavior of such equations under specific structural constraints and open potential for further investigation in generalizations with higher exponents or alternative parameterizations. From this study, we found that the integral solutions to the Diophantine equation $a^x + b^y = (a + 2m)^z$ for the case of $a \in \mathbb{Z}^+$ and $x, y, z \leq 4$ are as follow: $(a, b, m, x, y, z) = (1,(1 + 2m)^z - 1, m, 1, 1, z)$ and $(a,(a + 4)^2 - a, 2, 1, 1, 2)$. We also show that there exists no positive integral solutions to the equation when $a \in \mathbb{Z}^+$ and $(y, z) = (3, 3)$. For future research, we could extend the equation $a^x + b^y = (a + 2m)^z$ such as $a^x + b^y = (a + 2m)^{2z}$ for $x, y, z > 4$ or $a^x + b^y = (a + m)^z$ under the same conditions.

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