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A New Fifth Order Variable Step Size Block Backward Differentiation Formula with Off-Step Points for the Numerical Solution of Stiff Ordinary Differential Equations

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ABSTRACT

A new fully implicit two point variable step size based on block backward differentiation formula with two off-step points for the numerical integration of first order stiff ordinary differential equations in initial value problems is proposed. The methods are derived by introducing three different values of the step size ratio to the existing fifth order 2-point block backward differentiation formula with off-step points for solving stiff ordinary differential equations. The methods approximate two solutions values with two off-step points simultaneously at each step of the integration in block. The order, error constant, and consistency of the methods are presented. The stability analysis of the methods indicates that the methods are both zero and A-stable. The proposed methods are implemented in Microsoft Dev C++ compiler using Newton's iteration and the numerical comparison of results with existing algorithm of the same order shows that the proposed methods are better in terms of accuracy and compete with 3DIBBDF in terms of computation time. Hence, the proposed methods serve as alternative solver for stiff ODEs.

Keywords: Stiff, Variable Step Size, Block Backward Differentiation Formula, Off-Step, A-stability.

1 INTRODUCTION

In this paper, we shall consider the first order stiff initial value problems in ordinary differential equation of the form:

$$u' = f(x,u), \quad u(a) = u_0, \quad a \le x \le b \tag{1}$$

where the function $f(x,u)$ is assumed to satisfy the Lipschitz conditions for the existence and uniqueness of solutions which guarantees that the ordinary differential equation (1) has a uniquely continuous differentiable solutions [1]. The solution of such equation is characterized by the presence of transient and steady state terms which restricts the step length of many numerical integration schemes except those numerical methods with A-stability properties [2].

The system of ordinary differential equation (1) is said to be "stiff" when an extremely small step size is required to obtain correct numerical approximation. In other word, stiff problems are equations where certain implicit numerical methods, in particular backward differentiation formula (BDF) perform better than explicit numerical schemes [3]. Most interesting and physically relevant real world stiff problems are difficult to solve analytically rather an alternative numerical methods are used in determining approximate solutions to the problems. However, in dealing with stiff ODEs, the stiffness property restricts the conventional explicit numerical integration methods from handling the problems efficiently. The stiff IVPs occur in many fields of science, engineering and technology, they are particularly found in chemical kinetics, thermodynamics and heat flow, control systems, vibration of the strings, electrical circuits, nuclear radioactive decay, weather prediction and forecasting.

Implicit linear multistep methods are known to be best and suitable for the treatment of stiff ODEs, the backward differentiation formula was developed by [3], since then most of the improvements in the class of linear multistep methods are based on BDF, this is due to its special properties and better stability characteristics. Considerable recent research efforts have been made by the researchers to formulate fully and diagonally implicit block methods of both constant and variable step size technique with A-stability property suitable for the numerical solution of stiff ordinary differential equations such as [5],[6],[7],[8],[9],[10],[11],[12],[13],[15],[16],[17],[18],[19]. This paper focuses on the derivation, stability analysis and implementation of variable step size form of the formula in [14], by introducing different values of r but those that allow for the stability properties of the method will be used as they are the ones that can lead us to a reasonably accurate numerical approximation when implemented on first order stiff problems.

2 FORMULATION OF THE METHOD

In this section, we shall be concerned with the formulation of the proposed method we shall call fully implicit 2-point variable step size block backward differentiation formula with two off-step points for the numerical integration of stiff initial value problems.

Definition 2.1: The fully implicit 2-point variable step size block BDF with two off-step points for the numerical integration of stiff initial value problems is defined as:

$$\sum_{j=0}^{1} \alpha_{j,k,r} u_{n+j-1} + \sum_{j=0}^{3} \alpha_{j+2,k,r} u_{n+(j+1)/2} = h\beta_{k,r} f_{n+k}, \qquad k = \frac{1}{2}, 1, \frac{3}{2}, 2 \tag{2}$$

The method (2) approximate two solutions value with a step size h and two off-step points which are chosen when the step size is halved, the values of r that would be used in this paper are $r = 1, r = 2$ and $r = \frac{5}{8}$ corresponding to the step size control strategy of maintaining constant, doubling or multiplying the step size by a factor of 1.6 respectively. The Taylor's series expansion about x_n is used for the derivation of the method.

Definition 2.2: the linear difference operator L_i associated with the fully implicit 2-point variable step size BDF method with two off-step points is defined by

$$L_i[u(x_n), h] = \alpha_{0,k} u(x_n - rh) + \alpha_{1,k} u(x_n) + \alpha_{2,k} u\left(x_n + \frac{1}{2}h\right) + \alpha_{3,k} u(x_n + h)$$

$$+\alpha_{4,k}u\left(x_n+\frac{3}{2}h\right)+\alpha_{5,k}u(x_n+2h)-h\beta_k f(x_n+kh)=0,\quad k=\frac{1}{2},1,\frac{3}{2},2 \tag{3}$$

In this paper, we only consider the derivation of a first off-step point as similar procedures are used for the derivation of first, second off-step and second point respectively. We shall consider the derivation of first off-step point as case 1 as follows:

First Off-step Point: $k=\frac{1}{2}$

In deriving the first off-step point, the associated approximate relationship for the linear difference operator (3) is

$$L_{\frac{1}{2}}[u(x_n),h]=\alpha_{0,\frac{1}{2}}u(x_n-rh)+\alpha_{1,\frac{1}{2}}u(x_n)+\alpha_{2,\frac{1}{2}}u\left(x_n+\frac{1}{2}h\right)+\alpha_{3,\frac{1}{2}}u(x_n+h)$$

$$+\alpha_{4,\frac{1}{2}}u\left(x_n+\frac{3}{2}h\right)+\alpha_{5,\frac{1}{2}}u(x_n+2h)-h\beta_{\frac{1}{2}}f\left(x_n+\frac{1}{2}h\right)=0, \tag{4}$$

Expanding the functions in (4) as a Taylor's series about x_n, equating and collect the like terms gives

$$C_{0,\frac{1}{2}}u(x_n)+C_{1,\frac{1}{2}}hu'(x_n)+C_{2,\frac{1}{2}}h^2u''(x_n)+C_{3,\frac{1}{2}}h^3u'''(x_n)+\cdots=0, \tag{5}$$

where,

$$\begin{aligned}
C_{0,\frac{1}{2}} &= \alpha_{0,\frac{1}{2}}+\alpha_{1,\frac{1}{2}}+\alpha_{2,\frac{1}{2}}+\alpha_{3,\frac{1}{2}}+\alpha_{4,\frac{1}{2}}+\alpha_{5,\frac{1}{2}}=0,\\
C_{1,\frac{1}{2}} &= -r\alpha_{0,\frac{1}{2}}+\frac{1}{2}\alpha_{2,\frac{1}{2}}+\alpha_{3,\frac{1}{2}}+\frac{3}{2}\alpha_{4,\frac{1}{2}}+2\alpha_{5,\frac{1}{2}}-\beta_{\frac{1}{2}}=0,\\
C_{2,\frac{1}{2}} &= \frac{1}{2}r^2\alpha_{0,\frac{1}{2}}+\frac{1}{8}\alpha_{2,\frac{1}{2}}+\frac{1}{2}\alpha_{3,\frac{1}{2}}+\frac{9}{8}\alpha_{4,\frac{1}{2}}+2\alpha_{5,\frac{1}{2}}-\frac{1}{2}\beta_{\frac{1}{2}}=0,\\
C_{3,\frac{1}{2}} &= -\frac{1}{6}r^3\alpha_{0,\frac{1}{2}}+\frac{1}{48}\alpha_{2,\frac{1}{2}}+\frac{1}{6}\alpha_{3,\frac{1}{2}}+\frac{9}{16}\alpha_{4,\frac{1}{2}}+\frac{4}{3}\alpha_{5,\frac{1}{2}}-\frac{1}{8}\beta_{\frac{1}{2}}=0,\\
C_{4,\frac{1}{2}} &= \frac{1}{24}r^4\alpha_{0,\frac{1}{2}}+\frac{1}{384}\alpha_{2,\frac{1}{2}}+\frac{1}{24}\alpha_{3,\frac{1}{2}}+\frac{27}{128}\alpha_{4,\frac{1}{2}}+\frac{2}{3}\alpha_{5,\frac{1}{2}}-\frac{1}{48}\beta_{\frac{1}{2}}=0,\\
C_{5,\frac{1}{2}} &= -\frac{1}{120}r^5\alpha_{0,\frac{1}{2}}+\frac{1}{3840}\alpha_{2,\frac{1}{2}}+\frac{1}{120}\alpha_{3,\frac{1}{2}}+\frac{81}{1280}\alpha_{4,\frac{1}{2}}+\frac{4}{15}\alpha_{5,\frac{1}{2}}-\frac{1}{384}\beta_{\frac{1}{2}}=0.
\end{aligned} \tag{6}$$

In formulating the first off-step point $u_{n+\frac{1}{2}}$, the coefficient $\alpha_{3,\frac{1}{2}}$ is normalized to 1. Solving the set simultaneous equation (6) gives the values of $\alpha_{j,k}$ and β_k as:

$$\alpha_{0,\frac{1}{2}}=-\frac{9}{2(r+1)(10r^2+19r-2)(2r+3)r},\ \alpha_{1,\frac{1}{2}}=\frac{3}{4}\frac{4r^2+4r+1}{r(10r-1)},\ \alpha_{2,\frac{1}{2}}=-\frac{9}{2}\frac{4r^2+4r+1}{(r+1)(10r-1)},\ \alpha_{3,\frac{1}{2}}=1$$

$$\alpha_{4,\frac{1}{2}}=\frac{3(4r^2+4r+1)}{(2r+3)(10r-1)},\ \alpha_{5,\frac{1}{2}}=-\frac{1}{4}\frac{4r^2+4r+1}{10r^2+19r-2}\text{ and }\beta_{\frac{1}{2}}=-\frac{3(2r+1)}{10r-1}$$

The first off-step point is therefore obtained as follows

$$u_{n+\frac{1}{2}}=\frac{9}{2(r+1)(10r^2+19r-2)(2r+3)r}u_{n-1}-\frac{3}{4}\frac{4r^2+4r+1}{r(10r-1)}u_n+\frac{9}{2}\frac{4r^2+4r+1}{(r+1)(10r-1)}u_{n+1}-\frac{3(4r^2+4r+1)}{(2r+3)(10r-1)}u_{n+\frac{3}{2}}$$

$$+\frac{1}{4}\frac{4r^2+4r+1}{10r^2+19r-2}u_{n+2}-\frac{3(2r+1)}{10r-1}hf_{n+\frac{1}{2}} \tag{7}$$

The same technique is applied in formulating of first, second off-step and third points as in the derivation of the first off-step point. Therefore, fully implicit 2-point variable step size block backward differentiation formula with two off-step points (2BBDFO) is obtained as:

$$\begin{aligned}
u_{n+\frac{1}{2}} &= \frac{9}{2(r+1)(10r^2+19r-2)(2r+3)r}u_{n-1}-\frac{3}{4}\frac{4r^2+4r+1}{r(10r-1)}u_n+\frac{9}{2}\frac{4r^2+4r+1}{(r+1)(10r-1)}u_{n+1}-\frac{3(4r^2+4r+1)}{(2r+3)(10r-1)} \\
&\quad u_{n+\frac{3}{2}}+\frac{1}{4}\frac{4r^2+4r+1}{10r^2+19r-2}u_{n+2}-\frac{3(2r+1)}{10r-1}hf_{n+\frac{1}{2}}, \\
u_{n+1} &= \frac{1}{r(2r+1)(2r+3)(r+2)}u_{n-1}-\frac{1}{12}\frac{2r^2+4r+2}{r}u_n-\frac{1}{3}\frac{-8r^2-16r-8}{2r+1}u_{n+\frac{1}{2}}-\frac{1}{3}\frac{8r^2+16r+8}{2r+3}u_{n+\frac{3}{2}}+ \\
&\quad \frac{1}{12}\frac{2r^2+4r+2}{r+2}u_{n+2}+(r+1)hf_{n+1}, \\
u_{n+\frac{3}{2}} &= \frac{9}{2r(r+1)(-10r-21)(2r+1)(r+1)(r+2)}u_{n-1}-\frac{1}{4}\frac{4r^2+12r+9}{r(-10r-21)}u_n+\frac{12r^2+36r+27}{(2r+1)(-10r-21)}u_{n+\frac{1}{2}}- \\
&\quad \frac{3}{2}\frac{12+36r+27}{(r+1)(-10r-21)}u_{n+1}+\frac{1}{4}\frac{12r^2+36r+27}{(-10r-21)(r+2)}u_{n+2}-\frac{3(2r+3)}{-10r-21}hf_{n+\frac{3}{2}}, \\
u_{n+2} &= \frac{72}{r(-50r-112)(2r+1)(r+1)(2r+3)}u_{n-1}+\frac{6r^2+24r+24}{r(-50r-112)}u_n+\frac{4(16r^2+64r+64)}{(2r+1)(-50r-112)}u_{n+\frac{1}{2}}+ \\
&\quad \frac{18(4r^2+16r+16)}{(r+1)(-50r-112)}u_{n+1}-\frac{4(48r^2+192r+192)}{(2r+3)(-50r-112)}u_{n+\frac{3}{2}}-\frac{12(r+2)}{-50r-112}hf_{n+2}.
\end{aligned} \tag{8}$$

For the purpose of zero and absolute stability region of the method, three different values of $r=1$, $r=2$ and $r=\frac{5}{8}$ are selected as in [16]. By substituting the values of r in equation (8), thus fully implicit 2-point variable step size block backward differentiation formula with two off-step points becomes:

For $r=1$

$$\begin{aligned}
u_{n+\frac{1}{2}} &= \frac{1}{60}u_{n-1}-\frac{3}{4}u_n+\frac{9}{4}u_{n+1}-\frac{3}{5}u_{n+\frac{3}{2}}+\frac{1}{12}u_{n+2}-hf_{n+\frac{1}{2}}, \\
u_{n+1} &= \frac{1}{45}u_{n-1}-\frac{2}{3}u_n+\frac{32}{9}u_{n+\frac{1}{2}}-\frac{32}{15}u_{n+\frac{3}{2}}+\frac{2}{9}u_{n+2}+2hf_{n+1}, \\
u_{n+\frac{3}{2}} &= -\frac{1}{124}u_{n-1}+\frac{25}{124}u_n-\frac{25}{31}u_{n+\frac{1}{2}}+\frac{225}{124}u_{n+1}-\frac{25}{124}u_{n+2}+\frac{15}{31}hf_{n+\frac{3}{2}}, \\
u_{n+2} &= \frac{2}{135}u_{n-1}-\frac{1}{3}u_n+\frac{32}{27}u_{n+\frac{1}{2}}-2u_{n+1}+\frac{32}{15}u_{n+\frac{3}{2}}+\frac{2}{9}hf_{n+2}.
\end{aligned} \tag{9}$$

For $r=2$

$$\begin{aligned}
u_{n+\frac{1}{2}} &= \frac{3}{2128}u_{n-1}-\frac{75}{152}u_n+\frac{75}{38}u_{n+1}-\frac{75}{133}u_{n+\frac{3}{2}}+\frac{25}{304}u_{n+2}-\frac{15}{19}hf_{n+\frac{1}{2}}, \\
u_{n+1} &= \frac{1}{280}u_{n-1}-\frac{3}{4}u_n+\frac{24}{5}u_{n+\frac{1}{2}}-\frac{24}{7}u_{n+\frac{3}{2}}+\frac{3}{8}u_{n+2}+3hf_{n+1}, \\
u_{n+\frac{3}{2}} &= -\frac{3}{3280}u_{n-1}+\frac{49}{328}u_n-\frac{147}{205}u_{n+\frac{1}{2}}+\frac{147}{82}u_{n+1}-\frac{147}{656}u_{n+2}+\frac{21}{41}hf_{n+\frac{3}{2}}, \\
u_{n+2} &= \frac{3}{1855}u_{n-1}-\frac{12}{53}u_n+\frac{256}{265}u_{n+\frac{1}{2}}-\frac{96}{53}u_{n+1}+\frac{768}{371}u_{n+\frac{3}{2}}+\frac{12}{53}hf_{n+2}.
\end{aligned} \tag{10}$$

For $r=\frac{5}{8}$

$$\begin{aligned} u_{n+\frac{1}{2}} &= \frac{4096}{54145}u_{n-1} - \frac{81}{70}u_n + \frac{243}{91}u_{n+1} - \frac{81}{119}u_{n+\frac{3}{2}} + \frac{9}{98}u_{n+2} - \frac{9}{7}hf_{n+\frac{1}{2}}, \\ u_{n+1} &= \frac{1024}{16065}u_{n-1} - \frac{169}{240}u_n + \frac{169}{54}u_{n+\frac{1}{2}} - \frac{169}{102}u_{n+\frac{3}{2}} + \frac{169}{1008}u_{n+2} + \frac{13}{8}hf_{n+1}, \\ u_{n+\frac{3}{2}} &= -\frac{4096}{148785}u_{n-1} + \frac{289}{1090}u_n - \frac{289}{327}u_{n+\frac{1}{2}} + \frac{2601}{1417}u_{n+1} - \frac{289}{1526}u_{n+2} + \frac{51}{109}hf_{n+\frac{3}{2}}, \\ u_{n+2} &= \frac{32768}{633165}u_{n-1} - \frac{441}{955}u_n + \frac{784}{573}u_{n+\frac{1}{2}} - \frac{5292}{2483}u_{n+1} + \frac{7056}{3247}u_{n+\frac{3}{2}} + \frac{42}{191}hf_{n+2}. \end{aligned} \tag{11}$$

Thus, in this paper we shall be referring equation (9)-(11) as fully implicit 2-point variable step size block backward differentiation formula with two off-step points (2BBDFO) method. It should be noted that at constant step size $r=1$, the formula (8) reduces to the existing 2-point block BDF with off-step points for solving stiff ODEs of order five developed in [14].

3 ORDER, ERROR CONSTANT AND CONSISTENCY OF THE METHOD

In this section, we consider the derivation of order and error constant of the method corresponding to formulae in (9), (10) and (11) as in [20] and [16], by considering the formulae (9) when the step size ratio strategy is constant $r=2$. The formulae (9) can also rearrange as

$$\begin{aligned} -\frac{1}{60}u_{n-1} + \frac{3}{4}u_n + u_{n+\frac{1}{2}} - \frac{9}{4}u_{n+1} + \frac{3}{5}u_{n+\frac{3}{2}} - \frac{1}{12}u_{n+2} &= -hf_{n+\frac{1}{2}}, \\ -\frac{1}{45}u_{n-1} + \frac{2}{3}u_n - \frac{32}{9}u_{n+\frac{1}{2}} + u_{n+1} + \frac{32}{15}u_{n+\frac{3}{2}} - \frac{2}{9}u_{n+2} &= 2hf_{n+1}, \\ \frac{1}{124}u_{n-1} - \frac{25}{124}u_n + \frac{25}{31}u_{n+\frac{1}{2}} - \frac{225}{124}u_{n+1} + u_{n+\frac{3}{2}} + \frac{25}{124}u_{n+2} &= \frac{15}{31}hf_{n+\frac{3}{2}}, \\ -\frac{2}{135}u_{n-1} + \frac{1}{3}u_n - \frac{32}{27}u_{n+\frac{1}{2}} + 2u_{n+1} - \frac{32}{15}u_{n+\frac{3}{2}} + u_{n+2} &= \frac{2}{9}hf_{n+2}. \end{aligned} \tag{12}$$

The matrix formulation of equation (12) is given as

$$\begin{bmatrix} 0 & -\frac{1}{60} & 0 & \frac{3}{4} \\ 0 & -\frac{1}{40} & 0 & \frac{2}{3} \\ 0 & \frac{1}{124} & 0 & -\frac{25}{124} \\ 0 & -\frac{2}{135} & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} u_{n-\frac{3}{2}} \\ u_{n-1} \\ u_{n-\frac{1}{2}} \\ u_n \end{bmatrix} + \begin{bmatrix} 1 & -\frac{9}{4} & \frac{3}{5} & -\frac{1}{12} \\ -\frac{32}{9} & 1 & \frac{32}{15} & -\frac{2}{9} \\ \frac{25}{31} & -\frac{225}{124} & 1 & \frac{25}{124} \\ -\frac{32}{27} & 2 & -\frac{32}{15} & 1 \end{bmatrix} \begin{bmatrix} u_{n+\frac{1}{2}} \\ u_{n+1} \\ u_{n+\frac{3}{2}} \\ u_{n+2} \end{bmatrix} = h \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{bmatrix}$$

$$+\,h\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & \frac{15}{31} & 0\\ 0 & 0 & 0 & \frac{2}{9}\end{bmatrix}\begin{bmatrix}f_{n+\frac{1}{2}}\\ f_{n+1}\\ f_{n+\frac{3}{2}}\\ f_{n+2}\end{bmatrix} \tag{13}$$

Let

$$\rho_0=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},\ \rho_1=\begin{bmatrix}-\frac{1}{64}\\ -\frac{1}{40}\\ \frac{1}{124}\\ -\frac{2}{135}\end{bmatrix},\ \rho_2=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},\ \rho_3=\begin{bmatrix}\frac{3}{4}\\ \frac{2}{3}\\ -\frac{25}{124}\\ \frac{1}{3}\end{bmatrix},\ \rho_4=\begin{bmatrix}1\\ -\frac{32}{9}\\ \frac{25}{31}\\ -\frac{32}{27}\end{bmatrix},\ \rho_5=\begin{bmatrix}-\frac{9}{4}\\ 1\\ -\frac{225}{124}\\ 2\end{bmatrix},\rho_6=\begin{bmatrix}\frac{3}{5}\\ \frac{32}{15}\\ 1\\ -\frac{32}{15}\end{bmatrix},$$

$$\rho_7=\begin{bmatrix}-\frac{1}{12}\\ -\frac{2}{9}\\ \frac{25}{124}\\ 1\end{bmatrix},\ \sigma_0=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},\ \sigma_1=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},\ \sigma_2=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},\ \sigma_3=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},\ \sigma_4=\begin{bmatrix}-1\\0\\0\\0\end{bmatrix},\ \sigma_5=\begin{bmatrix}0\\2\\0\\0\end{bmatrix},\ \sigma_6=\begin{bmatrix}0\\0\\ \frac{15}{31}\\ 0\end{bmatrix},$$

$$\sigma_7=\begin{bmatrix}0\\0\\0\\ \frac{2}{9}\end{bmatrix}$$

Definition 3.1: The order of the block numerical scheme (9) and its associated linear operator given as

$$L[u(x),h]=\sum_{j=0}^{7}\left[\rho_j\left(x+j\frac{h}{2}\right)-h\sigma_j\left(x+j\frac{h}{2}\right)\right] \tag{14}$$

Is a unique integer α such that $A_q=0$, $q=0(1)\alpha$ and $A_{\alpha+1}\neq 0$; where A_q are the column vectors defined by

$$
\begin{gathered}
A_0 = \rho_0 + \rho_1 + \rho_2 + \cdots + \rho_k, \\
A_1 = \rho_1 + 2\rho_2 + \cdots + k\rho_k - 2(\sigma_0 + \sigma_1 + \sigma_2 + \cdots + \sigma_k), \\
\vdots \\
A_q = \frac{1}{q!}(\rho_1 + 2^q\rho_2 + \cdots + k^q\rho_k) - \frac{2}{(q-1)!}(\sigma_1 + 2^{q-1}\sigma_2 + \cdots + k^{q-1}\sigma_k). \\
q = 2,3, \ldots\ldots\ldots
\end{gathered}
\tag{15}
$$

For $q = 0(1)6$, we have

$$A_0 = \sum_{j=0}^{7} \rho_j = \rho_0 + \rho_1 + \rho_2 + \rho_3 + \rho_4 + \rho_5 + \rho_6 + \rho_7$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{64} \\ -\frac{1}{40} \\ \frac{1}{124} \\ -\frac{2}{135} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{3}{4} \\ \frac{2}{3} \\ -\frac{25}{124} \\ \frac{1}{3} \end{bmatrix} + \begin{bmatrix} 1 \\ -\frac{32}{9} \\ \frac{25}{31} \\ -\frac{32}{27} \end{bmatrix} + \begin{bmatrix} -\frac{9}{4} \\ 1 \\ -\frac{225}{124} \\ 2 \end{bmatrix} + \begin{bmatrix} \frac{3}{5} \\ \frac{32}{15} \\ 1 \\ -\frac{32}{15} \end{bmatrix} + \begin{bmatrix} -\frac{1}{12} \\ -\frac{2}{9} \\ \frac{25}{124} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A_1 = \sum_{j=0}^{7} (j\rho_j) - 2\sum_{j=0}^{7} \sigma_j = ((0)\rho_0 + (1)\rho_1 + (2)\rho_2 + (3)\rho_3 + (4)\rho_4 + (5)\rho_5 + (6)\rho_6 + (7)\rho_7)$$
$$-2(\sigma_0 + \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 + \sigma_6 + \sigma_7)$$

$$= \left[(0)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (1)\begin{bmatrix} -\frac{1}{64} \\ -\frac{1}{40} \\ \frac{1}{124} \\ -\frac{2}{135} \end{bmatrix} + (2)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (3)\begin{bmatrix} \frac{3}{4} \\ \frac{2}{4} \\ -\frac{25}{124} \\ \frac{1}{3} \end{bmatrix} + (4)\begin{bmatrix} 1 \\ -\frac{32}{9} \\ \frac{25}{31} \\ -\frac{32}{27} \end{bmatrix} + (5)\begin{bmatrix} -\frac{9}{4} \\ 1 \\ -\frac{225}{124} \\ 2 \end{bmatrix} + (6)\begin{bmatrix} \frac{3}{5} \\ \frac{32}{15} \\ 1 \\ -\frac{32}{15} \end{bmatrix} + (7)\begin{bmatrix} -\frac{1}{12} \\ -\frac{2}{9} \\ \frac{25}{124} \\ 1 \end{bmatrix} \right]$$

$$-2\left[\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{15}{31} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{2}{9} \end{bmatrix} \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A_2 = \sum_{j=0}^{7} \frac{(j^2\rho_j)}{2!} - 2\sum_{j=0}^{7} (j\sigma_j) = \frac{1}{2!}((0)^2\rho_0 + (1)^2\rho_1 + (2)^2\rho_2 + (3)^2\rho_3 + (4)^2\rho_4 + (5)^2\rho_5 + (6)^2\rho_6 + (7)^2\rho_7)$$
$$-2((0)\sigma_0 + (1)\sigma_1 + (2)\sigma_2 + (3)\sigma_3 + (4)\sigma_4 + (5)\sigma_5 + (6)\sigma_6 + (7)\sigma_7)$$

$$=\frac{1}{2!}\left[(0)^2\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^2\begin{bmatrix}-\frac{1}{64}\\-\frac{1}{40}\\\frac{1}{124}\\-\frac{2}{135}\end{bmatrix}+(2)^2\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^2\begin{bmatrix}\frac{3}{4}\\\frac{2}{4}\\-\frac{25}{124}\\\frac{1}{3}\end{bmatrix}+(4)^2\begin{bmatrix}1\\-\frac{32}{9}\\\frac{25}{31}\\-\frac{32}{27}\end{bmatrix}+(5)^2\begin{bmatrix}-\frac{9}{4}\\1\\-\frac{225}{124}\\2\end{bmatrix}+(6)^2\begin{bmatrix}\frac{3}{5}\\\frac{32}{15}\\1\\-\frac{32}{15}\end{bmatrix}+(7)^2\begin{bmatrix}-\frac{1}{12}\\-\frac{2}{9}\\\frac{25}{124}\\1\end{bmatrix}\right]$$

$$-2\left[(0)\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(2)\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(4)\begin{bmatrix}-1\\0\\0\\0\end{bmatrix}+(5)\begin{bmatrix}0\\2\\0\\0\end{bmatrix}+(6)\begin{bmatrix}0\\0\\\frac{15}{31}\\0\end{bmatrix}+(7)\begin{bmatrix}0\\0\\0\\\frac{2}{9}\end{bmatrix}\right]=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$A_3=\sum_{j=0}^{7}\frac{\left(j^3\rho_j\right)}{3!}-2\sum_{j=0}^{7}\frac{\left(j^2\sigma_j\right)}{2!}=\frac{1}{3!}\left((0)^3\rho_0+(1)^3\rho_1+(2)^3\rho_2+(3)^3\rho_3+(4)^3\rho_4+(5)^3\rho_5+(6)^3\rho_6+(7)^3\rho_7\right)$$

$$-\frac{2}{2!}\left((0)^2\sigma_0+(1)^2\sigma_1+(2)^2\sigma_2+(3)^2\sigma_3+(4)^2\sigma_4+(5)^2\sigma_5+(6)^2\sigma_6+(7)^2\sigma_7\right)$$

$$=\frac{1}{3!}\left[(0)^3\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^3\begin{bmatrix}-\frac{1}{64}\\-\frac{1}{40}\\\frac{1}{124}\\-\frac{2}{135}\end{bmatrix}+(2)^3\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^3\begin{bmatrix}\frac{3}{4}\\\frac{2}{4}\\-\frac{25}{124}\\\frac{1}{3}\end{bmatrix}+(4)^3\begin{bmatrix}1\\-\frac{32}{9}\\\frac{25}{31}\\-\frac{32}{27}\end{bmatrix}+(5)^3\begin{bmatrix}-\frac{9}{4}\\1\\-\frac{225}{124}\\2\end{bmatrix}+(6)^3\begin{bmatrix}\frac{3}{5}\\\frac{32}{15}\\1\\-\frac{32}{15}\end{bmatrix}+(7)^3\begin{bmatrix}-\frac{1}{12}\\-\frac{2}{9}\\\frac{25}{124}\\1\end{bmatrix}\right]$$

$$-\frac{2}{2!}\left[(0)^2\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^2\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(2)^2\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^2\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(4)^2\begin{bmatrix}-1\\0\\0\\0\end{bmatrix}+(5)^2\begin{bmatrix}0\\2\\0\\0\end{bmatrix}+(6)^2\begin{bmatrix}0\\0\\\frac{15}{31}\\0\end{bmatrix}+(7)^2\begin{bmatrix}0\\0\\0\\\frac{2}{9}\end{bmatrix}\right]=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$A_4=\sum_{j=0}^{7}\frac{\left(j^4\rho_j\right)}{4!}-2\sum_{j=0}^{7}\frac{\left(j^3\sigma_j\right)}{3!}=\frac{1}{4!}\left((0)^4\rho_0+(1)^4\rho_1+(2)^4\rho_2+(3)^4\rho_3+(4)^4\rho_4+(5)^4\rho_5+(6)^4\rho_6+(7)^4\rho_7\right)$$

$$-\frac{2}{3!}\left((0)^3\sigma_0+(1)^3\sigma_1+(2)^3\sigma_2+(3)^3\sigma_3+(4)^3\sigma_4+(5)^3\sigma_5+(6)^3\sigma_6+(7)^3\sigma_7\right)$$

$$=\frac{1}{4!}\left[(0)^4\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^4\begin{bmatrix}-\frac{1}{64}\\-\frac{1}{40}\\\frac{1}{124}\\-\frac{2}{135}\end{bmatrix}+(2)^4\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^4\begin{bmatrix}\frac{3}{4}\\\frac{2}{4}\\-\frac{25}{124}\\\frac{1}{3}\end{bmatrix}+(4)^4\begin{bmatrix}1\\-\frac{32}{9}\\\frac{25}{31}\\-\frac{32}{27}\end{bmatrix}+(5)^4\begin{bmatrix}-\frac{9}{4}\\1\\-\frac{225}{124}\\2\end{bmatrix}+(6)^4\begin{bmatrix}\frac{3}{5}\\\frac{32}{15}\\1\\-\frac{32}{15}\end{bmatrix}+(7)^4\begin{bmatrix}-\frac{1}{12}\\-\frac{2}{9}\\\frac{25}{124}\\1\end{bmatrix}\right]$$

$$-\frac{2}{3!}\left[(0)^3\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^3\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(2)^3\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^4\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(4)^3\begin{bmatrix}-1\\0\\0\\0\end{bmatrix}+(5)^3\begin{bmatrix}0\\2\\0\\0\end{bmatrix}+(6)^3\begin{bmatrix}0\\0\\\frac{15}{31}\\0\end{bmatrix}+(7)^3\begin{bmatrix}0\\0\\0\\\frac{2}{9}\end{bmatrix}\right]=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$A_5=\sum_{j=0}^{7}\frac{(j^5\rho_j)}{5!}-2\sum_{j=0}^{7}\frac{(j^4\sigma_j)}{4!}=\frac{1}{5!}\left((0)^5\rho_0+(1)^5\rho_1+(2)^5\rho_2+(3)^5\rho_3+(4)^5\rho_4+(5)^5\rho_5+(6)^5\rho_6+(7)^5\rho_7\right)$$

$$-\frac{2}{4!}\left((0)^4\sigma_0+(1)^4\sigma_1+(2)^4\sigma_2+(3)^4\sigma_3+(4)^4\sigma_4+(5)^4\sigma_5+(6)^4\sigma_6+(7)^4\sigma_7\right)$$

$$=\frac{1}{5!}\left[(0)^5\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^5\begin{bmatrix}-\frac{1}{64}\\-\frac{1}{40}\\\frac{1}{124}\\-\frac{2}{135}\end{bmatrix}+(2)^5\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^5\begin{bmatrix}\frac{3}{4}\\\frac{2}{4}\\-\frac{25}{124}\\\frac{1}{3}\end{bmatrix}+(4)^5\begin{bmatrix}1\\-\frac{32}{9}\\\frac{25}{31}\\-\frac{32}{27}\end{bmatrix}+(5)^5\begin{bmatrix}-\frac{9}{4}\\1\\-\frac{225}{124}\\2\end{bmatrix}+(6)^5\begin{bmatrix}\frac{3}{5}\\\frac{32}{15}\\1\\-\frac{32}{15}\end{bmatrix}+(7)^5\begin{bmatrix}-\frac{1}{12}\\-\frac{2}{9}\\\frac{25}{124}\\1\end{bmatrix}\right]$$

$$-\frac{2}{4!}\left[(0)^4\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^4\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(2)^4\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^4\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(4)^4\begin{bmatrix}-1\\0\\0\\0\end{bmatrix}+(5)^4\begin{bmatrix}0\\2\\0\\0\end{bmatrix}+(6)^4\begin{bmatrix}0\\0\\\frac{15}{31}\\0\end{bmatrix}+(7)^4\begin{bmatrix}0\\0\\0\\\frac{2}{9}\end{bmatrix}\right]=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$A_6=\sum_{j=0}^{7}\frac{(j^6\rho_j)}{6!}-2\sum_{j=0}^{7}\frac{(j^5\sigma_j)}{5!}=\frac{1}{6!}\left((0)^6\rho_0+(1)^6\rho_1+(2)^6\rho_2+(3)^6\rho_3+(4)^6\rho_4+(5)^6\rho_5+(6)^6\rho_6+(7)^6\rho_7\right)$$

$$-\frac{2}{5!}\left((0)^5\sigma_0+(1)^5\sigma_1+(2)^5\sigma_2+(3)^5\sigma_3+(4)^5\sigma_4+(5)^5\sigma_5+(6)^5\sigma_6+(7)^5\sigma_7\right)$$

$$
=\frac{1}{6!}\left[(0)^6\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^6\begin{bmatrix}-\frac{1}{64}\\-\frac{1}{40}\\\frac{1}{124}\\-\frac{2}{135}\end{bmatrix}+(2)^6\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^6\begin{bmatrix}\frac{3}{4}\\\frac{2}{4}\\-\frac{25}{124}\\\frac{1}{3}\end{bmatrix}+(4)^6\begin{bmatrix}1\\-\frac{32}{9}\\\frac{25}{31}\\-\frac{32}{27}\end{bmatrix}+(5)^6\begin{bmatrix}-\frac{9}{4}\\1\\-\frac{225}{124}\\2\end{bmatrix}+(6)^6\begin{bmatrix}\frac{3}{5}\\\frac{32}{15}\\1\\-\frac{32}{15}\end{bmatrix}+(7)^6\begin{bmatrix}-\frac{1}{12}\\-\frac{2}{9}\\\frac{25}{124}\\1\end{bmatrix}\right]
$$

$$
-\frac{2}{5!}\left[(0)^5\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^5\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(2)^5\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^5\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(4)^5\begin{bmatrix}-1\\0\\0\\0\end{bmatrix}+(5)^5\begin{bmatrix}0\\2\\0\\0\end{bmatrix}+(6)^5\begin{bmatrix}0\\0\\\frac{15}{31}\\0\end{bmatrix}+(7)^5\begin{bmatrix}0\\0\\0\\\frac{2}{9}\end{bmatrix}\right]=\begin{bmatrix}-\frac{1}{20}\\-\frac{4}{45}\\\frac{5}{124}\\-\frac{4}{45}\end{bmatrix}\neq\begin{bmatrix}0\\0\\0\\0\end{bmatrix}
$$

Therefore, in accordance with definition 3.1, we have shown that the method (9) is of order five with error constant given by

$$
A_6=\begin{bmatrix}-\frac{1}{20}\\-\frac{4}{45}\\\frac{5}{124}\\-\frac{4}{45}\end{bmatrix}
$$

The same procedure is applied to determine the order of the method (10) and (11) which indicates that their order is also five.

Definition 3.2: the block numerical schemes (9)-(11) are said to be consistent if it has order α at least one [9]. Hence, we conclude that the methods (9)-(11) are consistent since their order is five which is greater than one.

4 STABILITY ANALYSIS AND CONVERGENCE OF THE METHOD

In this section, we presents the zero stability, convergence and absolute stability properties of the method in (9), (10), and (11). We begin by the following definitions:

Definition 4.1: the block numerical scheme (9)-(11) is said to be zero stable if all the roots of first characteristic polynomial have modulus less than or equal to unity and those of modulus unity are simple [14].

Definition 4.2: the block numerical scheme (9)-(11) is said to be absolutely stable in a region R for a given $h\lambda$ if for that $h\lambda$, all the roots r_s of stability polynomial $\Omega(r,h\lambda)=\rho(r)-h\lambda\sigma(r)=0$ satisfy $r_s<1, s=1,2,\dots,k$ [20].

Definition 4.3: the block numerical scheme (9)-(11) is said to be A-stable if its stability region covers the entire negative half plane [20].

We shall begin with the absolute stability region of the method when $r = 2$ by applying the scalar test differential equation of the form $u' = \lambda u$, where λ is complex constant with negative real part. Letting the scalar test differential equation $u' = \lambda u$ in the formula (10) gives:

$$\begin{aligned} u_{n+\frac{1}{2}} &= \frac{3}{2128}u_{n-1} + \frac{75}{153}u_n + \frac{75}{38}u_{n+1} - \frac{75}{133}u_{n+\frac{3}{2}} + \frac{25}{304}u_{n+2} - \frac{15}{19}h\lambda u_{n+\frac{1}{2}}, \\ u_{n+1} &= \frac{1}{280}u_{n-1} - \frac{3}{4}u_n + \frac{24}{5}u_{n+\frac{1}{2}} - \frac{24}{7}u_{n+\frac{3}{2}} + \frac{3}{8}u_{n+2} + 3h\lambda u_{n+1}, \\ u_{n+\frac{3}{2}} &= -\frac{3}{3280}u_{n-1} + \frac{49}{328}u_n - \frac{147}{205}u_{n+\frac{1}{2}} + \frac{147}{82}u_{n+1} - \frac{147}{656}u_{n+2} + \frac{21}{41}h\lambda u_{n+\frac{3}{2}}, \\ u_{n+2} &= \frac{3}{1855}u_{n-1} - \frac{12}{53}u_n + \frac{256}{265}u_{n+\frac{1}{2}} - \frac{96}{53}u_{n+1} + \frac{768}{371}u_{n+\frac{3}{2}} + \frac{12}{53}h\lambda u_{n+2}. \end{aligned} \tag{16}$$

After collecting the like terms, the formula (16) becomes

$$\begin{aligned} u_{n+\frac{1}{2}} + \frac{15}{19}h\lambda u_{n+\frac{1}{2}} - \frac{75}{38}u_{n+1} + \frac{75}{133}u_{n+\frac{3}{2}} - \frac{25}{304}u_{n+2} &= \frac{3}{2128}u_{n-1} + \frac{75}{153}u_n, \\ -\frac{24}{5}u_{n+\frac{1}{2}} + u_{n+1} - 3h\lambda u_{n+1} + \frac{24}{7}u_{n+\frac{3}{2}} - \frac{3}{8}u_{n+2} &= \frac{1}{280}u_{n-1} - \frac{3}{4}u_n, \\ \frac{147}{205}u_{n+\frac{1}{2}} - \frac{147}{82}u_{n+1} + u_{n+\frac{3}{2}} - \frac{21}{41}h\lambda u_{n+\frac{3}{2}} + \frac{147}{656}u_{n+2} &= -\frac{3}{3280}u_{n-1} + \frac{49}{328}u_n, \\ -\frac{256}{265}u_{n+\frac{1}{2}} + \frac{96}{53}u_{n+1} - \frac{768}{371}u_{n+\frac{3}{2}} + u_{n+2} - \frac{12}{53}h\lambda u_{n+2} &= \frac{3}{1855}u_{n-1} - \frac{12}{53}u_n. \end{aligned} \tag{17}$$

The matrix formulation of the equations (17) is

$$\begin{bmatrix} \left(1+\frac{15}{19}\lambda h\right) & -\frac{75}{38} & \frac{75}{133} & -\frac{25}{304} \\ -\frac{24}{5} & (1-3\lambda h) & \frac{24}{7} & -\frac{3}{8} \\ \frac{147}{205} & -\frac{147}{82} & \left(1-\frac{21}{41}\lambda h\right) & \frac{147}{656} \\ -\frac{256}{265} & \frac{96}{53} & -\frac{768}{371} & \left(1-\frac{12}{53}\lambda h\right) \end{bmatrix} \begin{bmatrix} u_{n+\frac{1}{2}} \\ u_{n+1} \\ u_{n+\frac{3}{2}} \\ u_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{3}{2128} & 0 & \frac{75}{153} \\ 0 & \frac{1}{280} & 0 & -\frac{3}{4} \\ 0 & -\frac{3}{3280} & 0 & \frac{49}{328} \\ 0 & \frac{3}{1855} & 0 & -\frac{12}{53} \end{bmatrix} \begin{bmatrix} u_{n-\frac{3}{2}} \\ u_{n-1} \\ u_{n-\frac{1}{2}} \\ u_n \end{bmatrix}. \tag{18}$$

Putting $\bar{h} = \lambda h$ in equation (18), we have

$$\begin{bmatrix} \left(1+\frac{15}{19}\bar{h}\right) & -\frac{75}{38} & \frac{75}{133} & -\frac{25}{304} \\ -\frac{24}{5} & (1-3\bar{h}) & \frac{24}{7} & -\frac{3}{8} \\ \frac{147}{205} & -\frac{147}{82} & \left(1-\frac{21}{41}\bar{h}\right) & \frac{147}{656} \\ -\frac{256}{265} & \frac{96}{53} & -\frac{768}{371} & \left(1-\frac{12}{53}\bar{h}\right) \end{bmatrix} \begin{bmatrix} u_{n+\frac{1}{2}} \\ u_{n+1} \\ u_{n+\frac{3}{2}} \\ u_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{3}{2128} & 0 & \frac{75}{153} \\ 0 & \frac{1}{280} & 0 & -\frac{3}{4} \\ 0 & -\frac{3}{3280} & 0 & \frac{49}{328} \\ 0 & \frac{3}{1855} & 0 & -\frac{12}{53} \end{bmatrix} \begin{bmatrix} u_{n-\frac{3}{2}} \\ u_{n-1} \\ u_{n-\frac{1}{2}} \\ u_n \end{bmatrix}. \tag{19}$$

if m is the number of block and r is the number of points in the block, then $n = mr$. Here $r = 2$ and $n = 2m$. By [4], we let

$$U_m = \begin{bmatrix} u_{2m+\frac{1}{2}} \\ u_{2m+1} \\ u_{2m+\frac{3}{2}} \\ u_{2m+2} \end{bmatrix} = \begin{bmatrix} u_{n+\frac{1}{2}} \\ u_{n+1} \\ u_{n+\frac{3}{2}} \\ u_{n+2} \end{bmatrix}, U_{m-1} = \begin{bmatrix} u_{2(m-1)+\frac{1}{2}} \\ u_{2(m-1)+1} \\ u_{2(m-1)+\frac{3}{2}} \\ u_{2(m-1)+2} \end{bmatrix} = \begin{bmatrix} u_{2m+\frac{3}{2}} \\ u_{2m-1} \\ u_{2m-\frac{1}{2}} \\ u_{2m} \end{bmatrix} = \begin{bmatrix} u_{n-\frac{3}{2}} \\ u_{n-1} \\ u_{n-\frac{1}{2}} \\ u_{n} \end{bmatrix}.$$

Equation (19) can also be written in the following form

$$AU_m = BU_{m-1}, \tag{20}$$

where,

$$A = \begin{bmatrix} \left(1+\frac{15}{19}\bar{h}\right) & -\frac{75}{38} & \frac{75}{133} & -\frac{25}{304} \\ -\frac{24}{5} & \left(1-3\bar{h}\right) & \frac{24}{7} & -\frac{3}{8} \\ \frac{147}{205} & -\frac{147}{82} & \left(1-\frac{21}{41}\bar{h}\right) & \frac{147}{656} \\ -\frac{256}{265} & \frac{96}{53} & -\frac{768}{371} & \left(1-\frac{12}{53}\bar{h}\right) \end{bmatrix}, B = \begin{bmatrix} 0 & \frac{3}{2128} & 0 & \frac{75}{153} \\ 0 & \frac{1}{280} & 0 & -\frac{3}{4} \\ 0 & -\frac{3}{3280} & 0 & \frac{49}{328} \\ 0 & \frac{3}{1855} & 0 & -\frac{12}{53} \end{bmatrix}.$$

The stability polynomial of the method is obtained by evaluating the $det(At - B)$to obtain the stability polynomial as

$$\pi(t,\bar{h}) = \frac{9}{165148}\left(-5040t^4\bar{h}^4 + 27396t^4\bar{h}^3 + 5034t^3\bar{h}^3 - 84172t^4\bar{h}^2 + 31855t^3\bar{h}^2 + 151488t^4\bar{h} - 3t^2\bar{h}^2 + 98608t^3\bar{h} - 125248t^4 - 16t^2\bar{h} + 125232t^3 + 16t^2\right) = 0 \tag{21}$$

To show that the method (10) is zero stable, we substitute $\bar{h} = 0$ in equation (21) to obtain the first characteristics polynomial as:

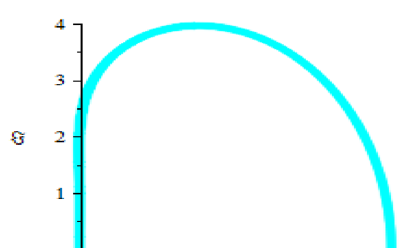
$$\pi(t,0) = \frac{9}{165148}(-125248t^4 - 16t^2 + 125232t^3 + 16t^2) = 0. \tag{22}$$

Solving equation (22) for t, we obtain the following roots as

$$t = 0, t = 0, t = 1, t = -\frac{1}{7828}.$$

Therefore the values of t above indicate that the method (10) is zero-stable since no magnitude of the root is greater than one and the root $t = 1$ is simple.

The boundary of the stability region of the method (10) is determined by inputting $t = e^{i\theta}$, $0 \le \theta \le 2\pi$ into equation (21). The graph of the stability region for the method (10) using Maple18 software is given below.



Stable Region Unstable Region Stable Region

Stable Region Unstable Region Stable Region

Figure 1: Stability region of the 2BBDFO when $r = 2$.

Thus, the region of absolute stability of the method (10) covers the entire negative half plane which indicates that the method is A-stable.

Next, we consider the stability of method (11) when $r = \frac{5}{8}$. The following are the formulae obtained when $r = \frac{5}{8}$ after the scalar test differential equation is substituted.

$$\begin{aligned}
u_{n+\frac{1}{2}} &= \frac{4096}{54145}u_{n-1} - \frac{81}{70}u_n + \frac{243}{91}u_{n+1} - \frac{81}{119}u_{n+\frac{3}{2}} + \frac{9}{98}u_{n+2} - \frac{9}{7}h\lambda u_{n+\frac{1}{2}}, \\
u_{n+1} &= \frac{1024}{16065}u_{n-1} - \frac{169}{240}u_n + \frac{169}{54}u_{n+\frac{1}{2}} - \frac{169}{102}u_{n+\frac{3}{2}} + \frac{169}{1008}u_{n+2} + \frac{13}{8}h\lambda u_{n+1}, \\
u_{n+\frac{3}{2}} &= -\frac{4096}{148785}u_{n-1} + \frac{289}{1090}u_n - \frac{289}{327}u_{n+\frac{1}{2}} + \frac{2601}{1417}u_{n+1} - \frac{289}{1526}u_{n+2} + \frac{51}{109}h\lambda u_{n+\frac{3}{2}}, \\
u_{n+2} &= \frac{32768}{633165}u_{n-1} - \frac{441}{955}u_n + \frac{784}{573}u_{n+\frac{1}{2}} - \frac{5292}{2483}u_{n+1} + \frac{7056}{3247}u_{n+\frac{3}{2}} + \frac{42}{191}h\lambda u_{n+2}.
\end{aligned} \tag{23}$$

The formula (23) can also be represented in matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} u_{n+\frac{1}{2}} \\ u_{n+1} \\ u_{n+\frac{3}{2}} \\ u_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{4096}{54145} & 0 & -\frac{81}{70} \\ 0 & \frac{1024}{16065} & 0 & -\frac{169}{240} \\ 0 & -\frac{4096}{148785} & 0 & \frac{289}{1090} \\ 0 & \frac{32768}{633165} & 0 & -\frac{441}{955} \end{bmatrix}\begin{bmatrix} u_{n-\frac{3}{2}} \\ u_{n-1} \\ u_{n-\frac{1}{2}} \\ u_n \end{bmatrix} + \begin{bmatrix} -\frac{9}{7}h\lambda & \frac{243}{91} & -\frac{81}{119} & \frac{9}{98} \\ \frac{169}{54} & \frac{13}{8}h\lambda & -\frac{169}{102} & \frac{169}{1008} \\ -\frac{289}{327} & \frac{2601}{1417} & \frac{51}{109}h\lambda & -\frac{289}{1526} \\ \frac{784}{573} & -\frac{5292}{2483} & \frac{7056}{3247} & \frac{42}{191}h\lambda \end{bmatrix}\begin{bmatrix} u_{n+\frac{1}{2}} \\ u_{n+1} \\ u_{n+\frac{3}{2}} \\ u_{n+2} \end{bmatrix}. \tag{24}$$

Equation (24) is equivalent to the following matrix equation:

$$\begin{bmatrix} \left(1+\frac{9}{7}h\lambda\right) & \frac{243}{91} & -\frac{81}{119} & \frac{9}{98} \\ \frac{169}{54} & \left(1-\frac{13}{8}h\lambda\right) & -\frac{169}{102} & \frac{169}{1008} \\ -\frac{289}{327} & \frac{2601}{1417} & \left(1-\frac{51}{109}h\lambda\right) & -\frac{289}{1526} \\ \frac{784}{573} & -\frac{5292}{2483} & \frac{7056}{3247} & \left(1-\frac{42}{191}h\lambda\right) \end{bmatrix} \begin{bmatrix} u_{n+\frac{1}{2}} \\ u_{n+1} \\ u_{n+\frac{3}{2}} \\ u_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{4096}{54145} & 0 & -\frac{81}{70} \\ 0 & \frac{1024}{16065} & 0 & -\frac{169}{240} \\ 0 & -\frac{4096}{148785} & 0 & \frac{289}{1090} \\ 0 & \frac{32768}{633165} & 0 & -\frac{441}{955} \end{bmatrix} \begin{bmatrix} u_{n-\frac{3}{2}} \\ u_{n-1} \\ u_{n-\frac{1}{2}} \\ u_{n} \end{bmatrix}. \quad (25)$$

Putting $\bar{h} = \lambda h$ in equation (25). This leads to

$$\begin{bmatrix} \left(1+\frac{9}{7}\bar{h}\right) & \frac{243}{91} & -\frac{81}{119} & \frac{9}{98} \\ \frac{169}{54} & \left(1-\frac{13}{8}\bar{h}\right) & -\frac{169}{102} & \frac{169}{1008} \\ -\frac{289}{327} & \frac{2601}{1417} & \left(1-\frac{51}{109}\bar{h}\right) & -\frac{289}{1526} \\ \frac{784}{573} & -\frac{5292}{2483} & \frac{7056}{3247} & \left(1-\frac{42}{191}\bar{h}\right) \end{bmatrix} \begin{bmatrix} u_{n+\frac{1}{2}} \\ u_{n+1} \\ u_{n+\frac{3}{2}} \\ u_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{4096}{54145} & 0 & -\frac{81}{70} \\ 0 & \frac{1024}{16065} & 0 & -\frac{169}{240} \\ 0 & -\frac{4096}{148785} & 0 & \frac{289}{1090} \\ 0 & \frac{32768}{633165} & 0 & -\frac{441}{955} \end{bmatrix} \begin{bmatrix} u_{n-\frac{3}{2}} \\ u_{n-1} \\ u_{n-\frac{1}{2}} \\ u_{n} \end{bmatrix}.$$

(26)

Equation (26) is equivalent to

$$AU_m = BU_{m-1}, \quad (27)$$

$$A = \begin{bmatrix} \left(1+\frac{9}{7}\bar{h}\right) & \frac{243}{91} & -\frac{81}{119} & \frac{9}{98} \\ \frac{169}{54} & \left(1-\frac{13}{8}\bar{h}\right) & -\frac{169}{102} & \frac{169}{1008} \\ -\frac{289}{327} & \frac{2601}{1417} & \left(1-\frac{51}{109}\bar{h}\right) & -\frac{289}{1526} \\ \frac{784}{573} & -\frac{5292}{2483} & \frac{7056}{3247} & \left(1-\frac{42}{191}\bar{h}\right) \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & \frac{4096}{54145} & 0 & -\frac{81}{70} \\ 0 & \frac{1024}{16065} & 0 & -\frac{169}{240} \\ 0 & -\frac{4096}{148785} & 0 & \frac{289}{1090} \\ 0 & \frac{32768}{633165} & 0 & -\frac{441}{955} \end{bmatrix}.$$

The zero and absolute stability region of 2-point variable step size block BDF with two off-step points method when $r = \frac{5}{8}$ is determined by solving the characteristic equation $\pi(t, \bar{h}) = det(At - B) = 0$ to obtain the stability polynomial as

$$\pi(t, \bar{h}) = \frac{1024}{728665}\Big(16t^2 + \frac{2582295}{8192}t^3\bar{h}^3 + \frac{7177243}{4096}t^3\bar{h}^2 + \frac{5419209}{1024}t^3\bar{h} - \frac{626535}{4096}t^4\bar{h}^4 + \frac{8173125}{8192}t^4\bar{h}^3 - \frac{14159475}{4096}t^4\bar{h}^2 + \frac{6913375}{1024}t^4\bar{h} - \frac{1513925}{256}t^4 - 3t^2\bar{h}^2 + \frac{1509829}{256}t^3 - 16t^2\bar{h}\Big) = 0 \quad (28)$$

For zero stability, we substitute $\bar{h} = \lambda h = 0$ in (28) to obtain the first characteristics polynomial as

$$\pi(t,0) = \frac{1024}{728665}\left(16t^2 - \frac{1513925}{256}t^4 + \frac{1509829}{256}t^3\right) = 0. \tag{29}$$

Solving the quadratic equation in (29) for t, we obtain the following roots as

$$t = 0, t = 0, t = 1, t = -\frac{4096}{1513925}.$$

The method (11) is also zero stable since it satisfies the root condition given in definition 4.1. Therefore, the stability region of the method (11) is given in Figure 2 as

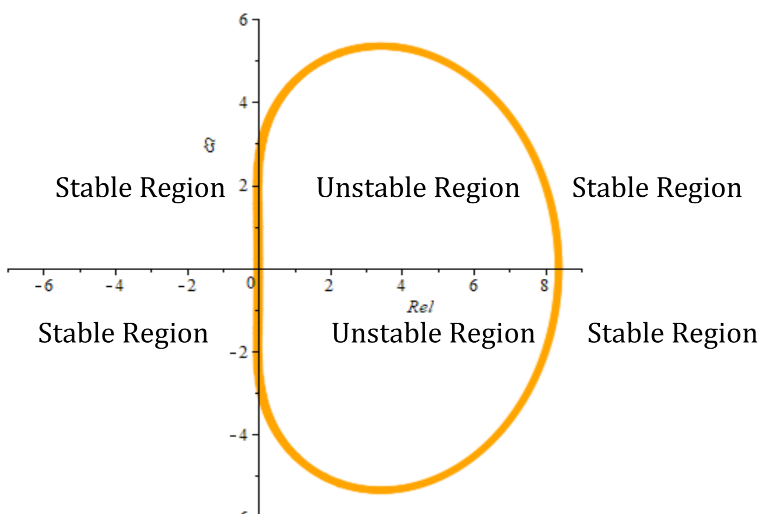


Figure 2: Stability region of the 2BBDF when $r = \frac{5}{8}$.

The stability region in figure 2 also shows that the method is also A-stable since the region covers the entire negative half plane.

Therefore, the detailed derivation, order, stability analysis, convergence and implementation of 2-point variable step size block backward differentiation formula with two off-step points of constant step size have been discussed in [14]. Using same procedure for the value of $r = 2$ as above, the stability polynomial is obtained as

$$\pi(t,\bar{h}) = -\frac{540}{2511}t^4\bar{h}^4 + \frac{3276}{2511}t^4\bar{h}^3 + \frac{804}{2511}t^3\bar{h}^3 - \frac{10779}{2511}t^4\bar{h}^2 + \frac{4702}{2511}t^3\bar{h}^2 + \frac{20272}{2511}t^4\bar{h} - \frac{3}{2511}t^2\bar{h}^2 + \frac{14304}{25111}t^3\bar{h} - \frac{17264}{2511}t^4 - \frac{16}{2511}t^2\bar{h} + \frac{17248}{2511}t^3 + \frac{16}{2511}t^2 = 0 \quad (30)$$

To show that the method (9) is zero stable, we put $\bar{h} = \lambda h = 0$ in (30) to obtain the first characteristics polynomial as:

$$\pi(t,0) = -\frac{17264}{2511}t^4 + \frac{17248}{2511}t^3 + \frac{16}{2511}t^2 = 0\,. \quad (31)$$

Solving (31) for t leads obtain the following roots as

$$t = 0, t = 0, t = 1, t = -\frac{1}{1079}.$$

Therefore the values of t above indicate that the method (9) is zero-stable since no magnitude of the root is greater than one and the root $t = 1$ is unique.

Using the technique of boundary locus, the boundary of the stability region of the method (9) is determined by substituting the set of points $t = e^{i\theta}, 0 \le \theta \le 2\pi$ for which $|t| \le 1$ into equation (30). The graph of the stability region for the method (9) is plotted using Maple18 software is given below.

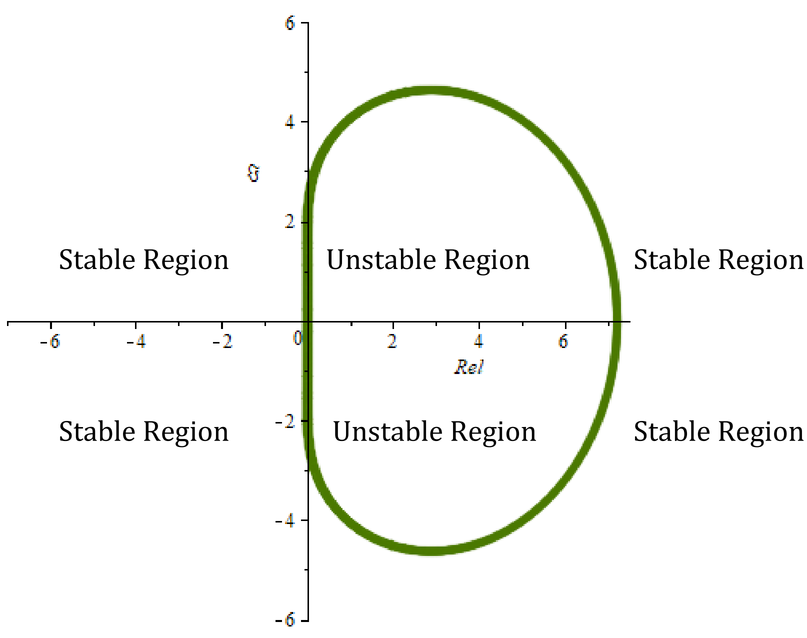


Figure 3: Stability region of the 2-point VSBBDF with two off-step points when $r = 1$.

Thus, the region of stability is the region outside the circular shape. It indicates that the method is A-stable since the region covers the entire negative half plane.

The above figures have shown that it is evident that all our block methods presented are A-stable and therefore suitable for the numerical integration of stiff initial value problems.

Definition 4.4: the necessary and sufficient conditions for the block numerical schemes (9)-(11) to be convergent are that it must be consistent and zero-stable [19].

The proposed methods (9)-(11) satisfied the requirement of consistency and that of zero-stability as stated in definition 4.4, therefore we conclude that the proposed methods (9)-(11) converge.

5 IMPLEMENTATION OF THE METHOD

In this section, Newton's iteration is used to implement the method, we begin by presenting the formulae (9)-(11) in the following form:

$$\begin{aligned} u_{n+\frac{1}{2}} &= \theta_1 u_{n+1} + \theta_2 u_{n+\frac{3}{2}} + \theta_3 u_{n+2} + \alpha_1 h f_{n+\frac{1}{2}} + \varepsilon_{\frac{1}{2}}, \\ u_{n+1} &= \theta_4 u_{n+\frac{1}{2}} + \theta_5 u_{n+\frac{3}{2}} + \theta_6 u_{n+2} + \alpha_2 h f_{n+1} + \varepsilon_1, \\ u_{n+\frac{3}{2}} &= \theta_7 u_{n+\frac{1}{2}} + \theta_8 u_{n+1} + \theta_9 u_{n+2} + \alpha_3 h f_{n+\frac{3}{2}} + \varepsilon_{\frac{3}{2}}, \\ u_{n+2} &= \theta_{10} u_{n+\frac{1}{2}} + \theta_{11} u_{n+1} + \theta_{12} u_{n+\frac{3}{2}} + \alpha_4 h f_{n+2} + \varepsilon_2. \end{aligned} \tag{32}$$

where $\varepsilon_{\frac{1}{2}}$, ε_1, $\varepsilon_{\frac{3}{2}}$ and ε_2 are the backvalues.
The formulae (32) can be cast in matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{n+\frac{1}{2}} \\ u_{n+1} \\ u_{n+\frac{3}{2}} \\ u_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & \theta_1 & \theta_2 & \theta_3 \\ \theta_4 & 0 & \theta_5 & \theta_6 \\ \theta_7 & \theta_8 & 0 & \theta_9 \\ \theta_{10} & \theta_{11} & \theta_{12} & 0 \end{bmatrix} \begin{bmatrix} u_{n+\frac{1}{2}} \\ u_{n+1} \\ u_{n+\frac{3}{2}} \\ u_{n+2} \end{bmatrix} + h \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_4 \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix} + \begin{bmatrix} \varepsilon_{\frac{1}{2}} \\ \varepsilon_1 \\ \varepsilon_{\frac{3}{2}} \\ \varepsilon_2 \end{bmatrix}. \tag{33}$$

Equation (33) can be represented as

$$(I - A)U = hBF_1 + \eta. \tag{34}$$

where,

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 0 & \theta_1 & \theta_2 & \theta_3 \\ \theta_4 & 0 & \theta_5 & \theta_6 \\ \theta_7 & \theta_8 & 0 & \theta_9 \\ \theta_{10} & \theta_{11} & \theta_{12} & 0 \end{bmatrix}, B = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_4 \end{bmatrix}, U = \begin{bmatrix} u_{n+\frac{1}{2}} \\ u_{n+1} \\ u_{n+\frac{3}{2}} \\ u_{n+2} \end{bmatrix}, F_1 = \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix}$$

$$\text{and } \eta = \begin{bmatrix} \varepsilon_{\frac{1}{2}} \\ \varepsilon_1 \\ \varepsilon_{\frac{3}{2}} \\ \varepsilon_2 \end{bmatrix}$$

Let

$$F = (I - A)U - hBF_1 - \eta = 0\,. \tag{35}$$

The Newton's iteration of 2-point variable step size block backward differentiation formula with two off-step points takes the form:

$$U^{(i+1)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2} - U^{(i)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2} = -\left[F'_j\left(U^{(i)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2}\right)\right]^{-1}\left[F_j\left(U^{(i)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2}\right)\right]. \tag{36}$$

Equation (36) is equivalent to

$$U^{(i+1)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2} - U^{(i)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2} = -\left[(I-A) - hB\frac{\partial F_1}{\partial U}\left(U^{(i)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2}\right)\right]^{-1} \times\left[(I-A)\left(U^{(i)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2}\right) - hBF_1 - \eta\right], \tag{37}$$

where ${\partial F_1}/{\partial U}\left(U^{(i)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2}\right)$ is the Jacobian matrix of $F_1$with respect to U. For the sake of comparison purposes, the maximum error is computed from the method developed. Let $U_i$and $U(x_i)$ be the approxximate and exact solution of first order ordinary differential equation (1). The absolute error is defined by

$$(error_i)_t = |(U_i)_t - (U(x_i))_t| \tag{38}$$

The maximum error is defined by:

$$MAXE = \underbrace{max}_{1\le i\le T}\,(\underbrace{max(error_i)_t}_{1\le i\le N}), \tag{39}$$

where, T is the total number of steps and N is the number of equations.

Let $U^{(i+1)}_{n+1}$ denote the $(i+1)th$ iterate and

$$E^{(i+1)}_{\frac{1}{2},1,\frac{3}{2},2} = U^{(i+1)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2} - U^{(i)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2}\,. \tag{40}$$

The equation (40) can be written as

$$E^{(i+1)}_{\frac{1}{2},1,\frac{3}{2},2} = \bar{A}^{-1}\bar{B}, \tag{41}$$

which is equivalent to

$$\bar{A}E^{(i+1)}_{\frac{1}{2},1,\frac{3}{2},2} = \bar{B}, \tag{42}$$

where

$$A=\left[(I-A)-hB\frac{\partial F_1}{\partial U}\left(U^{(i)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2}\right)\right]^{-1} \text{ and}$$
$$B=\left[(I-A)\left(U^{(i)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2}\right)-hBF_1-\eta\right]$$

Newton's iteration is therefore applied to solve the system (42). For different values of the step size ratio r

$$\bar{A}=\begin{pmatrix}\left(1-\alpha_1 h\frac{\partial f_{n+\frac{1}{2}}}{\partial u_{n+\frac{1}{2}}}\right) & -\theta_1 & \begin{matrix}-\theta_2 & -\theta_3\end{matrix} \\ -\theta_4 & \left(1-\alpha_2 h\frac{\partial f_{n+1}}{\partial u_{n+1}}\right) & \begin{matrix}-\theta_5 & -\theta_6\end{matrix} \\ \begin{matrix}-\theta_7 & -\theta_8\end{matrix} & \left(1-\alpha_3 h\frac{\partial f_{n+\frac{3}{2}}}{\partial u_{n+\frac{3}{2}}}\right) & -\theta_9 \\ \begin{matrix}-\theta_{10} & -\theta_{11}\end{matrix} & -\theta_{12} & \left(1-\alpha_4 h\frac{\partial f_{n+2}}{\partial u_{n+2}}\right)\end{pmatrix},$$

$$\bar{B}=\begin{pmatrix}-u^i_{n+\frac{1}{2}}+\theta_1 u^i_{n+1}+\theta_2 u^i_{n+\frac{3}{2}}+\theta_3 u^i_{n+2}+\alpha_1 hf^i_{n+\frac{1}{2}}+\varepsilon_{\frac{1}{2}} \\ -u^i_{n+1}+\theta_4 u^i_{n+\frac{1}{2}}+\theta_5 u^i_{n+\frac{3}{2}}+\theta_6 u^i_{n+2}+\alpha_2 hf^i_{n+1}+\varepsilon_1 \\ -u^i_{n+\frac{3}{2}}+\theta_7 u^i_{n+\frac{1}{2}}+\theta_8 u^i_{n+1}+\theta_9 u^i_{n+2}+\alpha_3 hf^i_{n+\frac{3}{2}}+\varepsilon_{\frac{3}{2}} \\ -u^i_{n+2}+\theta_{10} u^i_{n+\frac{1}{2}}+\theta_{11} u^i_{n+1}+\theta_{12} u^i_{n+\frac{3}{2}}+\alpha_4 hf^i_{n+2}+\varepsilon_2\end{pmatrix}.$$

When $r=1$

$$\bar{A}=\begin{pmatrix}\left(1+h\frac{\partial f_{n+\frac{1}{2}}}{\partial u_{n+\frac{1}{2}}}\right) & -\frac{9}{4} & \begin{matrix}\frac{3}{5} & -\frac{1}{12}\end{matrix} \\ -\frac{32}{9} & \left(1-2h\frac{\partial f_{n+1}}{\partial u_{n+1}}\right) & \begin{matrix}\frac{32}{15} & -\frac{2}{9}\end{matrix} \\ \begin{matrix}\frac{25}{31} & -\frac{225}{124}\end{matrix} & \left(1-\frac{15}{31}h\frac{\partial f_{n+\frac{3}{2}}}{\partial u_{n+\frac{3}{2}}}\right) & \frac{25}{124} \\ \begin{matrix}-\frac{32}{27} & 2\end{matrix} & -\frac{32}{15} & \left(1-\frac{2}{9}h\frac{\partial f_{n+2}}{\partial u_{n+2}}\right)\end{pmatrix},$$

$$\bar{B}=\begin{pmatrix}-u^i_{n+\frac{1}{2}}+\frac{9}{4}u^i_{n+1}-\frac{3}{5}u^i_{n+\frac{3}{2}}+\frac{1}{12}u^i_{n+2}-hf^i_{n+\frac{1}{2}}+\frac{1}{60}u_{n-1}-\frac{3}{4}u_n \\ -u^i_{n+1}+\frac{32}{9}u^i_{n+\frac{1}{2}}-\frac{32}{15}u^i_{n+\frac{3}{2}}+\frac{2}{9}u^i_{n+2}+2hf^i_{n+1}+\frac{1}{45}u_{n-1}-\frac{2}{3}u_n \\ -u^i_{n+\frac{3}{2}}-\frac{25}{31}u^i_{n+\frac{1}{2}}+\frac{225}{124}u^i_{n+1}-\frac{25}{124}u^i_{n+2}+\frac{15}{31}hf^i_{n+\frac{3}{2}}-\frac{1}{124}u_{n-1}+\frac{25}{124}u_n \\ -u^i_{n+2}+\frac{32}{27}u^i_{n+\frac{1}{2}}-2u^i_{n+1}+\frac{32}{15}u^i_{n+\frac{3}{2}}+\frac{2}{9}hf^i_{n+2}+\frac{2}{135}u_{n-1}-\frac{1}{3}u_n\end{pmatrix}.$$

When $r = 2$

$$\bar{A} = \begin{pmatrix} \left(1 + \frac{15}{19} h \frac{\partial f_{n+\frac{1}{2}}}{\partial u_{n+\frac{1}{2}}}\right) & -\frac{75}{38} & \frac{75}{133} & -\frac{25}{304} \\ -\frac{24}{5} & \left(1 - 3h \frac{\partial f_{n+1}}{\partial u_{n+1}}\right) & \frac{24}{7} & -\frac{3}{8} \\ \frac{147}{205} & -\frac{147}{82} & \left(1 - \frac{21}{41} h \frac{\partial f_{n+\frac{3}{2}}}{\partial u_{n+\frac{3}{2}}}\right) & \frac{147}{656} \\ -\frac{256}{265} & \frac{96}{53} & -\frac{768}{371} & \left(1 - \frac{12}{53} h \frac{\partial f_{n+2}}{\partial u_{n+2}}\right) \end{pmatrix},$$

$$\bar{B} = \begin{pmatrix} -u^i{}_{n+\frac{1}{2}} + \frac{75}{38} u^i{}_{n+1} - \frac{75}{133} u^i{}_{n+\frac{3}{2}} + \frac{25}{304} u^i{}_{n+2} - \frac{15}{19} h f^i{}_{n+\frac{1}{2}} + \frac{3}{2128} u_{n-1} - \frac{75}{152} u_n \\ -u^i{}_{n+1} + \frac{24}{5} u^i{}_{n+\frac{1}{2}} - \frac{24}{7} u^i{}_{n+\frac{3}{2}} + \frac{3}{8} u^i{}_{n+2} + 3h f^i{}_{n+1} + \frac{1}{280} u_{n-1} - \frac{3}{4} u_n \\ -u^i{}_{n+\frac{3}{2}} - \frac{147}{205} u^i{}_{n+\frac{1}{2}} + \frac{147}{82} u^i{}_{n+1} - \frac{147}{656} u^i{}_{n+2} + \frac{21}{41} h f^i{}_{n+\frac{3}{2}} - \frac{3}{3280} u_{n-1} + \frac{49}{328} u_n \\ -u^i{}_{n+2} + \frac{256}{265} u^i{}_{n+\frac{1}{2}} - \frac{96}{53} u^i{}_{n+1} + \frac{768}{371} u^i{}_{n+\frac{3}{2}} + \frac{12}{53} h f^i{}_{n+2} + \frac{3}{1855} u_{n-1} - \frac{12}{53} u_n \end{pmatrix}.$$

When $r = \frac{5}{8}$

$$\bar{A} = \begin{pmatrix} \left(1 + \frac{9}{7} h \frac{\partial f_{n+\frac{1}{2}}}{\partial u_{n+\frac{1}{2}}}\right) & -\frac{243}{91} & \frac{81}{119} & -\frac{9}{98} \\ -\frac{169}{54} & \left(1 - \frac{13}{8} h \frac{\partial f_{n+1}}{\partial u_{n+1}}\right) & \frac{169}{102} & -\frac{169}{1008} \\ \frac{289}{327} & -\frac{2601}{1417} & \left(1 - \frac{51}{109} h \frac{\partial f_{n+\frac{3}{2}}}{\partial u_{n+\frac{3}{2}}}\right) & \frac{289}{1526} \\ -\frac{784}{573} & \frac{5292}{2483} & -\frac{7056}{3247} & \left(1 - \frac{42}{191} h \frac{\partial f_{n+2}}{\partial u_{n+2}}\right) \end{pmatrix},$$

$$\bar{B} = \begin{pmatrix} -u^i{}_{n+\frac{1}{2}} + \frac{243}{91} u^i{}_{n+1} - \frac{81}{119} u^i{}_{n+\frac{3}{2}} + \frac{9}{98} u^i{}_{n+2} - \frac{9}{7} h f^i{}_{n+\frac{1}{2}} + \frac{4096}{54145} u_{n-1} - \frac{81}{70} u_n \\ -u^i{}_{n+1} + \frac{169}{54} u^i{}_{n+\frac{1}{2}} - \frac{169}{102} u^i{}_{n+\frac{3}{2}} + \frac{169}{1008} u^i{}_{n+2} + \frac{13}{8} h f^i{}_{n+1} + \frac{1024}{16065} u_{n-1} - \frac{169}{240} u_n \\ -u^i{}_{n+\frac{3}{2}} - \frac{289}{327} u^i{}_{n+\frac{1}{2}} + \frac{2601}{1417} u^i{}_{n+1} - \frac{289}{1526} u^i{}_{n+2} + \frac{51}{109} h f^i{}_{n+\frac{3}{2}} - \frac{4096}{148785} u_{n-1} + \frac{289}{1090} u_n \\ -u^i{}_{n+2} + \frac{784}{573} u^i{}_{n+\frac{1}{2}} - \frac{5292}{2483} u^i{}_{n+1} + \frac{7056}{3247} u^i{}_{n+\frac{3}{2}} + \frac{42}{191} h f^i{}_{n+2} + \frac{32768}{633165} u_{n-1} - \frac{441}{955} u_n \end{pmatrix}.$$

A computer code written in C programming language is used for the implementation of the methods.

6 PROBLEMS TESTED

The efficiency and performance of the proposed methods are validated by solving the following first order stiff initial value problems:

Problem 1:

$$\frac{du}{dx} = -5 + \cos x + 5\sin x,\ \ u(0) = 1,\ \ 0 \le x \le 0.1,$$

Exact solution: $u(x) = \sin x + e^{-5x}$.

Source: (Artificial)

Problem 2:

$$\frac{du}{dx} = -8(u - 2x) + 2,\ \ u(0) = 1,\ \ 0 \le x \le 0.01$$

Exact solution: $u(x) = 2x + e^{-8x}$.

Source: (Artificial)

Problem 3:

$$\frac{du}{dx} = -12u,\ \ u(0) = 1,\ 0 \le x \le 0.1,$$

Exact solution: $u(x) = e^{-12x}$.

Source: (Artificial)

The numerical results obtained for the tested problems are presented and compared with the existing diagonally implicit block backward differentiation formula of order five which is given as [6]:

$$y_{n+1} = \frac{2}{11}y_{n-2} - \frac{9}{11}y_{n-1} + \frac{18}{11}y_n + \frac{6}{11}hf_{n+1},$$
$$y_{n+2} = -\frac{3}{25}y_{n-2} + \frac{16}{25}y_{n-1} - \frac{36}{25}y_n + \frac{48}{25}y_{n+1} + \frac{12}{25}hf_{n+2},$$
$$y_{n+3} = \frac{12}{137}y_{n-2} - \frac{75}{137}y_{n-1} + \frac{200}{137}y_n - \frac{300}{137}y_{n+1} + \frac{300}{137}y_{n+2} + \frac{60}{137}hf_{n+3}.$$

For the purpose of illustrating the accuracy and computation time of the proposed methods, we denote the fully implicit block BDF method (9) which corresponds to the constant step size strategy $r = 1$ as 2BBDFO1 while those BBDF methods corresponding to halving and multiplying the step size by a factor of 1.6 given by the formulae in (10) and (11) are denoted by 2BBDFO2 and 2BBDFO3

respectively. The maximum error and computation time for the methods are presented and compared in the tables 1-3 below. However, the efficiency graphs of $Log_{10}(MAXE)$ against $Log_{10}H$ for each problem are plotted in order have a visual impact on the performance of the methods. The following notations apart from those defined earlier are used in tables below.

H: Step size.

$MAXE$: Maximum error.

$TIME$: Computation time in seconds.

Table 1: Numerical comparison between Exact and Approximate solution for Problem 1 with step size $H = 10^{-2}$

	Method

x_i	*Exact Solution*	$2BBDFO1$	$2BBDFO2$	$2BBDFO3$
0.00	1.0000000000	0.9983855500	0.9985411500	0.9962974500
0.01	0.9612292578	0.9596148078	0.9597704078	0.9575267078
0.02	0.9248360847	0.9232216347	0.9233772347	0.9211335347
0.03	0.8907034766	0.8890890266	0.8892446266	0.8870009266
0.04	0.8587200873	0.8571056373	0.8572612373	0.8550175373
0.05	0.8287799524	0.8271655024	0.8273211024	0.8250774024
0.06	0.8007822272	0.7991677772	0.7993233772	0.7970796772
0.07	0.7746309370	0.7730164870	0.7731720870	0.7709283870
0.08	0.7502347400	0.7486202900	0.7487758900	0.7465321900
0.09	0.7275067008	0.7258922508	0.7260478508	0.7238041508
0.10	0.7275067008	0.7047496264	0.7049052264	0.7026615264

Table 2: Numerical comparison between Exact and Approximate solution for Problem 2 with step size $H = 10^{-3}$

	Method			
x_i	*Exact Solution*	$2BBDFO1$	$2BBDFO2$	$2BBDFO3$
0.000	1.0000000000	0.9999527445	0.9999744530	0.9994566230
0.001	0.9940319148	0.9939846593	0.9940063678	0.9934885378
0.002	0.9881273201	0.9880800646	0.9881017731	0.9875839431
0.003	0.9822857098	0.9822384543	0.9822601628	0.9817423328
0.004	0.9765065821	0.9764593266	0.9764810351	0.9759632051
0.005	0.9707894392	0.9707421837	0.9707638922	0.9702460622
0.006	0.9651337871	0.9650865316	0.9651082401	0.9645904101
0.007	0.9595391359	0.9594918804	0.9595135889	0.9589957589
0.008	0.9540049995	0.9539577440	0.9539794525	0.9534616225
0.009	0.9485308958	0.9484836403	0.9485053488	0.9479875188
0.010	0.9431163464	0.9430690909	0.9430907994	0.9425729694

Table 3: Numerical comparison between Exact and Approximate solution for Problem 3 with step size $H = 10^{-2}$

	Method			
x_i	*Exact Solution*	$2BBDFO1$	$2BBDFO2$	$2BBDFO3$
0.00	1.0000000000	0.9925681300	0.9930093300	0.9896704000
0.01	0.8869204367	0.8794885667	0.8799297667	0.8765908367
0.02	0.7866278611	0.7791959911	0.7796371911	0.7762982611
0.03	0.6976763261	0.6902444561	0.6906856561	0.6873467261
0.04	0.6187833918	0.6113515218	0.6117927218	0.6084537918
0.05	0.5488116361	0.5413797661	0.5418209661	0.5384820361
0.06	0.4867522560	0.4793203860	0.4797615860	0.4764226560
0.07	0.4317105234	0.4242786534	0.4247198534	0.4213809234
0.08	0.3828928860	0.3754610160	0.3759022160	0.3725632860
0.09	0.3395955256	0.3321636556	0.3326048556	0.3292659256
0.10	0.3011942119	0.2937623419	0.2942035419	0.2908646119

These tables 1, 2, and 3 present the numerical comparison of exact and approximate solutions for different problems at three different step sizes. The methods used are labeled as 2BBDFO1, 2BBDFO2, 2BBDFO3, and 3DIBBDF. The results show that as the step size H decreases, the accuracy of the approximate solutions tends to improve. Furthermore, it can be observed that method 2BBDFO1 consistently exhibits better accuracy in comparison to the other methods, especially when H is smaller. This indicates the effectiveness of method 2BBDFO1 in approximating the exact solutions for these problems. However, the method 2BBDFO2 also provides reasonably accurate results and can be considered when computational efficiency is a priority.

Table 4: Numerical comparison of Maximum error and computation time between the methods for Problem 1

H	$METHOD$	$MAXE$	$TIME$
10^{-2}	2BBDFO1	1.61445e − 003	1.36700e − 001
	2BBDFO2	1.45885e − 003	1.69100e − 001
	2BBDFO3	3.70255e − 003	1.44200e − 001
	3DIBBDF	8.18728e − 001	1.30400e − 001
10^{-3}	2BBDFO1	1.86340e − 005	1.43100e − 001
	2BBDFO2	3.83801e − 004	1.47100e − 001
	2BBDFO3	3.11823e − 003	1.37400e − 001
	3DIBBDF	9.80199e − 001	1.43800e − 001
10^{-4}	2BBDFO1	1.89018e − 007	1.50000e − 001
	2BBDFO2	4.01272e − 004	1.50400e − 001
	2BBDFO3	3.16549e − 003	1.54600e − 001
	3DIBBDF	9.98002e − 001	1.42400e − 001
10^{-5}	2BBDFO1	1.89287e − 009	2.04800e − 001
	2BBDFO2	4.01967e − 004	2.04400e − 001
	2BBDFO3	3.17112e − 003	2.11400e − 001
	3DIBBDF	9.99800e − 001	1.43200e − 001
10^{-6}	2BBDFO1	1.89313e − 011	5.26800e − 001
	2BBDFO2	4.02026e − 004	5.31800e − 001
	2BBDFO3	3.17170e − 003	5.45700e − 001
	3DIBBDF	9.99980e − 001	2.19400e − 001

Table 5: Numerical comparison of Maximum error and computation time between the methods for Problem 1

H	$METHOD$	$MAXE$	$TIME$
10^{-2}	2BBDFO1	3.75727e − 003	1.46700e − 001
	2BBDFO2	3.52947e − 003	1.55500e − 001
	2BBDFO3	5.26696e − 003	1.52300e − 001
	3DIBBDF	7.26149e − 001	1.51500e − 001
10^{-3}	2BBDFO1	4.72555e − 005	1.54300e − 001
	2BBDFO2	2.55470e − 005	1.54500e − 001
	2BBDFO3	5.43377e − 004	1.67200e − 001
	3DIBBDF	9.68507e − 001	1.75800e − 001
10^{-4}	2BBDFO1	4.83430e − 007	1.66200e − 001
	2BBDFO2	9.39008e − 005	1.69900e − 001
	2BBDFO3	7.46901e − 004	1.80000e − 001
	3DIBBDF	9.96805e − 001	1.79200e − 001

10^{-5}	$2BBDFO1$ $2BBDFO2$ $2BBDFO3$ 3DI$BBDF$	$4.84530e-009$ $9.73565e-005$ $7.70730e-004$ $9.99680e-001$	$1.67200e-001$ 1.74500e − 001 1.80100$e-001$ 1.70000$e-001$
10^{-6}	$2BBDFO1$ $2BBDFO2$ $2BBDFO3$ 3DI$BBDF$	$4.84638e-011$ $9.76614e-005$ $7.73146e-004$ $9.99968e-001$	$1.78600e-001$ 1.82500e − 001 $1.89000e-001$ $1.71900e-001$

Table 6: Numerical comparison of Maximum error and computation time between the methods for Problem 3

H	$METHOD$	$MAXE$	$TIME$
10^{-2}	$2BBDFO1$ $2BBDFO2$ $2BBDFO3$ 3DI$BBDF$	$7.43187e-003$ $6.99067e-003$ $1.03296e-002$ $6.18783e-001$	1.54100e − 001 1.59000e − 001 1.83000$e-001$ 1.54800$e-001$
10^{-3}	$2BBDFO1$ $2BBDFO2$ $2BBDFO3$ 3DI$BBDF$	$1.04988e-004$ $6.20482e-004$ $5.17892e-003$ $9.53134e-001$	1.74600e − 001 1.67400e − 001 1.89500$e-001$ 1.63800$e-001$
10^{-4}	$2BBDFO1$ $2BBDFO2$ $2BBDFO3$ 3DI$BBDF$	$1.08634e-006$ $6.58066e-004$ $5.17148e-003$ $9.95212e-001$	1.84600e − 001 1.69400$e-001$ 1.60400$e-001$ 1.68300$e-001$
10^{-5}	$2BBDFO1$ $2BBDFO2$ $2BBDFO3$ 3DI$BBDF$	$1.09005e-008$ $6.58438e-004$ $5.17434e-003$ $9.99520e-001$	1.85200e − 001 1.81600e − 001 1.97800$e-001$ 1.69500$e-001$
10^{-6}	$2BBDFO1$ $2BBDFO2$ $2BBDFO3$ 3DI$BBDF$	$1.09042e-010$ $6.58440e-004$ $5.17467e-003$ $9.99952e-001$	4.29600e − 001 3.99000e − 001 4.45400$e-001$ 1.74000$e-001$

Similarly, the tables 4, 5, and 6 provide a comparison of the maximum error (MAXE) and computation time (TIME) for the different methods and step sizes. As expected, the maximum error generally decreases as the step size decreases. However, it's worth noting that method 2BBDFO1 consistently outperforms the other methods in terms of maximum error, indicating its accuracy.

In terms of computation time, the tables show that the 3DIBBDF method generally has the lowest computation time across different step sizes, suggesting it is the most computationally efficient method. However, method 2BBDFO1 offers a trade-off between accuracy and computation time, with reasonably low computation times while maintaining good accuracy.

Overall, these tables illustrate the trade-offs between accuracy and computation time when using different numerical methods for solving the stiff problems. Method 2BBDFO1 appears to be a strong contender, offering good accuracy even with small step sizes, making it a promising choice for applications where precision is essential. However, the selection of the most suitable method may depend on the specific requirements of the problem, considering the balance between accuracy and computational efficiency.

In order to visually highlight the performance of the 2BBDFO1, 2BBDFO2, 2BBDFO3, and 3DIBBDF methods, we have generated graphs in the Matlab software environment. These graphs depict $Log_{10}(MAXE)$ against $Log_{10}H$ for all the tested problems and also the efficiency graphs between exact and approximate solutions have been plotted.

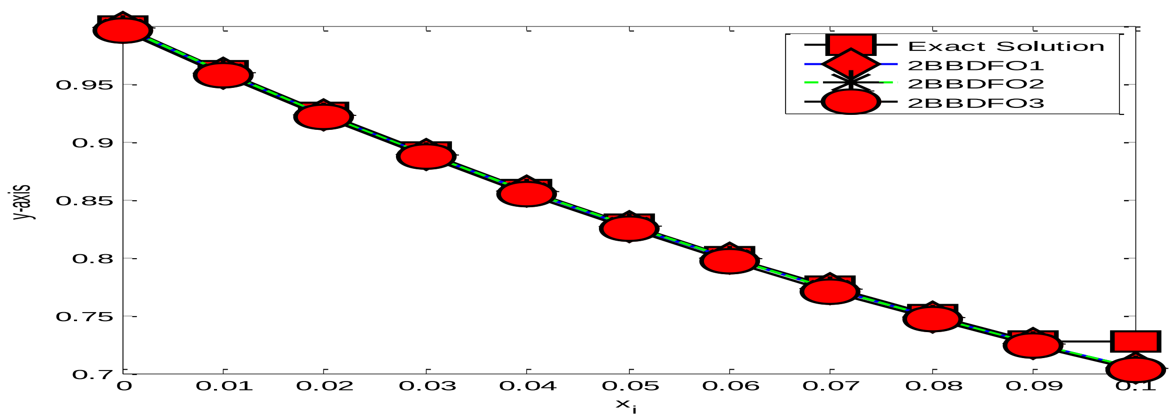


Figure 4: Efficiency graph between exact and approximate solutions for Problem 1.

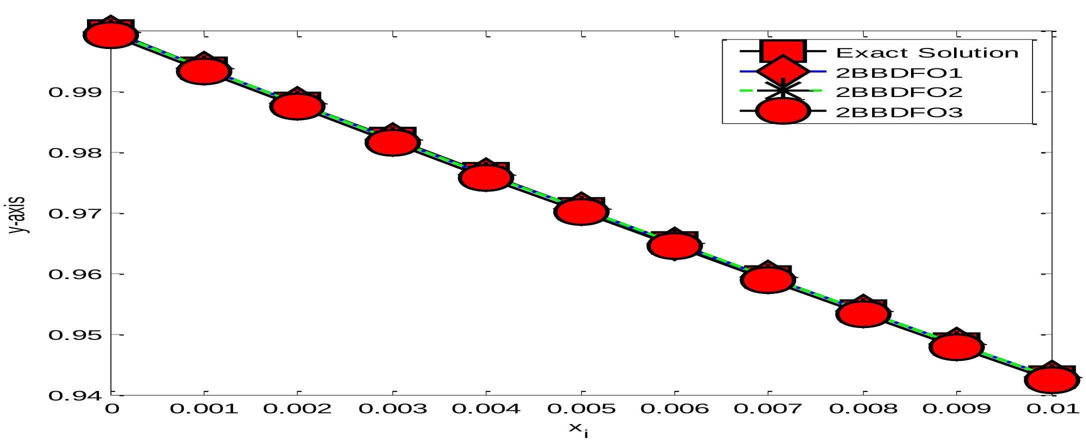


Figure 5: Efficiency graph between exact and approximate solutions for Problem 2.

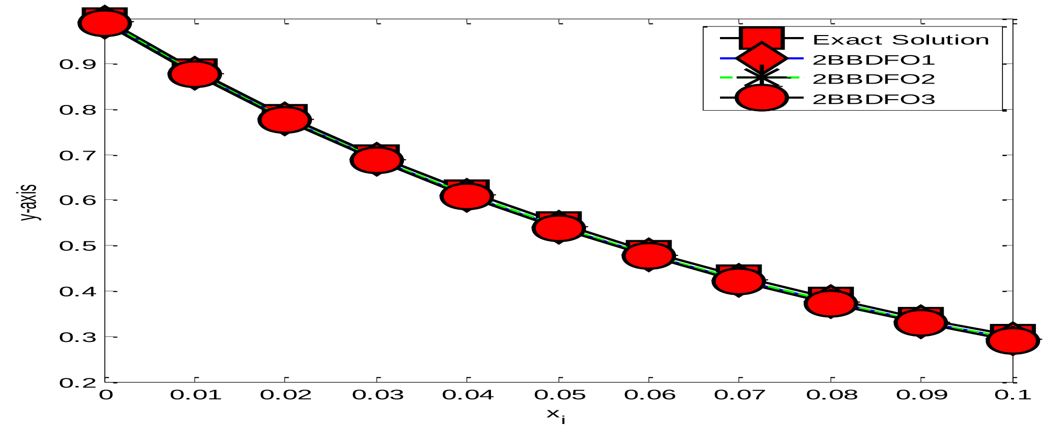


Figure 6: Efficiency graph between exact and approximate solutions for Problem 3.

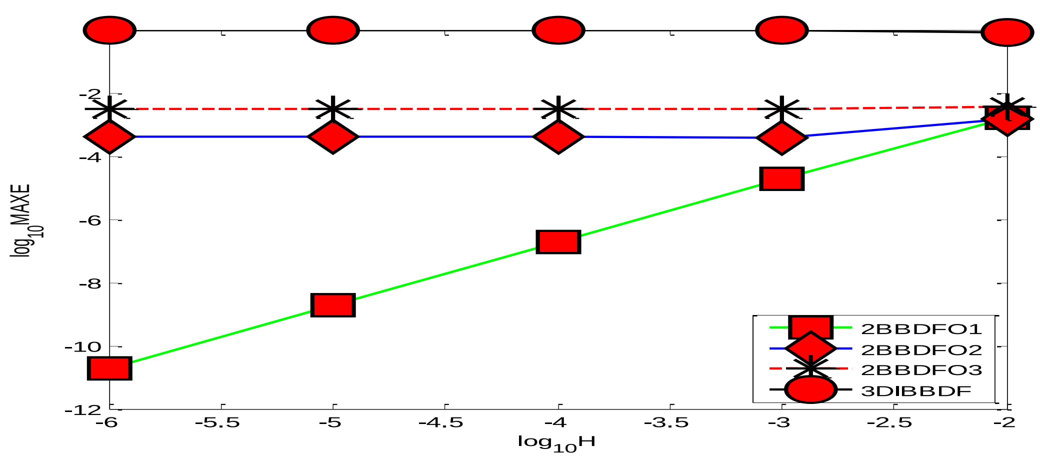


Figure 7: Efficiency graph of $Log_{10}MAXE$ against $Log_{10}H$ for Problem 1.

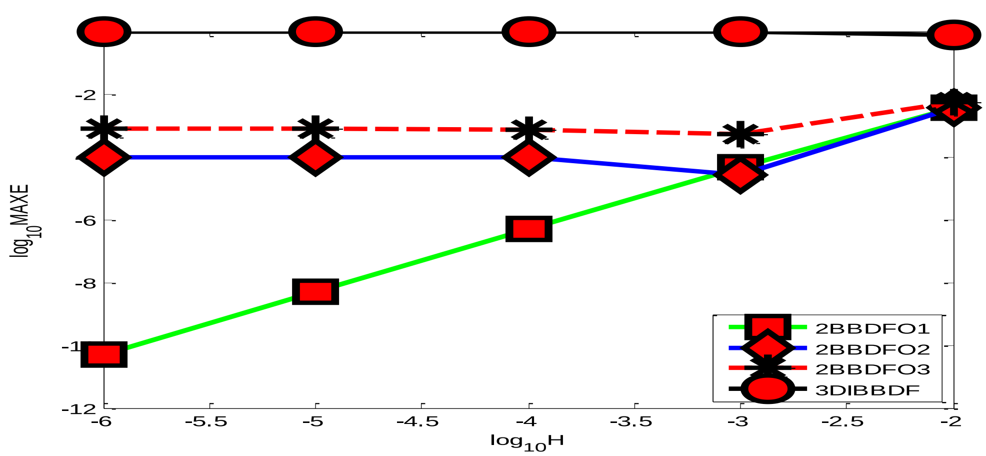


Figure 8: Efficiency graph of $Log_{10}MAXE$ against $Log_{10}H$ for Problem 2.

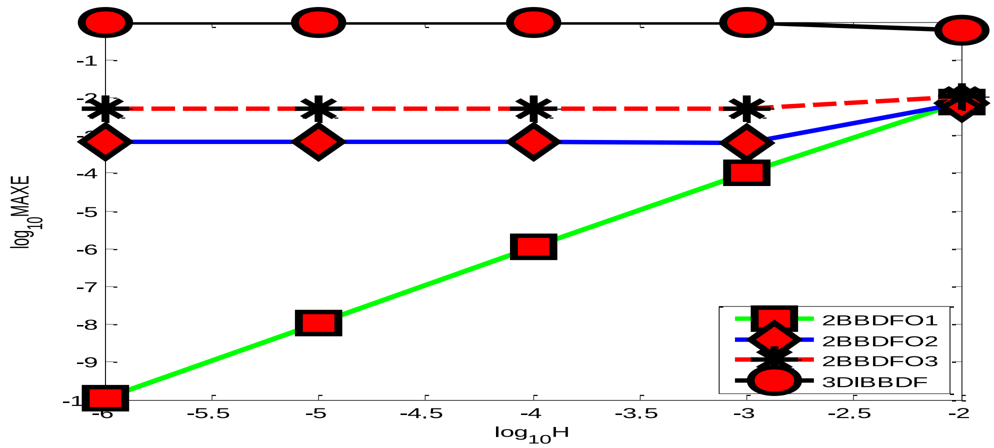


Figure 9: Efficiency graph of $Log_{10}MAXE$ against $Log_{10}H$ for Problem 3.

The new methods (2BBDFO1, 2BBDFO2, and 2BBDFO3) tend to provide more accurate results compared to 3DIBBDF for the given problems. This is evident from the smaller maximum error values in most of the cases.

7. CONCLUSION

A new fully implicit 2-point variable step size block backward differentiation formula with two off-step points for solving first order stiff ordinary differential equations has been developed. The method is derived to be of order five. The stability analysis of the methods indicates that the methods are both zero and A-stable. To demonstrate the performance of the methods, some first order stiff initial value problems are solved and compared with existing diagonally implicit 3-point block backward differentiation formula in terms of accuracy and computation time. The numerical results obtained show that all the proposed methods performed better than 3DIBBDF in terms of accuracy. In addition, it can also be seen that the 3DIBBDF method is better than the proposed methods in terms of computation time.

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REFERENCES

[1] J. D. Lambert, Computaional Methods in Ordinary Differential Equations, John Wiley & Sons, New York, 1973.

[2] M. B. Sulaiman, H. Musa, F. Ismail, N. Senu, and Z. B. Ibrahim, "A new super class of block backward differentiation formula for stiff ODEs," Asian European Journal of Mathematics, vol.

7, no. 1, 2014.

[3] C.F. Curtiss and J.O. Hirschfelder, “Integration of Stiff Equations,” in Proceedings - 1952 of the National Academy of Sciences, 1952, vol. 38, pp. 235-243.

[4] H. Musa, A. Buhari, and N. Abasi, “Diagonally implicit 2-point Block Backward Differentiation Formula with Two Off-step Points for Solving Stiff Initial Value Problems,” Nigerian Journal of Mathematics and Applications. vol. 32 (II), pp. 108-119, 2022.

[5] Z. B. Ibrahim, N. M. Noor, and K. I. Othman,” Fixed Coefficient $A(\alpha)$ Stable Block Backward Differentiation Formulas For Stiff Ordinary Differential Equations”, MDPI, Symmetry, pp.1-12, 2019.

[6] N. Bala, B. Alhassan, and H. Musa,” A New 3-point Implicit Block Numerical Method For Solving First Order Stiff Initial Value Problems,” Timbou-Africa Academics Publications, International Journal of Science Research and Technology, vol. 9, no 9, pp. 219-231,2022.

[7] S. J. Aksah, Z. B. Ibrahim, and I. S. M. Zawawi, “Stability Analysis of Singly Diagonally Implicit Block Backward Differentiation Formulas for Stiff Ordinary Differential Equations,” Mathematics, vol. 7, pp. 211, 2019.

[8] Z. B. Ibrahim and A. A. Nasarudin, “A class of hybrid multistep block methods with A-stability for the numerical solution of stiff ordinary differential equations,” Mathematics, vol. 8, pp. 914, 2020.

[9] A. Buhari, M. Hamisu, and N. Abasi,” Convergence and Order of the 2-point Diagonally Implicit Block Backward Differentiation Formula with Two Off-step Points” UMYU Scientifica, vol. 1, no 1, pp. 30-38, 2022.

[10] H. Musa, M. B. Suleiman, and N. Senu, “Fully implicit 3-point block extended backward differentiation formula for stiff initial value problems,” Applied Mathematical Sciences, vol. 6, pp. 4211-4228, 2012.

[11] N. I. N. Ismail, Z. A. Majid, and N.Senu, “Hybrid multistep block method for solving neutral delay differential equations,” Sains Malaysiana, vol. 49, pp. 929-940, 2020.

[12] Z. B. Ibrahim, K. Othman, and M.B. Suleiman, “Implicit r-point block backward differentiation formula for solving first- order stiff ODEs,” Applied Mathematics and Computation, vol. 186, pp. 558-565, 2007.

[13] H. Musa, M. B. Suleiman, F. Ismail, N. Senu, Z. A. Majid, and Z. B. Ibrahim, “A new fifth order implicit block method for solving first order stiff ordinary differential equations,” Malaysian Journal of Mathematical Sciences, vol. 8, pp. 45-59, 2014.

[14] N. Abasi, M. Suleiman, N. Abbasi, and H. Musa,” 2-Point Block BDF Method with Off-step Points for Solving Stiff ODEs”, Journal of Soft Computing and Applications, vol. 2014, pp.1-15, 2014.

[15] Z. B. Ibrahim, N. Zainuddin, K. I. Othman, M. Suleiman, and I. S. M. Zawawi, “Variable order block method for solving second order ordinary differential equations,” Sains Malays., vol. 48,

pp. 1761–1769, 2019.

[16] M. B. Suleiman, H. Musa, F. Ismail, and N. Senu, "A new variable step size block backward differentiation formula for solving stiff initial value problems," International Journal of Computer Mathematics, vol. 90, pp. 2391-2408, 2013.

[17] I. S. M. Zawawi, Z. B. Ibrahim, and K. I. Othman, "Derivation of Diagonally Implicit Block Backward Differentiation Formulas for solving Stiff Initial Value Problems," Hindawi Publishing Coperation, Mathematical Problems in Engineering, Article ID 179231 pp. 1-13, 2015.

[18] A. Buhari and M. Hamisu, "Diagonally Implicit Extended 2-Point Super Class of Block Backward Differentiation Formula for solving First Order Stiff Initial Value Problems," Applied Mathematics and Computational Intelligence, vol 12, no 1, pp. 101-124, 2023.

[19] A. A. Nasarudin, Z. B. Ibrahim, and H. Rosali," On the Integration of Stiff ODEs Using Block Backward Differentiation Formulas of Order Six", MDPI, Symmetry, pp. 1-13, 2020.

[20] N. Abasi, M. B. Suleiman, F. Ismail, Z. B. Ibrahim, H. Musa and N. Abbasi, "A new formulae of variable step 3-point block BDF method for solving stiff ODEs", Journal of Pure and Applied Mathematics: Advances and Applications, vol. 12, no. 1, pp. 49-76 (2014).