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Half-Sweep Approximation for Nonlinear Diffusion Equation In Two-Dimensional Porous Medium

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ABSTRACT

This paper investigates the efficacy of the half-sweep approximation to solve the nonlinear diffusion equation in the two-dimensional porous medium. The half-sweep approximation is systematically formulated, and its stability properties are analysed based on its iterative form. The system of equations corresponding to the approximation equation to the two-dimensional nonlinear diffusion equation is solved using the developed half-sweep Newton-Gauss-Seidel algorithm. The numerical experiment uses several initial boundary value problems in natural science to illustrate the efficacy of the proposed approximation. This study finds that the half-sweep approximation is more efficient than the implicit finite difference approximation in numerical computation. The numerical convergence of the approximation is presented to show the potential of the half-sweep approximation to solve different types of nonlinear diffusion equations in a two-dimensional porous medium.

Keywords: nonlinear diffusion, porous medium, finite difference method, Newton-Gauss-Seidel, iterative method.

1 INTRODUCTION

The finite difference (FD) method is a versatile numerical discretisation method to approximate the solution of a multidimensional nonlinear partial differential equation (NPDE). Then, the derivation of the FD schemes can be easily made using the expansion of the Taylor series. The simplicity of deriving and implementing FD to solve NPDE has attracted the interest of many researchers. Thus, many FD schemes for solving NPDE have been introduced to the literature, such as stochastic non-standard FD [1], meshless generalised FD [2], fourth-order standard compact FD [3], sixth-order FD [4], radial basis function-generated FD [5], space-time generalised FD [6], semi-implicit FD [7], and many more. Generally, the choice of FD scheme to solve NPDE depends on the balance of approximation accuracy and computational cost. The scheme with higher order accuracy in time produces more accurate approximations for the time derivatives. However, additional function evaluations are required and thus increase the computational cost. Similarly, the scheme with higher order accuracy in space suffers from higher computational expense. It requires a finer grid or more grid points to calculate for a more accurate approximation of the spatial derivatives.

The ongoing development of an FD scheme to solve NPDE has motivated this study to investigate a computationally efficient FD scheme that can overcome the complexity of computing a complex nonlinear system modelled from a discretised NPDE. This study focuses on a computationally efficient FD scheme called the half-sweep FD. Half-sweep FD is a type of FD scheme that can reduce the complexity of numerical computation of a large-sized system of equations [8]. Besides that, the accuracy of the half-sweep FD approximation is almost like the implicit FD but has a lower computational cost. Several researchers reported the advantage of using half-sweep FD over the standard implicit FD. For example, Sunarto *et al.* [9] showed the superiority of the half-sweep FD with a preconditioned successive over-relaxation over the implicit preconditioned successive over-relaxation regarding computational complexity to solve the space-fractional diffusion equation. Then, Xu *et al.* [10] and Xu *et al.* [11] studied a rational half-sweep FD with a composite trapezoidal approach to approximate the solutions of Fredholm integro-differential equation in first and second-order problems, respectively. They showed that the half-sweep FD could reduce the number of iterations and elapsed time less than the implicit FD in solving the system of equations from first- and second-order linear Fredholm integro-differential equations. Next, Sunarto *et al.* [12] showed that the half-sweep FD with a preconditioned accelerated over relaxation could be one of the best computational approaches to solve the time-fractional diffusion equation, compared to the implicit preconditioned accelerated over relaxation method. Besides that, Agarwal *et al.* [13] introduced an efficient half-sweep accelerated overrelaxation iterative method with a new preconditioning matrix to solve a one-dimensional linear space-fractional diffusion equation. They showed that the new numerical method solves the problem with lesser iterations and faster computation time than the preconditioned accelerated overrelaxation and implicit Euler method. Recently, Xu *et al.* [14] established the three-point newly half-sweep linear rational finite difference-quadrature discretisation scheme for solving the first-order linear Fredholm integro-differential equation. Half-sweep FD has a lot of success in solving integro-differential and fractional differential equations efficiently. However, the information about the efficacy of half-sweep FD to solve NDPE is still limited, especially for the two-dimensional problem. Thus, we aim to evaluate the effectiveness of the half-sweep FD in solving a two-dimensional NPDE.

The contribution of this paper is to derive a two-dimensional half-sweep FD approximation for solving NPDE and to present the efficiency and convergence rate of the approximation. A porous medium-type differential equation is used to illustrate the efficacy of the half-sweep FD approximation. This differential equation is one of the essential NPDE of the parabolic type. It famously appears in modelling natural phenomena related to nonlinear liquid or gas diffusion and heat propagation [15-22]. The solution of this equation is essential to understand the mentioned phenomena and leads to a realisation of a more complex mathematical model. The exact solution is challenging to obtain when a porous medium model considers many assumptions. Hence, the numerical solution from an efficient numerical method becomes the better option.

The structure of this paper contains: Section 2 shows the formulation of the half-sweep FD approximation to the two-dimensional porous medium-type differential equation. Then, the stability of the approximation equation is analysed based on the iterative formula. Next, the iterative procedure using Gauss-Seidel's method to solve the arisen nonlinear system is discussed. Section 3 describes the numerical experiment using several initial boundary value problems (IBVP) in natural sciences. The finding from implementing the half-sweep FD approximation to the proposed problems is reported. The last section concludes this paper with the future research direction.

2 METHODS

2.1 Two-Dimensional Nonlinear Porous Medium-type Differential Equation

This section presents a half-sweep FD approximation to solve a two-dimensional nonlinear porous medium-type differential equation. Let consider the main equation be defined as follows [23],

$$\frac{\partial u}{\partial t} = \alpha\left[\frac{\partial}{\partial x}\left(u^m \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(u^m \frac{\partial u}{\partial y}\right)\right], \tag{1}$$

where $m \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$. Eq. (1) describes the evolution of a scalar field $u = u(x, y, t)$ over time t in two spatial dimensions (x, y). The equation constitutes nonlinear diffusion terms represented by $u^m \frac{\partial u}{\partial x}$ and $u^m \frac{\partial u}{\partial y}$ where both m and α are constant. Eq. (1) can be understood as a mathematical model for a process where the field u undergoes diffusion in a porous medium that is influenced by the nonlinearity with varied magnitude m. When $m > 0$, the field u undergoes slow diffusion, while $m < 0$ is where the field u undergoes fast diffusion. The diffusion speed or the value of m can be determined experimentally by considering various variables, including porosity, permeability, concentration gradient and molecules sizes [24].

2.2 Half-sweep FD Scheme

To formulate a half-sweep FD approximation to Eq. (1), we define an initial condition $u(x, y, 0), 0 \leq x, y \leq 1$ and boundary conditions $u(0, y, t), u(1, y, t), u(x, 0, t)$, and $u(x, 1, t)$ at the time interval $0 \leq t \leq 1$. Then, we define the approximate solutions of Eq. (1) as $U_{p,q,n} = U(x_p, y_q, t_n)$, that correspond to exact solutions $u(x_p, y_q, t_n)$ where $p, q \in \mathbb{Z}$ such that $1 \leq p, q \leq M - 1$ and $n \in \mathbb{Z}$ such that $1 \leq n \leq N$. Note that M and N represent the number of grid points (or size of matrix) and total time level, respectively. The distribution of grid points is equidistant with $x_p = ph, y_q = qh$, and spatial steps $h = 1/M$. The time step is also uniform with $t_n = nk$ and $k = 1/N$.

Now, we derive the half-sweep FD schemes using the Taylor series expansion as follows. Let us consider

$$U_{p,q,n+1} = U_{p,q,n} + k\frac{\partial}{\partial t}U_{p,q,n+1} + \cdots, \tag{2}$$

$$U_{p+1,q,n+1} = U_{p,q,n+1} + h\frac{\partial}{\partial x}U_{p,q,n+1} + \frac{h^2}{2}\left(\frac{\partial}{\partial x}U_{p,q,n+1}\right)^2 + \cdots, \tag{3}$$

$$U_{p-1,q,n+1} = U_{p,q,n+1} - h\frac{\partial}{\partial x}U_{p,q,n+1} + \frac{h^2}{2}\left(\frac{\partial}{\partial x}U_{p,q,n+1}\right)^2 + \cdots, \tag{4}$$

$$U_{p,q+1,n+1} = U_{p,q,n+1} + h\frac{\partial}{\partial y}U_{p,q,n+1} + \frac{h^2}{2}\left(\frac{\partial}{\partial y}U_{p,q,n+1}\right)^2 + \cdots, \tag{5}$$

and

$$U_{p,q-1,n+1} = U_{p,q,n+1} - h\frac{\partial}{\partial y}U_{p,q,n+1} + \frac{h^2}{2}\left(\frac{\partial}{\partial y}U_{p,q,n+1}\right)^2 + \cdots. \tag{6}$$

Then, Eq. (2) can be rearranged to approximate the first-order derivative to t, which is

$$\frac{\partial u}{\partial t} = \frac{U_{p,q,n+1} - U_{p,q,n}}{k}. \tag{7}$$

The standard approximation to first order derivative to x can be derived using the difference of Eq. (3) and (4) into

$$\frac{\partial u}{\partial x} = \frac{U_{p+1,q,n+1} - U_{p-1,q,n+1}}{2h}. \tag{8}$$

We modify Eq. (8) for the diagonal direction from $U_{p+1,q+1,n+1}$ to $U_{p-1,q-1,n+1}$ and obtain the new distance between these two points using the Pythagoras formula, which yields the following half-sweep FD scheme for the first-order derivative to x,

$$\frac{\partial u}{\partial x} = \frac{U_{p+1,q+1,n+1} - U_{p-1,q-1,n+1}}{2\sqrt{2}h}. \tag{9}$$

Next, the standard approximation to the first order derivative to y can be derived using the difference of Eq. (5) and (6) into

$$\frac{\partial u}{\partial y} = \frac{U_{p,q+1,n+1} - U_{p,q-1,n+1}}{2h}. \tag{10}$$

The half-sweep FD scheme for the first-order derivative to y is derived by modifying the approximation for the diagonal direction from $U_{p-1,q+1,n+1}$ to $U_{p+1,q-1,n+1}$ with a new distance to become

$$\frac{\partial u}{\partial y} = \frac{U_{p-1,q+1,n+1} - U_{p+1,q-1,n+1}}{2\sqrt{2}h}. \tag{11}$$

The half-sweep FD schemes for the second-order derivatives in both x and y can be derived similarly. The sum of Eq. (3) and (4) produce the standard approximation to the second-order derivative to x as follows,

$$\frac{\partial^2 u}{\partial x^2} = \frac{U_{p+1,q,n+1} - 2U_{p,q,n+1} + U_{p-1,q,n+1}}{h^2}. \tag{12}$$

Then, Eq. (12) is modified for the diagonal direction from $U_{p+1,q+1,n+1}$ to $U_{p-1,q-1,n+1}$ with the new distance into

$$\frac{\partial^2 u}{\partial x^2} = \frac{U_{p+1,q+1,n+1} - 2U_{p,q,n+1} + U_{p-1,q-1,n+1}}{2h^2}. \tag{13}$$

Next, the sum of Eq. (5) and (6) produce the standard approximation to the second-order derivative to y,

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_{p,q+1,n+1} - 2U_{p,q,n+1} + U_{p,q-1,n+1}}{h^2}, \tag{14}$$

which we then modify for the diagonal direction from $U_{p-1,q+1,n+1}$ to $U_{p+1,q-1,n+1}$ with the new distance into

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_{p-1,q+1,n+1} - 2U_{p,q,n+1} + U_{p+1,q-1,n+1}}{2h^2}. \tag{15}$$

Thus, using the schemes shown in Eqs. (7), (9), (11), (13) and (15), the half-sweep FD approximation equation to Eq. (1) can be formulated and yields,

$$\begin{aligned} F_{p,q}\left(\widehat{U}_{n+1}\right) = {} & U_{p,q,n+1} - A_1 U_{p,q,n+1}^m U_{p+1,q+1,n+1} - A_1 U_{p,q,n+1}^m U_{p-1,q+1,n+1} \\ & +4A_1 U_{p,q,n+1}^m U_{p,q,n+1} - A_1 U_{p,q,n+1}^m U_{p+1,q-1,n+1} - A_1 U_{p,q,n+1}^m U_{p-1,q-1,n+1} \\ & -A_2 m U_{p,q,n+1}^{m-1} U_{p+1,q+1,n+1}^2 - A_2 m U_{p,q,n+1}^{m-1} U_{p-1,q+1,n+1}^2 \\ & +2A_2 m U_{p,q,n+1}^{m-1} U_{p+1,q+1,n+1} U_{p-1,q-1,n+1} + 2A_2 m U_{p,q,n+1}^{m-1} U_{p-1,q+1,n+1} U_{p+1,q-1,n+1} \\ & -A_2 m U_{p,q,n+1}^{m-1} U_{p+1,q-1,n+1}^2 - A_2 m U_{p,q,n+1}^{m-1} U_{p-1,q-1,n+1}^2 - U_{p,q,n}, \end{aligned} \tag{16}$$

where $1 \le p,q \le M-1, 0 \le n \le N$, $A_1 = \alpha k/2h^2$, $A_2 = \alpha k/8h^2$, and

$\widehat{U}_{n+1} = (U_{p+1,q+1,n+1}, U_{p-1,q+1,n+1}, U_{p,q,n+1}, U_{p+1,q-1,n+1}, U_{p-1,q-1,n+1})$.

Using Eq. (16) on the specified solution domain, a nonlinear system that is typically large-sized and possesses high computational complexity can be formed as follows,

$$\hat{F}\left(\widehat{U}_{n+1}\right) = 0, \tag{17}$$

where $\hat{F}(.) = \Big(f_{1,1}(.), \dots, f_{M-1,1}(.), \dots, f_{1,M-1}(.), \dots, f_{M-1,M-1}(.)\Big)$.

A second-order Newton's method is used to solve the nonlinear system shown by Eq. (17). Then, through a linearisation approach using Newton's method, the corresponding linear system for each time level $n+1$ can be generalised as,

$$J_{n+1}\widehat{H}_{n+1} = -\widehat{F}_{n+1}, \tag{18}$$

where J_{n+1} is a penta-diagonal Jacobian matrix with the following form,

$$J_{n+1} = \begin{bmatrix} T_1 & V_1 & & & \\ L_2 & T_2 & \ddots & & \\ & \ddots & \ddots & & \\ & & & T_{M-2} & V_{M-2} \\ & & & L_{M-1} & T_{M-1} \end{bmatrix}, \tag{19}$$

where T, L and V are tri-diagonal groups, lower groups, and upper groups of the coefficient matrix, respectively. Based on Eq. (19), $\widehat{H}_{n+1}$ represents the solution corrector, which can lead to the approximate solution $\widehat{U}_{n+1}$ by iteration by the following equation,

$$\widehat{U}_{n+1}^{(l+1)} = \widehat{U}_{n+1}^{(l)} + \widehat{H}_{n+1}^{(l+1)}, \tag{20}$$

where l is the iteration index.

2.3 Stability Analysis

Theorem 2.1 Suppose that $\widehat{U}_{n+1}$ be the solution. The half-sweep FD approximation equation shown by Eq. (16) is unconditionally stable and satisfies the eigenvalue,

$$\left[\frac{1}{1+\lambda}\right] < 1. \tag{21}$$

Proof.

Let us consider the iterative form given by

$$(I + A)\widehat{U}_{n+1} = \widehat{U}_n - \widehat{F}_{n+1}, \tag{22}$$

where I is an identity matrix, and A is a coefficient matrix. Then, we define the approximation errors as $\hat{\epsilon}_{n+1}$ and $\hat{\epsilon}_n$ which correspond to $\widehat{U}_{n+1}$ and $\widehat{U}_n$ respectively. By substituting the errors into $\widehat{U}_{n+1}$ and $\widehat{U}_n$, Eq. (22) can be rewritten as

$$(I + A)\hat{\epsilon}_{n+1} = \hat{\epsilon}_n - \widehat{F}_{n+1}. \tag{23}$$

According to the Neumann series theorem, if A is invertible, then $(I + A)$ is also invertible [25]. Thus, the growth of approximation errors by Eq. (16) can be expressed in the following equation,

$$\underset{\sim}{\epsilon}_n = (I + A)^{-1} I \, \underset{\sim}{\epsilon}_{n-1}, \tag{24}$$

and $\widehat{F}_{n+1} \to 0$.

Next, according to Gerschgorin's theorem [26], the eigenvalues of A can be found inside the disks centred at each diagonal entry. Since the diagonal entries for the coefficient matrix in Eq. (18) are always positive, this implies that A has an eigenvalue λ. If A has an eigenvalue λ, then every eigenvalue of the matrix $(I+A)$ has a radius larger than unity.

Also, if $(I+A)$ is invertible, then $(I+A)$ has an eigenvalue of $1+\lambda$. Consequently, $(I+A)^{-1}I$ has an eigenvalue of $1/(1+\lambda)$. Therefore, it can be concluded that $1/(1+\lambda)$ is always smaller than 1, which implies that Eq. (16) is unconditionally stable. □

2.4 Half-Sweep Newton-Gauss-Seidel Algorithm

Next, to develop the half-sweep Newton-Gauss-Seidel (hs-NGS) algorithm for solving the two-dimensional porous medium-type equation, let considers a type of matrix decomposition as follows,

$$J_{n+1} = T_{n+1} + L_{n+1} + V_{n+1}, \tag{25}$$

where $T_{n+1} = (T_1, T_2, \dots, T_{M-2}, T_{M-1})_{n+1}$, $L_{n+1} = (L_2, \dots, L_{M-1})_{n+1}$ and $V_{n+1} = (V_1, \dots, V_{M-2})_{n+1}$.

Hence, using Eq. (25) and substituting it into Eq. (18), the iterative formula of hs-NGS can be expressed as

$$\widehat{H}_{n+1}^{(l+1)} = (T_{n+1} + L_{n+1})^{-1}\left(-V_{n+1}\widehat{H}_{n+1}^{(l)} - \widehat{F}_{n+1}\right), \tag{26}$$

and the algorithm is provided as follows,

(i) Define $u(x,y,0), u(0,y,t), u(1,y,t), u(x,0,t)$, and $u(x,1,t)$.

(ii) For $1 \le n \le N$, set $\widehat{U}_{n+1}^{(0)} = 1.000$ and $\widehat{H}_{n+1}^{(0)} = 0$.

(iii) For $l = 1,2, \dots$, compute Eq. (26).

(iv) Convergence test $\left|\widehat{H}_{n+1}^{(l+1)} - \widehat{H}_{n+1}^{(l)}\right| \le 10^{-10}$.

(v) For $l = 1,2, \dots$, compute Eq. (20).

(vi) Check the convergence $\left|\widehat{U}_{n+1}^{(l+1)} - \widehat{U}_{n+1}^{(l)}\right| \le 10^{-10}$.

(vii) Display outputs.

3 RESULTS AND DISCUSSION

This section evaluates the efficacy of the half-sweep FD approximation to the porous medium-type differential equation based on the number of iterations, computer time and absolute errors. The three mentioned criteria are analysed using different spatial and temporal step sizes. Several experiments are presented to show the results of the research. Three porous medium-type natural science problems are used in the experiment. Below are the proposed problems used in the experiment.

3.1 Proposed Problem 1

Consider the following IBVP of porous medium type:

$$\frac{\partial u}{\partial t}=\frac{1}{5}\left[\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial y}\right)\right], \tag{27}$$

subjects to the conditions, $u(x,y,0)=x+y$, $u(0,y,t)=y+0.4t$, $u(1,y,t)=1+y+0.4t$, $u(x,0,t)=x+0.4t$ and $u(x,1,t)=x+1+0.4t$. The exact solution is given by $u(x,y,t)=x+y+0.4t$. Eq. (27) is important to simulate a two-dimensional unsteady flow of groundwater that involves the presence of a free surface. The solutions to be computed determine the groundwater pressure in the porous medium over a certain period [23].

3.2 Proposed Problem 2

Consider an IBVP of the porous medium type that is given by:

$$\frac{\partial u}{\partial t}=\frac{1}{5}\left[\frac{\partial}{\partial x}\left(u^2\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(u^2\frac{\partial u}{\partial y}\right)\right], \tag{28}$$

with the initial $u(x,y,0)=\sqrt{5x+5y}$ and the boundaries $u(0,y,t)=\sqrt{5y+5t}$, $u(1,y,t)=\sqrt{5+5y+5t}$, $u(x,0,t)=\sqrt{5x+5t}$ and $u(x,1,t)=\sqrt{5x+5+5t}$. The exact solution is $u(x,y,t)=\sqrt{5x+5y+5t}$. Eq. (28) can be used to model a slow diffusion process such as two-dimensional melting metals [27].

3.3 Proposed Problem 3

Let the IBVP of porous medium type be expressed as:

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u^5\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(u^5\frac{\partial u}{\partial y}\right), \tag{29}$$

subjects to the initial value $u(x,y,0)=\sqrt[4]{0.8x+0.8y}$. The solutions are based on boundary conditions of $u(0,y,t)=\sqrt[4]{0.8y+1.6t}$, $u(1,y,t)=\sqrt[4]{0.8+0.8y+1.6t}$, $u(x,0,t)=\sqrt[4]{0.8x+1.6t}$ and $u(x,1,t)=\sqrt[4]{0.8x+0.8+1.6t}$. The exact solution is $u(x,y,t)=\sqrt[4]{0.8x+0.8y+1.6t}$ [28].

The numerical outputs are compared against the implicit finite difference scheme with NGS iterative formula. The efficiency of the hs-NGS iterative method is studied based on the iteration numbers and computer time in seconds against several sizes of matrices. The comparison results for each IBVP are shown in Tables 1 to 3.

Table 1 : Number of iterations and computer time required to solve Problem 1

	Iterations		Time (seconds)	
Sizes of matrices	**NGS**	**hs-NGS**	**NGS**	**hs-NGS**
16×16	136	80	0.76	0.40
32×32	436	245	2.93	2.68
64×64	1525	829	19.59	9.12
128×128	5462	2901	360.89	201.13
256×256	19,404	10,313	4586.69	2958.93

Table 2 : Number of iterations and computer time required to solve Problem 2

	Iterations		Time (seconds)	
Sizes of matrices	**NGS**	**hs-NGS**	**NGS**	**hs-NGS**
16×16	130	77	0.97	0.55
32×32	400	221	2.70	1.90
64×64	1380	738	18.26	8.56
128×128	4901	2593	248.82	183.43
256×256	17,458	9243	4243.79	1965.08

Table 3 : Number of iterations and computer time required to solve Problem 3

	Iterations		Time (seconds)	
Sizes of matrices	**NGS**	**hs-NGS**	**NGS**	**hs-NGS**
16×16	739	405	0.98	0.85
32×32	2630	1402	9.51	8.39
64×64	9478	5005	113.63	63.64
128×128	34,098	18,002	1653.85	530.82
256×256	121,649	64,469	29,234.80	13,038.50

Based on the results shown in Tables 1 to 3, it can be observed that the number of iterations and the computer time required by the hs-NGS method to solve the selected problems are lesser and shorter than the implicit NGS method. It can be said that the half-sweep FD approximation is superior to the implicit FD approximation in terms of computing efficiency. Besides that, the numerical convergence test of the solutions obtained using the hs-NGS method to solve the problems is carried out using spatial steps, $h = 1/16$, $1/32$, $1/64$, $1/128$ and $1/256$, and temporal steps, $k = 0.01$, 0.001, and 0.0001, see Figures 1 until 3.

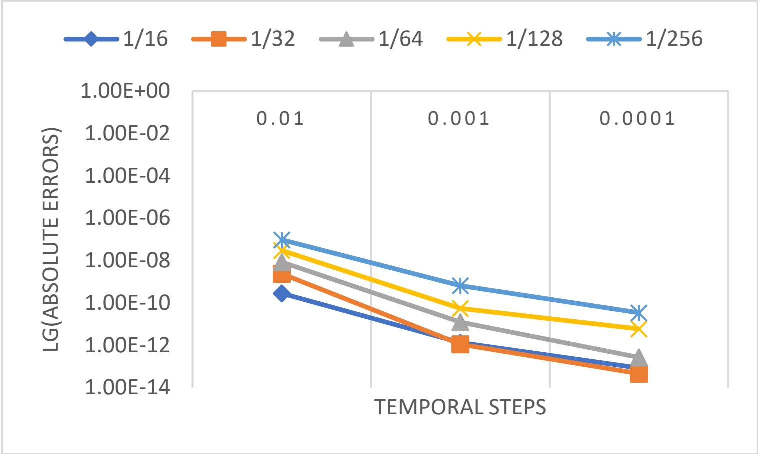

Figure 1 : Numerical convergence of hs-NGS method to solve Problem 1

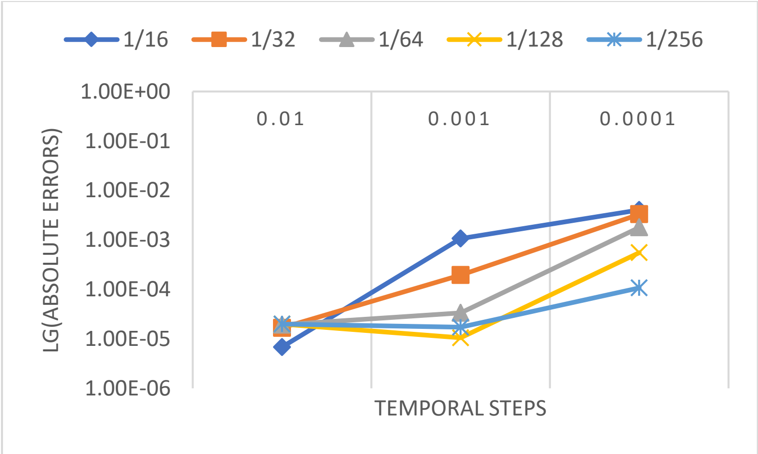

Figure 2 : Numerical convergence of hs-NGS method to solve Problem 2

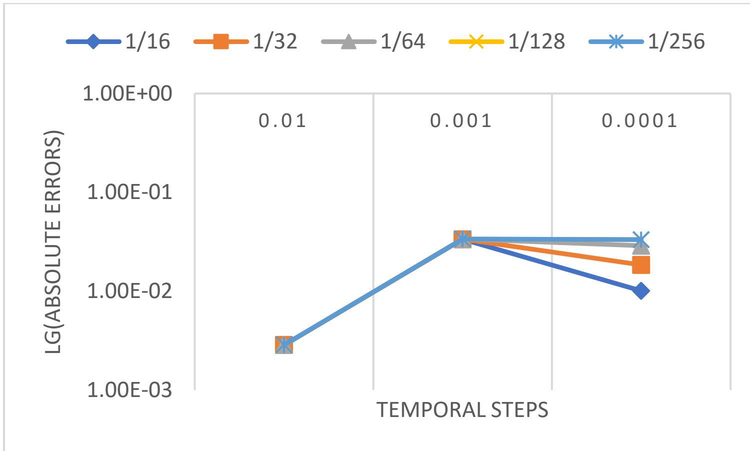

Figure 3 : Numerical convergence of hs-NGS method to solve Problem 3

Based on Figure 1, the absolute errors decrease as the temporal step decreases, although absolute errors grow slightly as the spatial step decrease. This result indicates the hs-NGS method can solve Problem 1 accurately in smaller temporal steps. Then, as in Figures 2 and 3, absolute errors decrease as the spatial step decreases, but the errors grow when the temporal step decrease. These errors behaviour show that the hs-NGS method can accurately obtain the solution to Problems 2 and 3 using smaller spatial steps.

Based on the numerical results of solving nonlinear porous medium-type equations using the hs-NGS, this study finds that the half-sweep FD can reduce the complexity of numerical computation of a large-sized system of equations even after linearising the nonlinear system. The findings of this study are compared against the findings from Sunarto *et al.* [9], Xu *et al.* [10], Xu *et al.* [11], Sunarto *et al.* [12], Agarwal *et al.* [13] and Xu *et al.* [14], and all agree that the half-sweep FD is more computationally efficient the implicit FD regardless the use of linear solvers. In addition, this study offers some information about the half-sweep FD schemes for solving two-dimensional nonlinear problems, including the derivations and stability analysis. We believe that the presented derivations, analysis, and results can be the benchmark for the researchers to explore the efficient solution for nonlinear integro-differential and fractional differential equations.

4 CONCLUSION

In conclusion, the efficiency of the half-sweep FD approximation to solve the two-dimensional nonlinear porous medium-type differential equations is better than the implicit FD approximation. The presented efficiency results agree with the literature that the iteration number and computer time required to solve the large-sized system of equations can be reduced using the half-sweep FD method. The numerical convergence tests using several IBVPs of porous medium-type differential equations are provided to illustrate the absolute error behaviours using different spatial and temporal steps. The application of a half-sweep FD to solve a more variety of NPDE, including the more realistic mathematical problems, will be investigated in future.

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