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Implicit Block and Runge- Kutta type Methods for Solution of Second-Order Ordinary Differential Equations

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ABSTRACT

In this research, implicit discrete schemes which form our block integrators were developed for solving Initial Valued Problems of Ordinary Differential Equations of the form $y'' = f(x, y, y')$. The equivalent second order Runge-Kutta type Methods (RKTM) were also constructed for the same purpose. Both methods were demonstrated on linear and nonlinear problems of Ordinary Differential Equations. Numerical results obtained from RKTM show that the method is competitive with the existing one.

Keywords: Block implicit methods, Implicit Runge-Kutta type method, Linear and nonlinear Ordinary Differential Equations.

1 INTRODUCTION

The general second-order ordinary differential equations (ODEs) is of the form

$$y'' = f(x, y, y') \ , \qquad y(x_0) = y_0 \, , \quad y'(x_0) = y'_0 \tag{1}$$

Several real-life and physical phenomenon could be modeled to nonlinear second-order ODEs, some of which do not possesses analytical solutions. Hence, numerical methods are implored in getting their approximate solutions. These numerical approaches often fall into two different categories; Linear Multistep Methods (LMM) and Runge-Kutta Type Methods (RKTM). Both methods have been widely used to approximate the solutions of higher order be over emphasized. ODEs see [1] and [2]. Many researchers have worked in this area such as [3], [4] and [5] to mention a few. In a similar vein, [6], [7] and [8] developed methods of solving higher or cannot brought predictor-corrector approach, [9] indicated that this method is not self-starting and tediousness in developing corrector schemes of the same order cannot be over emphasized.

Definition 1 Linear Multi-step method (LMM)

A LMM of second order ODEs with k-step size have the form

$$\sum_{j=1}^{m+t-1} \alpha_j y_{n+j} = \ h^2 \sum_{j=0}^{m+t-1} \beta_j f_{n+j} \quad j = 0,1,..k \tag{2}$$

where α_j and β_j are constants, t and m are points of interpolation and collocations.

Equation (2) becomes implicit scheme if $\beta_k \neq 0$, otherwise explicit [9].

Definition 2 Order and Error Constant

A LMM of (2) associated with a linear differential operator of the

$$L[y(x);h] = \sum_{j=0}^{k} \alpha_j y(x+jh) = h^2 \beta_j y''(x+jh) \tag{3}$$

Expanding (3) in Taylor's series expansion about the point x and collecting like terms as

$$L[y(x);h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \cdots + C_q h^q y^q \tag{4}$$

where

$$C_0 = \sum_{j=0}^{q} \alpha_j \ y(x), \qquad C_1 = \sum_{j=1}^{q} j\alpha_j \, h \, y'(x), \qquad C_2 = \left(\sum_{j=1}^{q} \frac{j^2 \alpha_j}{2!} - \sum_{j=0}^{q} \beta_j \right) h^2 y''(x) \,,$$

$$C_3 = \left(\sum_{j=1}^{q} \frac{j^3 \alpha_j}{3!} - \sum_{j=1}^{q} j\beta_j \right) h^3 y'''(x) \ldots \quad , \qquad C_q = \left(\sum_{j=1}^{q} \frac{j^q \alpha_j}{q!} - \sum_{j=1}^{q} \frac{j^{(q-2)} \beta_j}{(q-2)!} \right) h^q y^q(x)$$

$\forall \ q = 2,3..$

The method is of order P if $C_0 = C_1 = C_2 = \cdots = C_p = C_{p+1} = 0$ but $C_{p+2} \neq$ and $C_{p+2} h^{p+2} y^{p+2}$ is called error constant.

Definition 3 Zero Stability

The LMM (2) is said to satisfy the root conditions if all the roots of the first characteristics polynomial have modulus less than or equal to unity and those of modulus unity are simple [10].

2 RUNGE-KUTTA METHOD FOR SECOND-ORDER ODES

An S-stage R-K methods for direct integration of general-second order ODEs is of the form

$$y_{n+1} = y_n + \alpha_i h y'_n + h^2 \sum_{j=1}^{s} a_{ij} k_j \tag{5}$$

$$y'_{n+1} = y'_n + h \sum_{j=1}^{s} \overline{a}_{ij} k_j \qquad , \quad i = 1, \dots, s$$

$$k_i = f\left(x_n + \alpha_j h, y_n + \alpha_i y'_n + h^2 \sum_{j=1}^{s} a_{ij} k_j,\ y'_n + h \sum_{j=1}^{s} \overline{\alpha}_{ij} k_j \right)$$

where $\alpha_j, k_i, a_{ij}, \overline{\alpha}_{ij}$ defined the method above and its Butcher array [11] is of the form

$$\begin{array}{c|c|c} \alpha & \overline{A} & A \\ \hline \overline{b}^T & b & \end{array} \tag{6}$$

$A = a_{ij} = \beta^2,\ \ \overline{A} = \overline{a_{ij}} = \beta, \quad \beta = \beta e,\ \ \overline{b} = W^T, \quad b = W^T\beta$. This method is consistent if summation $b_i = \frac{1}{2}$ for $i =, \dots, s$.

3 SPECIFICATION OF THE NEW LMM METHOD

Given a series of the form

$$y(z) = \sum_{j=0}^{t+m-1} \alpha_j(z) P_n(z) = y_{n+j} \tag{7}$$

$$y'(z) = \sum_{j=1}^{t+m-1} \alpha_j(z) P'_n(z) = f_{n+j} \tag{8}$$

$$y''(z) = \sum_{j=2}^{t+m-1} \alpha_j(z) P''_n(z) = g_{n+j} \tag{9}$$

Also t and m are interpolation and collocation points respectively, $(t + m - 1)$ is the degree of polynomials, $\alpha_j(z)$ is the unknown polynomial functions to be determined and $P_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} [(z^2 - 1)^n]$, which is a Legendre Polynomial basis with z a positive integer taken from the range $P_0(z)$ to $P_6(z)$ as

$P_0(z) = 1$

$P_1(z) = z$

$P_2(z) = \frac{1}{2}(3z^2 - 1)$

$P_3(z) = \frac{1}{2}(5z^3 - 3z)$

$P_4(z) = \frac{1}{8}(35z^4 - 30z^2 + 3)$

$P_5(z) = \frac{1}{8}(63z^5 - 70z^3 + 15z)$

$P_6(z) = \frac{1}{16}(231z^6 - 315z^4 + 105z^2 - 5)$

. .
..

4 DERIVATION OF IMPLICIT LINEAR MULTISTEP METHOD FOR FIRST-ORDER ODES

Interpolate (7) at $z = z_{n+j}$, $j = (0,1)$ and collocate (8) at $j = (1, \frac{3}{2}, 2, \frac{5}{2}, 3)$ to form the D-matrix as

$$D = \begin{bmatrix} 1 & z_n & \cdots & \frac{1}{16}(231(z_n)^6 - 315(z_n)^4 + 105(z_n)^2 - 5) \\ 1 & z_{n+1} & \cdots & \frac{1}{16}(231(z_{n+1})^6 - 315(z_{n+1})^4 + 105(z_{n+1})^2 - 5) \\ 0 & 1 & \cdots & \frac{1}{16}1386(z_{n+1})^5 - 1260(z_{n+1})^3 + 210(z_{n+1}) \\ 0 & 1 & \cdots & \frac{1}{16}1386(z_{n+\frac{3}{2}})^5 - 1260\left(z_{n+\frac{3}{2}}\right)^3 + 210(z_{n+\frac{3}{2}}) \\ 0 & 1 & \cdots & \frac{1}{16}1386(z_{n+2})^5 - 1260(z_{n+2})^3 + 210(z_{n+2}) \\ 0 & 1 & \cdots & \frac{1}{16}1386(z_{n+\frac{5}{2}})^5 - 1260\left(z_{n+\frac{5}{2}}\right)^3 + 210(z_{n+\frac{5}{2}}) \\ 0 & 1 & \cdots & \frac{1}{16}1386(z_{n+3})^5 - 1260(z_{n+3})^3 + 210(z_{n+3}) \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \beta_1 \\ \beta_{\frac{3}{2}} \\ \beta_2 \\ \beta_{\frac{5}{2}} \\ \beta_3 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+1} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \end{bmatrix} \quad (10)$$

The inverse of (10) is obtained using Mathematical software called Maple. Each of the rows of the inverse matrix is multiplied by the column vector $[y_n, y_{n+1}, f_{n+1}, f_{n+\frac{3}{2}}, f_{n+2}, f_{n+\frac{5}{2}}, f_{n+3}]^T$ to obtain

$$\alpha_0 = \sum_{i=1}^{6} C_{1i}[L_k(Z)]^T \ , \alpha_1 = \sum_{i=1}^{6} C_{2i}[L_k(Z)]^T \ , \alpha_2 = \sum_{i=1}^{6} C_{3i}[L_k(Z)]^T \ , \alpha_3 = \sum_{i=1}^{6} C_{4i}[L_k(Z)]^T \ ,$$

$$\alpha_4 = \sum_{i=1}^{6} C_{5i}[L_k(Z)]^T , \alpha_5 = \sum_{i=1}^{6} C_{6i}[L_k(Z)]^T , \alpha_6 = \sum_{i=1}^{6} C_{7i}[L_k(Z)]^T$$

where $[L_k(Z)]^T = [y_n, y_{n+1}, f_{n+1}, f_{n+\frac{3}{2}}, f_{n+2}, f_{n+\frac{5}{2}}, f_{n+3}]^T$ and C_{ij} values are the coefficient of inverse matrix. The values of the parameters $\alpha_0, \alpha_1, \beta_1, \beta_{\frac{3}{2}}, \beta_2, \beta_{\frac{5}{2}}, \beta_3$ were substituted into (7) after algebraic manipulations which gives the continuous formulation as

$$y(z) = \alpha_0 y_n + \alpha_1 y_{n+1} = h\left[\beta_0 f_{n+1} + \beta_{\frac{3}{2}} f_{n+\frac{3}{2}} + \beta_2 f_{n+2} + \beta_{\frac{5}{2}} f_{n+\frac{5}{2}} + \beta_3 f_{n+3}\right] \tag{11}$$

Specifically, we have

$$y(z) = y_n\left(\frac{1}{2079}\frac{2079h^6 + 3654h^4 + 651h^2 + 8}{h^6} - \frac{4}{693}\frac{(630h^5 + 406h^3 + 24h)z}{h^6}\right.$$

$$+ \frac{4}{6237}\frac{(5481h^4 + 1395h^2 + 20)\left(\frac{3}{2}z^2 - \frac{1}{2}\right)}{h^6} - \frac{8(174h^3 + 16h)\left(\frac{5}{2}z^3 - \frac{3}{2}z\right)}{891h^6}$$

$$+ \frac{8}{7623}\frac{(341h^2 + 8)\left(\frac{35}{8}z^4 - \frac{15}{4}z^2 + \frac{3}{8}\right)}{h^6} - \frac{256}{6237}\frac{\left(\frac{63}{8}z^5 - \frac{35}{4}z^3 + \frac{15}{8}z\right)}{h^5}$$

$$\left. + \frac{128}{68607}\frac{\left(\frac{231}{16}z^6 - \frac{315}{16}z^4 + \frac{105}{16}z^2 - \frac{5}{16}\right)}{h^6}\right)$$

$$+ y_{n+1}\left(-\frac{3654h^4 + 651h^2 + 8}{2079h^6} + \frac{4(630h^5 + 406h^3 + 24h)z}{693h^6}\right.$$

$$- \frac{4(5481h^4 + 1395h^2 + 20)\left(\frac{3}{2}z^2 - \frac{1}{2}\right)}{6237h^6} + \frac{8(174h^3 + 16h)\left(\frac{5}{2}z^3 - \frac{3}{2}z\right)}{891h^6}$$

$$- \frac{8(341h^2 + 8)\left(\frac{35}{8}z^4 - \frac{15}{4}z^2 + \frac{3}{8}\right)}{7623h^6} + \frac{256\left(\frac{63}{8}z^5 - \frac{35}{4}z^3 + \frac{15}{8}z\right)}{6237h^5}$$

$$\left. - \frac{128\left(\frac{231}{16}z^6 - \frac{315}{16}z^4 + \frac{105}{16}z^2 - \frac{5}{16}\right)}{68607h^6}\right) + \left(\frac{2165121h^4 + 590163h^2 + 8632}{374220h^5}\right.$$

$$-\frac{1}{62370}\frac{(423990h^5+628747h^3+48228h)z}{h^5}+\frac{(6495363h^4+2529270h^2+43160)\left(\frac{3}{2}z^2-\frac{1}{2}\right)}{561330h^5}$$

$$-\frac{1}{40095}\frac{(269463h^3+32152h)\left(\frac{5}{2}z^3-\frac{3}{2}z\right)}{h^5}+\frac{2}{343035}\frac{(309133h^2+8632)\left(\frac{35}{8}z^4-\frac{15}{4}z^2+\frac{3}{8}\right)}{h^5}$$

$$\left.-\frac{1}{280665}\frac{64304\left(\frac{63}{8}z^5-\frac{35}{4}z^3+\frac{15}{8}z\right)}{h^4}+\frac{34528\left(\frac{231}{16}z^6-\frac{315}{16}z^4+\frac{105}{16}z^2-\frac{5}{16}\right)}{3087315h^5}\right)f_{n+1}$$

$$+f_{n+2}\left(\frac{213318h^4+88116h^2+1696}{31185h^5}-\frac{(66465h^5+155099h^3+16788h)z}{10395h^5}\right.$$

$$+\frac{2(639954h^4+377640h^2+8480)\left(\frac{3}{2}z^2-\frac{1}{2}\right)}{93555h^5}-\frac{1}{13365}\frac{2(66471h^3+11192h)\left(\frac{5}{2}z^3-\frac{3}{2}z\right)}{h^5}$$

$$+\frac{1}{114345}\frac{32(11539h^2+424)\left(\frac{35}{8}z^4-\frac{15}{4}z^2+\frac{3}{8}\right)}{h^5}-\frac{1}{93555}\frac{44768\left(\frac{63}{8}z^5-\frac{35}{4}z^3+\frac{15}{8}z\right)}{h^4}$$

$$\left.+\frac{27136\left(\frac{231}{16}z^6-\frac{315}{16}z^4+\frac{105}{16}z^2-\frac{5}{16}\right)}{1029105h^5}\right)$$

$$+f_{n+3}\left(\frac{1}{374220}\frac{182511h^4+87801h^2+2152}{h^5}-\frac{1}{62370}\frac{(27090h^5+70819h^3+9348h)z}{h^5}+\right.$$

$$\frac{1}{561330}\frac{(547533h^4+376290h^2+10760)\left(\frac{3}{2}z^2-\frac{1}{2}\right)}{h^5}-\frac{(30351h^3+6232h)\left(\frac{5}{2}z^3-\frac{3}{2}z\right)}{40095h^5}+$$

$$\frac{2}{343035}\frac{(45991h^2+2152)\left(\frac{35}{8}z^4-\frac{15}{4}z^2+\frac{3}{8}\right)}{h^5}-\frac{12464\left(\frac{63}{8}z^5-\frac{35}{4}z^3+\frac{15}{8}z\right)}{280665h^4}$$

$$\left.+\frac{8608\left(\frac{231}{16}z^6-\frac{315}{16}z^4+\frac{105}{16}z^2-\frac{5}{16}\right)}{3087315h^5}\right)+\left(\frac{-8(99477h^4+35490h^2+599)}{93555h^5}\right.$$

$$+\frac{134336}{280665}\frac{\left(\frac{63}{8}z^5-\frac{35}{4}z^3+\frac{15}{8}z\right)}{h^4}+\frac{8(32760h^5+67543h^3+6297h)z}{31185h^5}-$$

$$\frac{16(298431h^4+152100h^2+2995)\left(\frac{3}{2}z^2-\frac{1}{2}\right)}{280665h^5}+\frac{16}{40095}\frac{(28947h^3+4198h)\left(\frac{5}{2}z^3-\frac{3}{2}z\right)}{h^5}-$$

$$\frac{64}{343035}\frac{(18590h^2+599)\left(\frac{35}{8}z^4-\frac{15}{4}z^2+\frac{3}{8}\right)}{h^5}-\frac{76672}{3087315}\frac{\left(\frac{231}{16}z^6-\frac{315}{16}z^4+\frac{105}{16}z^2-\frac{5}{16}\right)}{h^5}\Bigg)f_{n+\frac{3}{2}}$$

$$+\Bigg(-\frac{8}{13365}\frac{4761h^4+2154h^2+47}{h^5}+\frac{8}{31185}\frac{(10080h^5+25207h^3+3057h)z}{h^5}-$$

$$\frac{16}{280665}\frac{(99981h^4+64620h^2+1645)\left(\frac{3}{2}z^2-\frac{1}{2}\right)}{h^5}+\frac{16(10803h^3+2038h)\left(\frac{5}{2}z^3-\frac{3}{2}z\right)}{40095h^5}-$$

$$\frac{64(7898h^2+329)\left(\frac{35}{8}z^4-\frac{15}{4}z^2+\frac{3}{8}\right)}{343035h^5}+\frac{65216\left(\frac{63}{8}z^5-\frac{35}{4}z^3+\frac{15}{8}z\right)}{280665h^4}$$

$$-\frac{6016\left(\frac{231}{16}z^6-\frac{315}{16}z^4+\frac{105}{16}z^2-\frac{5}{16}\right)}{441045h^5}\Bigg)f_{n+\frac{5}{2}} \tag{12}$$

Equation (12) is evaluated at $z = z_{n+\frac{3}{2}}, z_{n+2}, z_{n+\frac{5}{2}},\ z_{n+3}$ and its first derivative evaluated at $z = z_{n+1}$ which produces five hybrid discrete schemes as follows

$$y_{n+\frac{3}{2}}-\frac{175}{176}y_{n+1}-\frac{1}{176}y_n=\frac{6601}{31680}hf_{n+1}-\frac{17}{165}hf_{n+2}-\frac{149}{31680}hf_{n+3}+\frac{1477}{3960}hf_{n+\frac{3}{2}}+\frac{127}{3960}hf_{n+\frac{5}{2}}$$

$$y_{n+2}-\frac{296}{297}y_{n+1}-\frac{1}{297}y_n=\frac{2423}{13365}hf_{n+1}+\frac{806}{4455}hf_{n+2}-\frac{7}{13365}hf_{n+3}+\frac{8608}{13365}hf_{n+\frac{3}{2}}-\frac{32}{13365}hf_{n+\frac{5}{2}}$$

$$y_{n+\frac{5}{2}}-\frac{175}{176}y_{n+1}-\frac{1}{176}y_n=\frac{1285}{6336}hf_{n+1}+\frac{35}{66}hf_{n+2}-\frac{65}{6336}hf_{n+3}+\frac{445}{792}hf_{n+\frac{3}{2}}+\frac{175}{792}hf_{n+\frac{5}{2}}$$

$$y_{n+3}-y_{n+1}=\frac{7}{45}hf_{n+1}+\frac{4}{15}hf_{n+2}+\frac{7}{45}hf_{n+3}+\frac{32}{45}hf_{n+\frac{3}{2}}+\frac{32}{45}hf_{n+\frac{5}{2}}$$

$$-360y_{n+1}+360y_n=-99hf_n-673hf_{n+1}-633hf_{n+2}-43hf_{n+3}+832hf_{n+\frac{3}{2}}+256hf_{n+\frac{5}{2}} \tag{13}$$

4.1 Derivation of Implicit Linear Multistep Method for Second-Order ODEs

Here, we interpolate (2.0) at $z = z_{n+j}$, $j = (0,1)$ and collocate (2.2) at $j = (1,\frac{3}{2},2,\frac{5}{2},3)$ to form the D-matrix of this method. The continuous formulation of this method gives

$$y(z)=\alpha_0 y_n+\alpha_1 y_{n+1}=h^2\left[\beta_1 g_{n+1}+\beta_{\frac{3}{2}}g_{n+\frac{3}{2}}+\beta_2 g_{n+2}+\beta_{\frac{5}{2}}g_{n+\frac{5}{2}}+\beta_3 g_{n+3}\right] \tag{14}$$

which can be expressed as

$$y(z)=y_n\left(1-\frac{z}{h}\right)+\left(\frac{y_{n+1}}{h}\right)z+\Bigg(\frac{(6300h^4+833h^2+8)}{2520h^4}-\frac{(10395h^5+7182h^3+324h)z}{2520h^4}+$$

$$\frac{(1890h^4+357h^2+4)}{378h^4}\left(\frac{3}{2}z^2-\frac{1}{2}\right)-\frac{1}{270}\frac{(513h^3+36h)}{h^4}\left(\frac{5}{2}z^3-\frac{3}{2}z\right)+\frac{1}{3465}\frac{(1309h^2+24)}{h^4}$$
$$\left(\frac{35}{8}z^4-\frac{15}{4}z^2+\frac{3}{8}\right)-\frac{4}{105}\frac{1}{h^3}\left(\frac{63}{8}z^5-\frac{35}{4}z^3+\frac{15}{8}z\right)+\frac{16}{10395}\frac{1}{h^4}$$
$$\left(\frac{231}{16}z^6-\frac{315}{16}z^4+\frac{105}{16}z^2-\frac{5}{16}\right)\Bigg)g_{n+1}\quad+\Bigg(\frac{1}{420}\frac{(3150h^4+637h^2+8)}{h^4}-\frac{1}{420}\frac{(4459h^5+4536h^3+288h)z}{h^4}+$$
$$\frac{1}{63}\frac{(945h^4+273h^2+4)}{h^4}\left(\frac{3}{2}z^2-\frac{1}{2}\right)-\frac{2}{45}\frac{(162h^3+16h)}{h^4}\left(\frac{5}{2}z^3-\frac{3}{2}z\right)+\frac{2}{1155}\frac{(1001h^2+24)}{h^4}\left(\frac{35}{8}z^4-\right.$$
$$\left.\frac{15}{4}z^2+\frac{3}{8}\right)-\frac{64}{315}\frac{1}{h^3}\left(\frac{63}{8}z^5-\frac{35}{4}z^3+\frac{15}{8}z\right)+\frac{32}{3465}\frac{1}{h^4}\left(\frac{231}{16}z^6-\frac{315}{16}z^4+\frac{105}{16}z^2-\right.$$
$$\left.\left.\frac{5}{16}\right)\right)g_{n+2}\quad+\Bigg(\frac{1}{2520}\frac{(2100h^4+497h^2+8)}{h^4}-\frac{(2863h^5+3234h^3+252h)z}{2520h^4}+$$
$$\frac{(630h^4+213h^2+4)}{378h^4}\left(\frac{3}{2}z^2-\frac{1}{2}\right)-\frac{(231h^3+28h)}{270h^4}\left(\frac{5}{2}z^3-\frac{3}{2}z\right)+\frac{1}{3465}\frac{(781h^2+24)}{h^4}\left(\frac{35}{8}z^4-\right.$$
$$\left.\frac{15}{4}z^2+\frac{3}{8}\right)-\frac{4}{135}\frac{1}{h^3}\left(\frac{63}{8}z^5-\frac{35}{4}z^3+\frac{15}{8}z\right)+\frac{16}{10395}\frac{1}{h^4}\left(\frac{231}{16}z^6-\frac{315}{16}z^4+\frac{105}{16}z^2-\right.$$
$$\left.\left.\frac{5}{16}\right)\right)g_{n+3}+\Bigg(\frac{-(2100h^4+364h^2+4)}{315h^4}+\frac{1}{315}\frac{(3101h^5+2814h^3+153h)z}{h^4}-$$
$$\frac{4}{189}\frac{(630h^4+156h^2+2)}{h^4}\left(\frac{3}{2}z^2-\frac{1}{2}\right)+\frac{4}{135}\frac{(201h^3+17h)}{h^4}\left(\frac{5}{2}z^3-\frac{3}{2}z\right)-$$
$$\frac{8}{3465}\frac{(572h^2+12)}{h^4}\left(\frac{35}{8}z^4-\frac{15}{4}z^2+\frac{3}{8}\right)+\frac{136}{945}\frac{1}{h^3}\left(\frac{63}{8}z^5-\frac{35}{4}z^3+\frac{15}{8}z\right)-$$
$$\frac{64}{10395}\frac{1}{h^4}\left(\frac{231}{16}z^6-\frac{315}{16}z^4+\frac{105}{16}z^2-\frac{5}{16}\right)\Bigg)g_{n+\frac{3}{2}}+\Bigg(-\frac{1}{315}\frac{(1260h^4+280h^2+4)}{h^4}+$$
$$\frac{4}{315}\frac{(1743h^5+1890h^3+135h)z}{h^4}-\frac{4}{189}\frac{(378h^4+120h^2+2)}{h^4}\left(\frac{3}{2}z^2-\frac{1}{2}\right)+\frac{4}{135}\frac{(135h^3+15h)}{h^4}\left(\frac{5}{2}z^3-\right.$$
$$\left.\frac{3}{2}z\right)-\frac{8}{3465}\frac{(440h^2+12)}{h^4}\left(\frac{35}{8}z^4-\frac{15}{4}z^2+\frac{3}{8}\right)+\frac{8}{63}\frac{1}{h^3}\left(\frac{63}{8}z^5-\frac{35}{4}z^3+\frac{15}{8}z\right)-$$
$$\frac{64}{10395}\frac{1}{h^4}\left(\frac{231}{16}z^6-\frac{315}{16}z^4+\frac{105}{16}z^2-\frac{5}{16}\right)\Bigg)g_{n+\frac{5}{2}}$$

(15)

Evaluating (15) at points $z=(z_{n+\frac{3}{2}},z_{n+2},z_{n+\frac{5}{2}},\ z_{n+3})$ and its first derivative at $z=(z_n)$ gives a Block hybrid scheme of the form

$$y_{n+3}-3y_{n+1}+2y_n=\frac{81}{20}h^2g_{n+1}+\frac{73}{10}h^2g_{n+2}+\frac{43}{60}h^2g_{n+3}-\frac{88}{15}h^2g_{n+\frac{3}{2}}-\frac{16}{5}h^2g_{n+\frac{5}{2}}$$

$$y_{n+\frac{3}{2}}-\frac{3}{2}y_{n+1}+\frac{1}{2}y_n=\frac{639}{640}h^2g_{n+1}+\frac{547}{320}h^2g_{n+2}+\frac{337}{1920}h^2g_{n+3}-\frac{787}{480}h^2g_{n+\frac{3}{2}}-\frac{139}{160}h^2g_{n+\frac{5}{2}}$$

$$y_{n+2}-2y_{n+1}+y_n=\frac{121}{60}h^2g_{n+1}+\frac{103}{30}h^2g_{n+2}+\frac{7}{20}h^2g_{n+3}-\frac{46}{15}h^2g_{n+\frac{3}{2}}-\frac{26}{15}h^2g_{n+\frac{5}{2}}$$

$$y_{n+\frac{5}{2}}-\frac{5}{2}y_{n+1}+\frac{3}{2}y_n=\frac{1165}{384}h^2g_{n+1}+\frac{343}{64}h^2g_{n+2}+\frac{67}{128}h^2g_{n+3}-\frac{143}{32}h^2g_{n+\frac{3}{2}}-\frac{247}{96}h^2g_{n+\frac{5}{2}} \tag{16}$$

Also, evaluating the first derivative of (15) at points $z=(z_n, z_{n+1}, z_{n+\frac{3}{2}}, z_{n+2}, z_{n+\frac{5}{2}}, z_{n+3})$ gives the other six derivative discrete schemes as

$$-360y'_n=1485h^2g_{n+1}+3822h^2g_{n+2}+409h^2g_{n+3}-3544h^2g_{n+\frac{3}{2}}-1992h^2g_{n+\frac{5}{2}}$$
$$+360y_n-360y_{n+1}$$

$$360y'_{n+1}=673h^2g_{n+1}+1266h^2g_{n+2}+129h^2g_{n+3}-1248h^2g_{n+\frac{3}{2}}-640h^2g_{n+\frac{5}{2}}-360y_n+360y_{n+1}$$

$$1440y'_{n+\frac{3}{2}}=2943h^2g_{n+1}+4800h^2g_{n+2}+497h^2g_{n+3}-4346h^2g_{n+\frac{3}{2}}-2454h^2g_{n+\frac{5}{2}}-1440y_n+1440y_{n+1}$$

$$360y'_{n+2}=731h^2g_{n+1}+1314h^2g_{n+2}+127h^2g_{n+3}-1000h^2g_{n+\frac{3}{2}}-632h^2g_{n+\frac{5}{2}}-360y_n+360y_{n+1}$$

$$1440y'_{n+\frac{5}{2}}=2935h^2g_{n+1}+5712h^2g_{n+2}+489h^2g_{n+3}-4074h^2g_{n+\frac{3}{2}}-2182h^2g_{n+\frac{5}{2}}-1440y_n+1440y_{n+1}$$

$$360y'_{n+3}=729h^2g_{n+1}+1362h^2g_{n+2}+185h^2g_{n+3}-992h^2g_{n+\frac{3}{2}}-384h^2g_{n+\frac{5}{2}}-360y_n$$
$$+360y_{n+1} \tag{17}$$

The block methods of (16) and (17) are of uniform order 5 with their error constants as $\left[\frac{-11}{480},\frac{353}{61440},\frac{11}{960},\frac{211}{12288}\right]^T$ and $\left[\frac{-1609}{112},\frac{-235}{56},\frac{-7331}{448},\frac{-463}{112},\frac{-7331}{448},\frac{-235}{56}\right]^T$, hence shows that the methods are consistent since the order is>1

To illustrate the zero stability of the methods, we arrange the discrete scheme (13) in matrix equation form as follows:

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 1 \\ \frac{-175}{176} & 1 & 0 & 0 & 0 \\ \frac{-296}{297} & 0 & 1 & 0 & 0 \\ \frac{-176}{176} & 0 & 0 & 1 & 0 \\ -360 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \\ y_{n+\frac{5}{2}} \\ y_{n+3} \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{176} \\ 0 & 0 & 0 & 0 & \frac{1}{297} \\ 0 & 0 & 0 & 0 & \frac{1}{176} \\ 0 & 0 & 0 & 0 & -360 \end{pmatrix}\begin{pmatrix} y_{n-2} \\ y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{pmatrix}=$$

$$\begin{pmatrix} \frac{7}{45} & \frac{32}{45} & \frac{4}{15} & \frac{32}{45} & \frac{7}{45} \\ \frac{6601}{31680} & \frac{1477}{3960} & \frac{-17}{165} & \frac{127}{3960} & \frac{-149}{31680} \\ \frac{2423}{13365} & \frac{8608}{13365} & \frac{806}{4455} & \frac{-32}{13365} & \frac{-7}{13365} \\ \frac{1285}{6336} & \frac{445}{792} & \frac{36}{66} & \frac{175}{792} & \frac{-65}{6336} \\ -673 & 832 & -633 & 256 & -43 \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -99 \end{pmatrix} \begin{pmatrix} f_{n-2} \\ f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{pmatrix} \tag{18}$$

Let $A^0 = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 \\ \frac{-175}{176} & 1 & 0 & 0 & 0 \\ \frac{-296}{297} & 0 & 1 & 0 & 0 \\ \frac{-176}{176} & 0 & 0 & 1 & 0 \\ -360 & 0 & 0 & 0 & 0 \end{pmatrix}$ then $(A^0)^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{-1}{360} \\ 0 & 1 & 0 & 0 & \frac{-35}{12672} \\ 0 & 0 & 1 & 0 & \frac{-37}{13365} \\ 0 & 0 & 0 & 1 & \frac{-35}{12672} \\ 1 & 0 & 0 & 0 & -\frac{1}{360} \end{pmatrix}$

Multiply equation (17) through by $(A^0)^{-1}$ to obtain the following

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \\ y_{n+\frac{5}{2}} \\ y_{n+3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-2} \\ y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{pmatrix} +$$

$$\begin{pmatrix} \frac{673}{360} & \frac{-104}{45} & \frac{211}{120} & \frac{-32}{45} & \frac{43}{360} \\ \frac{1323}{640} & \frac{-77}{40} & \frac{1053}{640} & \frac{-27}{40} & \frac{73}{640} \\ \frac{92}{45} & \frac{-224}{135} & \frac{29}{15} & \frac{-32}{45} & \frac{16}{135} \\ \frac{2375}{1152} & \frac{-125}{72} & \frac{875}{384} & \frac{-35}{72} & \frac{125}{1152} \\ \frac{81}{40} & \frac{-8}{5} & \frac{81}{40} & 0 & \frac{11}{40} \end{pmatrix} \begin{pmatrix} y_{n-2} \\ y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{11}{40} \\ 0 & 0 & 0 & 0 & \frac{35}{128} \\ 0 & 0 & 0 & 0 & \frac{37}{135} \\ 0 & 0 & 0 & 0 & \frac{35}{128} \\ 0 & 0 & 0 & 0 & \frac{11}{40} \end{pmatrix} \begin{pmatrix} f_{n-2} \\ f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{pmatrix} \tag{19}$$

$$\rho(R) = de\,t \left[R \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right] = 0$$

$$\rho(R)=\left[\begin{pmatrix} R & 0 & 0 & 0 & -1 \\ 0 & R & 0 & 0 & -1 \\ 0 & 0 & R & 0 & -1 \\ 0 & 0 & 0 & R & -1 \\ 0 & 0 & 0 & 0 & R-1 \end{pmatrix}\right]=R^5-R^4=0$$

This implies $R_1 = R_2 = R_3 = R_4 = 0$ and $R_5 = 1$. This shows the method is zero stable by definition (3).

4.2 Reformulation into Runge – Kutta Nystrom type Methods

Rearranged (19) to the Butcher form of (6) for R-K method for second-order ODEs which gives

C	$\bar{A}$						A					
0	0	0	0	0	0	0	0	0	0	0	0	0
$\frac{1}{3}$	$\frac{11}{120}$	$\frac{673}{1080}$	$-\frac{104}{135}$	$\frac{211}{360}$	$-\frac{32}{135}$	$\frac{43}{1080}$	$\frac{82}{3645}$	$\frac{235}{1944}$	$-\frac{707}{3645}$	$\frac{271}{1620}$	$-\frac{91}{1215}$	$\frac{401}{29160}$
$\frac{1}{2}$	$\frac{35}{384}$	$\frac{441}{640}$	$-\frac{77}{120}$	$\frac{351}{640}$	$-\frac{9}{40}$	$\frac{73}{1920}$	$\frac{29}{768}$	$\frac{297}{1280}$	$-\frac{5}{16}$	$\frac{333}{1280}$	$-\frac{9}{80}$	$\frac{77}{3840}$
$\frac{2}{3}$	$\frac{37}{405}$	$\frac{92}{135}$	$-\frac{224}{405}$	$\frac{29}{45}$	$-\frac{32}{135}$	$\frac{16}{405}$	$\frac{193}{3645}$	$\frac{421}{1215}$	$-\frac{1496}{3645}$	$\frac{29}{81}$	$-\frac{184}{1215}$	$\frac{97}{3645}$
$\frac{5}{6}$	$\frac{35}{384}$	$\frac{2375}{3456}$	$-\frac{125}{216}$	$\frac{875}{1152}$	$-\frac{35}{216}$	$\frac{125}{3456}$	$\frac{12725}{186624}$	$\frac{28625}{62208}$	$-\frac{5875}{11664}$	$\frac{9875}{20736}$	$-\frac{725}{3888}$	$\frac{6125}{186624}$
1	$\frac{11}{120}$	$\frac{27}{40}$	$-\frac{8}{15}$	$\frac{27}{40}$	0	$\frac{11}{120}$	$\frac{1}{12}$	$\frac{23}{40}$	$-\frac{3}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	$\frac{1}{24}$
1	$\frac{11}{120}$	$\frac{27}{40}$	$-\frac{8}{15}$	$\frac{27}{40}$	0	$\frac{11}{120}$	$\frac{1}{12}$	$\frac{23}{40}$	$-\frac{3}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	$\frac{1}{24}$

(20)

The equation (20) is consistent since

$$\sum_{j=1}^{s} b_i = \frac{1}{2}\ , \qquad for\ i = 1, \ldots, s$$

Therefore, an implicit 6 Stages Block R-K Method for second-order ODEs is obtained as

$$y_{n+1} = y_n + hy'_n + h^2\left[\tfrac{1}{12}k_1 + \tfrac{23}{40}k_2 - \tfrac{3}{5}k_3 + \tfrac{3}{5}k_4 - \tfrac{1}{5}k_5 + \tfrac{1}{24}k_6\right]$$

$$y'_{n+1} = y'_n + h\left[\tfrac{11}{120}k_1 + \tfrac{27}{40}k_2 - \tfrac{8}{15}k_3 + \tfrac{27}{40}k_4 + 0k_5 + \tfrac{11}{120}k_6\right] \qquad (21)$$

where

$k_1 = f(x_n, y_n, y'_n)$

$k_2 = f[x_n + \frac{1}{3}h,\ y_n + \frac{1}{3}y'_n + h^2\left(\frac{82}{3645}k_1 + \frac{235}{1944}k_2 - \frac{707}{3645}k_3 + \frac{271}{1620}k_4 - \frac{91}{1215}k_5 + \frac{401}{29160}k_6\right),\ y'_n + h\left(\frac{11}{120}k_1 + \frac{673}{1080}k_2 - \frac{104}{135}k_3 + \frac{211}{360}k_4 - \frac{32}{135}k_5 + \frac{43}{1080}k_6\right)]$

$k_3 = f[x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}y'_n + h^2\left(\frac{29}{768}k_1 + \frac{297}{1280}k_2 - \frac{5}{16}k_3 + \frac{333}{1280}k_4 - \frac{9}{80}k_5 + \frac{77}{3840}k_6\right),\ y'_n + h\left(\frac{35}{384}k_1 + \frac{441}{640}k_2 - \frac{77}{120}k_3 + \frac{351}{640}k_4 - \frac{9}{40}k_5 + \frac{73}{1920}k_6\right)]$

$k_4 = f[x_n + \frac{2}{3}h,\ y_n + \frac{2}{3}y'_n + h^2\left(\frac{193}{3645}k_1 + \frac{421}{1215}k_2 - \frac{1496}{3645}k_3 + \frac{29}{81}k_4 - \frac{184}{1215}k_5 + \frac{97}{3645}k_6\right),\ y'_n + h\left(\frac{37}{405}k_1 + \frac{92}{135}k_2 - \frac{224}{405}k_3 + \frac{29}{45}k_4 - \frac{32}{135}k_5 + \frac{16}{405}k_6\right)]$

$k_5 = f[x_n + \frac{5}{6}h,\ y_n + \frac{5}{6}y'_n + h^2\left(\frac{12725}{186624}k_1 + \frac{28625}{62208}k_2 - \frac{5875}{11664}k_3 + \frac{9875}{20736}k_4 - \frac{725}{3888}k_5 + \frac{6125}{186624}k_6\right),\ y'_n + h\left(\frac{35}{384}k_1 + \frac{2375}{3456}k_2 - \frac{125}{216}k_3 + \frac{875}{1152}k_4 - \frac{35}{216}k_5 + \frac{125}{3456}k_6\right)]$

$k_6 = f[x_n + h,\ y_n + y'_n + h^2\left(\frac{1}{12}k_1 + \frac{23}{40}k_2 - \frac{3}{5}k_3 + \frac{3}{5}k_4 - \frac{1}{5}k_5 + \frac{1}{24}k_6\right),\quad y'_n + h\left(\frac{11}{120}k_1 + \frac{27}{40}k_2 - \frac{8}{15}k_3 + \frac{27}{40}k_4 + 0k_5 + \frac{11}{120}k_6\right)]$

5 NUMERICAL EXPERIMENTS

The following IVPs for a second-order ODEs are used to ascertain the efficiency of the two newly derived methods

1) $y'' - 3y' = 8e^{2xy}$, $y(0) = 1$, $y'(0) = 1$, $h = 0.005$ and 0.0005

   (No exact solution)

2) $y'' - 3y' = 8e^{2x}$, $y(0) = 1$, $y'(0) = 1$, $h = 0.005$,

   $y(x) = -4e^{2x} + 3e^{3x} + 2$

3) $y'' + \dfrac{6}{x}y' + \dfrac{4}{x^2}y = 0$, $y(1) = 1$, $y'(1) = 1$, $h = \dfrac{0.1}{32}$

   $y(x) = \dfrac{5}{3x} - \dfrac{2}{3x^4} \qquad x > 0$

4) $y'' - xy' + 4y = 0$, $y(0) = 3$, $y'(0) = 0$, $h = 0.1$,

   $y(x) = x^4 - 6x^2 + 3$

6 IMPLEMENTATION STRATEGY

The implementation of problem 1 with the method (16) is as follows

$$y_{n+3}-3y_{n+1}+2y_n=\frac{81}{20}h^2((8e^{2x_ny_{n+1}})+3y'_{n+1})+\frac{73}{10}h^2((8e^{2x_ny_{n+2}})+3y'_{n+2})+$$
$$\frac{43}{60}h^2((8e^{2x_ny_{n+3}})+3y'_{n+3})-\frac{88}{15}h^2(\left(8e^{2x_ny_{n+\frac{3}{2}}}\right)+3y'_{n+\frac{3}{2}})-\frac{16}{5}h^2(\left(8e^{2x_ny_{n+\frac{5}{2}}}\right)+3y'_{n+\frac{5}{2}})$$

$$y_{n+\frac{3}{2}}-\frac{3}{2}y_{n+1}+\frac{1}{2}y_n=\frac{639}{640}h^2((8e^{2x_ny_{n+1}})+3y'_{n+1})+\frac{547}{320}h^2((8e^{2x_ny_{n+2}})+3y'_{n+2})+$$
$$\frac{337}{1920}h^2((8e^{2x_ny_{n+3}})+3y'_{n+3})-\frac{787}{480}h^2(\left(8e^{2x_ny_{n+\frac{3}{2}}}\right)+3y'_{n+\frac{3}{2}})-\frac{139}{160}h^2(\left(8e^{2x_ny_{n+\frac{5}{2}}}\right)+3y'_{n+\frac{5}{2}})$$

$$y_{n+2}-2y_{n+1}+y_n=\frac{121}{60}h^2((8e^{2x_ny_{n+1}})+3y'_{n+1})+\frac{103}{30}h^2((8e^{2x_ny_{n+2}})+3y'_{n+2})+$$
$$\frac{7}{20}h^2((8e^{2x_ny_{n+3}})+3y'_{n+3})-\frac{46}{15}h^2(\left(8e^{2x_ny_{n+\frac{3}{2}}}\right)+3y'_{n+\frac{3}{2}})-\frac{26}{15}h^2(\left(8e^{2x_ny_{n+\frac{5}{2}}}\right)+3y'_{n+\frac{5}{2}})$$

$$y_{n+\frac{5}{2}}-\frac{5}{2}y_{n+1}+\frac{3}{2}y_n=\frac{1165}{384}h^2((8e^{2x_ny_{n+1}})+3y'_{n+1})+\frac{343}{64}h^2((8e^{2x_ny_{n+2}})+3y'_{n+2})+$$
$$\frac{67}{128}h^2((8e^{2x_ny_{n+3}})+3y'_{n+3})-\frac{143}{32}h^2(\left(8e^{2x_ny_{n+\frac{3}{2}}}\right)+3y'_{n+\frac{3}{2}})-\frac{247}{96}h^2(\left(8e^{2x_ny_{n+\frac{5}{2}}}\right)+3y'_{n+\frac{5}{2}})$$

$$-360y_n+360y_{n+1}=360hy'_n+1485h^2((8e^{2x_ny_{n+1}})+3y'_{n+1})+3822h^2((8e^{2x_ny_{n+2}})+3y'_{n+2})$$
$$+409h^2((8e^{2x_ny_{n+3}})+3y'_{n+3})-3544h^2(\left(8e^{2x_ny_{n+\frac{3}{2}}}\right)+3y'_{n+\frac{3}{2}})-1992h^2(\left(8e^{2x_ny_{n+\frac{5}{2}}}\right)+3y'_{n+\frac{5}{2}}$$

when $n=0,3,6$ and 9 at $y(0)=1$, $y'(0)=1$ and $h=0.005$, the results for $y_1,y_2,y_3,y_4,y_5,y_6,y_7,y_8,y_9,y_{10}$ and y_{11} are obtained at once.

6.1 Implementation strategy of R-K method at K= 3 for second-order ODEs

Also, illustrating the implementation of R-K method (21) at K = 3 with problem 1 as

$$k_1=3y'_n+8e^{2x_ny_n}$$

$$k_2=3\left(y'_n+h\left(\frac{11}{120}k_1+\frac{673}{1080}k_2-\frac{104}{135}k_3+\frac{211}{360}k_4-\frac{32}{135}k_5+\frac{43}{1080}k_6\right)\right)$$

$$+(8e^{2\left(x_n+\frac{1}{3}h\right)*\left(y_n+\frac{1}{3}hy'_n\right)+h^2\left(\frac{82}{3645}k_1+\frac{235}{1944}k_2-\frac{707}{3645}k_3+\frac{271}{1620}k_4-\frac{91}{1215}k_5+\frac{401}{29160}k_6\right)})$$

$$k_3 = 3\left(y'_n + h\left(\frac{35}{384}k_1 + \frac{441}{640}k_2 - \frac{77}{120}k_3 + \frac{351}{640}k_4 - \frac{9}{40}k_5 + \frac{73}{1920}k_6\right)\right)$$

$$+(8e^{2\left(x_n+\frac{1}{2}h\right)*(y_n+\frac{1}{2}hy'_n)+h^2\left(\frac{29}{768}k_1+\frac{297}{1280}k_2-\frac{5}{16}k_3+\frac{333}{1280}k_4-\frac{9}{80}k_5+\frac{77}{3840}k_6\right)})$$

$$k_4 = 3\left(y'_n + h\left(\frac{37}{405}k_1 + \frac{92}{135}k_2 - \frac{224}{405}k_3 + \frac{29}{45}k_4 - \frac{32}{135}k_5 + \frac{16}{405}k_6\right)\right)$$

$$+(8e^{2\left(x_n+\frac{2}{3}h\right)*(y_n+\frac{2}{3}hy'_n)+h^2\left(\frac{193}{3645}k_1+\frac{421}{1215}k_2-\frac{1496}{3645}k_3+\frac{29}{81}k_4-\frac{184}{1215}k_5+\frac{97}{3645}k_6\right)})$$

$$k_5 = 3\left(y'_n + h\left(\frac{35}{384}k_1 + \frac{2375}{3456}k_2 - \frac{125}{216}k_3 + \frac{875}{1152}k_4 - \frac{35}{216}k_5 + \frac{125}{3456}k_6\right)\right)$$

$$+(8e^{2\left(x_n+\frac{5}{6}h\right)*\left(y_n+\frac{5}{6}hy'_n\right)+h^2\left(\frac{12725}{186624}k_1+\frac{28625}{62208}k_2-\frac{5875}{11664}k_3+\frac{9875}{20736}k_4-\frac{725}{3888}k_5+\frac{6125}{186624}k_6\right)})$$

$$k_6 = 3\left(y'_n + h\left(\frac{11}{120}k_1 + \frac{27}{40}k_2 - \frac{8}{15}k_3 + \frac{27}{40}k_4 + 0k_5 + \frac{11}{120}k_6\right)\right) +$$

$$\left(8e^{2(x_n+h)*(y_n+hy'_n)+h^2\left(\frac{1}{12}k_1+\frac{23}{40}k_2-\frac{3}{5}k_3+\frac{3}{5}k_4-\frac{1}{5}k_5+\frac{1}{24}k_6\right)}\right) \tag{22}$$

when $n = 0$ in (22) the values of $k_i's$, $i = 1 \ldots 6$, were obtained which then substituted in (21) as

$$y_{n+1} = y_n + hy'_n + h^2\left[\frac{1}{12}k_1 + \frac{23}{40}k_2 - \frac{3}{5}k_3 + \frac{3}{5}k_4 - \frac{1}{5}k_5 + \frac{1}{24}k_6\right]$$

$$y'_{n+1} = y'_n + h\left[\frac{11}{120}k_1 + \frac{27}{40}k_2 - \frac{8}{15}k_3 + \frac{27}{40}k_4 + 0k_5 + \frac{11}{120}k_6\right]$$

in order to obtain the values of y_1 and y'_1 respectively. This process continues repeatedly at a single iteration when n=1,2,3,…,9 to obtain y_1, y_2, y_3,…, y_{10} accordingly.

Table 1: Approximate solutions and Absolute error of Problem 1 with LMM at K=3

X	[12] at h = 0.005 at K = 4 (Order 8)	New LMM at K = 3, h = 0.005, (Order 5)	New LMM at K = 3, h = 0.0005, (Order 5)	Absolute error $\|0.005 - 0.0005\|$
0.005	1.0051388451	1.005138526365790	1.005138526365310	4.8000×10^{-13}
0.01	1.0105569851	1.010558255783450	1.010558255782340	1.1100×10^{-12}
0.015	1.0162568611	1.016265516755490	1.016265516753740	1.7500×10^{-12}
0.02	1.0222409615	1.022266778971230	1.022266778968240	2.9900×10^{-12}
0.025	1.0285346996	1.028568658274570	1.028568658270140	4.4300×10^{-12}
0.03	1.035118000	1.035177921871660	1.035177921865770	5.8900×10^{-12}
0.035	-	1.042101493788640	1.042101493780530	8.8100×10^{-11}
0.04	1.049257509	1.049346460591710	1.049346460581120	1.0590×10^{-11}
0.045	-	1.056920077391750	1.056920077378640	1.3110×10^{-11}
0.05	-	1.064829774145990	1.064829774129390	1.6600×10^{-11}

Table 2: Approximation results of Problem 1 with Equivalent R-K methods at K = 3

X	New R-K method at K = 3, h = 0.005
0.005	1.005138526365310
0.01	1.010558255782350
0.015	1.016265516753750
0.02	1.022266778968250
0.025	1.028568658270150
0.03	1.035177921865780
0.035	1.042101493780530
0.04	1.049346460581130
0.045	1.056920077378650
0.05	1.064829774129400

Table 3: Approximate results of Problem 2 at h = 0.005 with both new Methods

X	Theoretical Solution	[12] at K = 4 (Order 8)	[13] at K = 3 (Order 8)	New LMM at K = 3 with p = 5	New R-K method at K = 3 with p = 6
0.005	1.005138525510480	1.005138368	1.0051388419	1.00513852551050	1.00513852551048
0.01	1.010558241753520	1.010555066	1.0105569711	1.01055824175357	1.01055824175352
0.015	1.016265443912080	1.016252503	1.0162567886	1.01626544391216	1.01626544391208
0.02	1.022266542866520	1.022247320	1.0222407282	1.02226654286665	1.02226654286652
0.025	1.028568067149810	1.028527886	1.02853411642	1.02856806714998	1.02856806714980
0.03	1.035176664934190	1.035154590	1.03511676083	1.03517666493443	1.03517666493420
0.035	1.042099106050250	-	-	1.04209910605057	1.04209910605026
0.04	1.049342284038300	1.049432200	1.04925342567	1.04934228403868	1.04934228403830
0.045	1.056913218233090	-	-	1.05691321823357	1.05691321823311
0.05	1.064819055882240	-	-	1.06481905588284	1.06481905588227

Table 4: Absolute error of Problem 2 at h = 0.005 with both new methods

X	Error [12] at K = 4 (Order 8)	Error [13] at K = 3 (Order 8)	Error of New LMM at K = 3 with p = 5	Error of New R-K method at K = 3 with p = 6
0.005	3.159E(-07)	5.8849E(-07)	2.0000×10^{-14}	0.00
0.01	1.2709E(-06)	1.03675E(-06)	5.0000×10^{-14}	0.00
0.015	8.6554E(-06)	1.03759E(-05)	8.0000×10^{-14}	0.00
0.02	2.59148E(-05)	3.95659E(-05)	1.3000×10^{-13}	0.00
0.025	3.395058E(-05)	5.97171E(-05)	1.7000×10^{-13}	1.0000×10^{-14}
0.03	5.990417E(-05)	1.66006E(-04)	2.4000×10^{-13}	1.0000×10^{-14}
0.035	-	-	3.2000×10^{-13}	1.0000×10^{-14}
0.04	8.885833E(-05)	4.13483E(-04)	3.8000×10^{-13}	0.00
0.045	-		4.8000×10^{-13}	2.0000×10^{-14}
0.05	-		6.0000×10^{-13}	3.0000×10^{-14}

Table 5: Approximate results of Problem 3 at h = $\frac{0.1}{32}$ with both new methods

X	Theoretical Solution	[4] at k = 5 with (Order 6)	[6] at k = 4 with (Order 6)	New LMM at K = 3 with (Order 5)	New R-K method at K = 3 with (Order 5)
3.125×10^{-3}	1.003076525857700	1.003114880	1.003076525	1.00307652585775	1.003076525857700
6.25×10^{-3}	1.006057503083520	1.006132507	1.006057499	1.00605750308360	1.006057503083520
9.375×10^{-3}	1.008944995088840	1.009050915	-	1.00894499508902	1.008944995088840
1.25×10^{-2}	1.011741018167980	1.011876494	1.011740982	1.01174101816825	1.011741018167990
1.5625×10^{-2}	1.014447542686410	1.014603110	1.014447461	1.01444754268680	1.014447542686410
1.875×10^{-2}	1.017066494235670	1.017252866	-	1.01706649423615	1.017066494235660
2.1875×10^{-2}	1.019599754756290	1.019795810	-	1.01959975475689	1.019599754756270
2.5×10^{-2}	1.022049163629430	1.022270209	1.022049012	1.02204916363016	1.022049163629410
2.8125×10^{-2}	1.024416518738400	1.024622147	-	1.02441651873929	1.024416518738380
3.125×10^{-2}	1.026703577500810	1.026981486	-	1.02670357750185	1.026703577500780

Table 6: Absolute error of Problem 3 with both methods

X	Error of [4] at k = 5 with (Order 6)	Error of [6] at k = 4 with (Order 6)	Error of New LMM at K = 3 with p = 5	Error of New R-K method at K = 3 with p = 6
3.125×10^{-3}	3.8354E(-05)	1.40E(-09)	5.0000×10^{-14}	0.00
6.25×10^{-3}	7.5004E(-05)	4.09E(-09)	8.0000×10^{-14}	0.00
6.375×10^{-3}	1.0592E(-04)	-	1.8000×10^{-13}	0.00
1.25×10^{-2}	1.35476E(-04)	3.63E(-08)	2.7000×10^{-13}	1.0000×10^{-14}
1.5625×10^{-2}	1.55567E(-04)	8.18E(-08)	3.9000×10^{-13}	0.00
1.875×10^{-2}	1.86372E(-04)	-	4.8000×10^{-13}	1.0000×10^{-14}
2.1875×10^{-2}	1.96055E(-04)	-	6.0000×10^{-13}	2.0000×10^{-14}
2.5×10^{-2}	2.21045E(-04)	1.52E(-07)	7.3000×10^{-13}	2.0000×10^{-14}
2.8125×10^{-2}	2.05628E(-04)	-	8.9000×10^{-13}	2.0000×10^{-14}
3.125×10^{-2}	2.77908E(-04)	-	1.0400×10^{-13}	3.0000×10^{-14}

Table 7: Approximate results of Problem 4 at h = 0.1 with both methods

X	Theoretical Solution	[14] at, K =4 with (Order 5)	New LMM of Order 5 at K = 3	New R-K method at K = 3 with p = 6
0.1	2.940100000000000	2.940100001	2.940100000000020	2.940100000000000
0.2	2.761600000000000	2.761600000	2.761600000000050	2.761600000000000
0.3	2.468100000000000	2.468100001	2.468100000000010	2.468100000000000
0.4	2.065600000000000	2.065599998	2.065600000000030	2.065600000000000
0.5	1.562500000000000	-	1.562500000000050	1.562500000000000
0.6	0.969600000000018	-	0.969600000000000	0.969600000000000
0.7	0.300100000000000	-	0.300100000000015	0.300100000000002
0.8	0.430400000000000	-	0.430399999999990	0.430399999999996
0.9	1.203900000000000	-	1.203899999999930	1.203899999999990
1.0	2.000000000000000	-	1.999999999999940	2.999999999999990

Table 8: Absolute error of Problem 4 at h = 0.1 for both New LMMs and R-K methods

X	Error of [14] at, K =4 with (Order 5)	Error of New LMM of Order 5 at K = 3	Error of New R-K method at K = 3 with p = 6
0.1	1.0000 x 10^{-9}	2.0000 x 10^{-14}	0.00
0.2	1.0000 x 10^{-9}	5.0000 x 10^{-14}	0.00
0.3	1.0000 x 10^{-9}	2.0656 x 10^{-14}	0.00
0.4	2.0000 x 10^{-9}	3.0000 x 10^{-14}	0.00
0.5	-	5.0000 x 10^{-14}	0.00
0.6	-	1.8000 x 10^{-14}	1.8000 x 10^{-14}
0.7	-	1.5000 x 10^{-14}	2.0000 x 10^{-15}
0.8	-	1.0000 x 10^{-14}	4.0000 x 10^{-15}
0.9	-	7.0000 x 10^{-14}	1.0000 x 10^{-14}
1.0	-	6.0000 x 10^{-14}	1.0000 x 10^{-14}

7 CONCLUSION

Two Linear and Nonlinear IVPs for a second-order ODEs were experimented with the newly derived LMM and R-K Methods of step length of k=3. The numerical results for both newly derived methods were compared with the result of [12] at K = 4 under the same condition of step size 0.005 for Problem 1 and the absolute error differences in the mesh refinement with only the newly LMM at $h = 0.005$ and $h = 0.0005$. In Problem 2, numerical results of the newly derived methods were compared with [4], [12] and [13] at K=5.

In conclusion all the newly derived LMM and the R-K Methods show their superiority over existing methods as seen from all the Tables. But the R-K Method converges faster with the exact solution because of its single step nature.

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