

Mathematical Modeling of Illicit Drug Use Dynamics: Examining the Impact of Recycling Recovered Individuals into the Population

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ABSTRACT

This study focuses on the pervasive issue of illicit drug use, which poses a substantial threat to public health and society. It aims to create a comprehensive mathematical model that captures the dynamics of drug use, including susceptible individuals, exposed individuals, addicts, and those in recovery. The model considers that recovered individuals may not remain resistant to drugs and that exposed individuals cannot introduce new users. By using this model, the study seeks to deepen our understanding of drug use behaviours and patterns, offering valuable insights into drug-related research. It also utilizes the next-generation matrix method to calculate the effective reproduction number and conducts sensitivity analysis to identify effective control measures. The findings emphasize the urgent need for intensified efforts in combating drug abuse, and the study also highlights the significance of considering bifurcation phenomena for local stability analysis. This research contributes to ongoing efforts to address the complex challenges of illicit drug use, informing policies and interventions for community well-being.

Keywords: Illicit drugs, Mathematical modeling, Recycling, Threshold dynamics

1 INTRODUCTION

The term drug is used to refer to any natural or processed substance that when used can cause some changes to the systems of living beings. Substances are said to be abused when taken without a physician's prescription, these changes are always supposed to be for the betterment of a whole or any part of the system i.e., prevention from diseases (vaccination), seeking a cure from diseases (medication), sometimes to invigorate the system (energizers), slimming tablets, tranquillizers, cough mixtures the worst of all is taking the drugs (narcotics) to alter one's mood, perception, or behaviour. In other words, it can be defined as excessive use of or dependence on any psychoactive chemical substance to establish harmful effects on users' physical, social, psychological, and mental health. According to recent studies conducted by [1,2,3,4], some drug users get initiated through prescription rather than experimentation and later misuse the opportunity. In the works of [5,6,7], the results revealed that public tolerance of drug usage, with limited consideration of the adverse

health and socio-economic consequences can influence initiation as well as continuous usage. Teenagers and adolescents are more vulnerable to the use of psychoactive substances [8,9,17]. According to a prior study, Polypharmacy was observed to become more widespread among the elderly [18]. Based on the research conducted by Sarkar and Parmar [15,16], there is a comprehensive analysis of the widespread use of psychoactive substances among elderly individuals in India. The research findings indicate a notable decline in the utilization of psychoactive drugs/substances once individuals reach the age of fifty. Abuse of drugs can harm living beings in diverse ways thereby increasing the population of physically disabled and mentally imbalanced individuals within societies. Rehabilitation is the most effective way of curbing the prevalence of the global epidemic of drug abuse, the investigation revealed that limited rehabilitation facilities can increase the accessibility and rate of intake of psychoactive drugs [11]. Consumption of psychoactive substances varies across countries and depends on individual needs and sociocultural purposes. Globally, 9% of the youth population has engaged in the usage of one psychoactive substance or the other [14,15]. Up to 58% of the adolescent population abuse drugs [12]. A report of studies conducted across Sub-Saharan Africa discovered that the overall prevalence of drug abuse is 41.6% [19]. In Nigeria, drug abuse among the adolescent population ranges from approximately 33% to 69% [13,16,17].

A descriptive cross-sectional study was conducted, involving 1304 adolescent students from a Nigerian high school. The study employed various assessment tools, including a sociodemographic questionnaire, the Drug Abuse Screening Test (DAST) consisting of 10 items, the Problematic Internet Use (PIU) Questionnaire, the Mini International Neuropsychiatric Interview Suicidality (MINI) module, the Hospital Anxiety and Depression Scale (HADS), and the Rosenberg Self-Esteem Scale (RSES) [10,11]. The findings revealed a high prevalence rate of drug abuse, reaching 49.8% among the surveyed students.

There is a habit of 'addiction' which develops over time and regularly provides an appetite for the use and misuse of substances. Excessive drug usage can lead to continued serious drug dependence despite the harmful consequences. [20,21] views the habit of addiction as a biopsychosocial condition resulting from variations in biological predispositions, psychological processes and environmental influence. The work argued that intra and inter-individual dynamics need to be incorporated into addiction models to unravel intricacies and deepen understanding of addiction phenomenon.

In their recent research, [29,30] employed mathematical modelling techniques to investigate the dynamics of illicit drug use and the banditry population. Within the results section, they emphasized significant findings, including an evaluation of the effects of optimal control strategies and a thorough cost-effectiveness analysis. Additionally, [31] observed that their model displayed a forward bifurcation property, indicating the presence of a unique and locally stable equilibrium associated with illicit drug presence. However, it's noteworthy that the models did not address the potential consequences of reintroducing illicit drug users into the population. The mathematical model developed by [32-34] is applied to conduct numerical simulations. These simulations yield illustrative results, aiding in the comprehension of the spread of illicit drug use within communities.

The primary aim of this study is to enhance our comprehension of the dynamics of drug abuse through the development and analysis of a comprehensive mathematical model. This model will encompass various aspects of drug abuse, including susceptible individuals, exposed individuals, drug addicts, and individuals in recovery, challenging previously held assumptions.

Through this research, the study seeks to achieve several key objectives: Utilize the mathematical model to analyze the dynamics of drug abuse, uncover multiple equilibria and potential backward bifurcation points for specific parameter values, conduct a sensitivity analysis to evaluate the effectiveness of preventive measures, such as coping strategies and resistance to temptations, in addressing drug abuse, offer actionable recommendations to policymakers based on the findings, enabling them to implement effective strategies to combat drug abuse in the region and enhance our understanding of the complex dynamics and patterns associated with illicit drug use, making a substantial contribution to the field of drug-related research.

2 MATERIAL AND METHODS

In this paper, we focus on several populations of interest: Susceptible (S), Exposed (E), Drug users (D), individuals in prison (P), and those who have recovered (R)(SEDPRE). Susceptible individuals join the population through an input rate denoted as π . When susceptible individuals come into contact with drug addicts, there is a probability of adopting the drug ideology, denoted as β . However, embracing the ideology does not automatically make them extremists, except for a proportion with a chance of α , which leads them to join the drug group. Some exposed individuals may transition to the recovered class without engaging in drug use, occurring at a rate denoted as ξ . The drug group can either be cured and move to the recovered class at a rate of γ , or they may be intercepted and imprisoned at a rate of ρ . Those in detention facilities have the possibility of recovering and joining the recovered class at a rate of ϑ . Occasionally, individuals in the recovered class may become exposed again and relapse into drug use at a rate of $(1 - \delta)\beta$. The parameter d represents the induced death rate resulting from drug-related activities, while μ represents the natural death rate within each compartment.

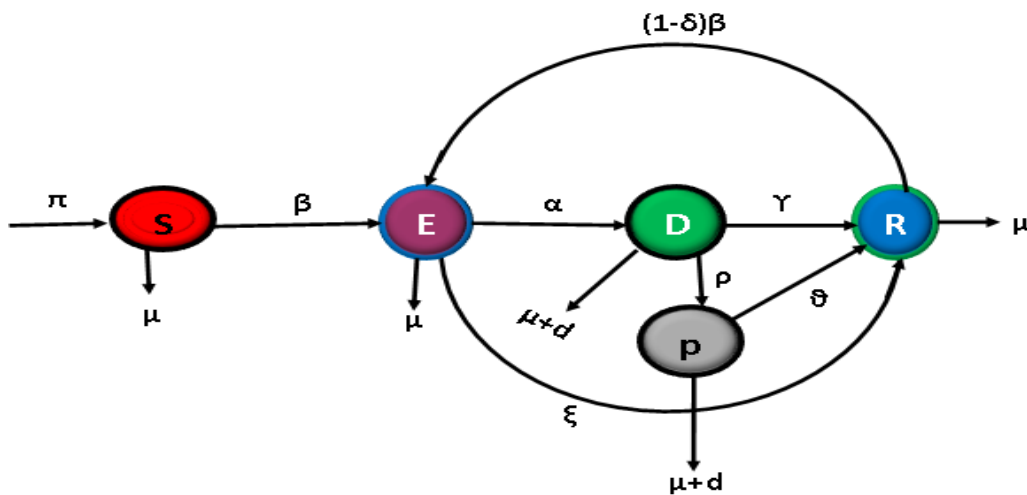


Figure 1 : The flow diagram of the model.

The system of ordinary differential equation

$$\left. \begin{aligned} \frac{dS}{dt} &= \pi - \beta SD - \mu S \\ \frac{dE}{dt} &= \beta SD + (1 - \delta)\beta DR - (\alpha + \mu + \xi)E \\ \frac{dD}{dt} &= \alpha E - (\gamma + \rho + \mu + d)D \\ \frac{dP}{dt} &= \rho D - (\vartheta + \mu + d)P \\ \frac{dR}{dt} &= \xi E + \gamma D + \vartheta P - (\mu + (1 - \delta)\beta D)R \end{aligned} \right\} \quad (1)$$

Table 1 : Parameters of the model and their description

S/no	Parameter	Description
1	π	Recruit rate
2	μ	Natural death
3	β	Contact rate
4	d	Drug-induced death rate
5	$(1 - \delta)\beta$	Rate of loss of drug resistivity
6	δ	The scaling factor of rate loss of drug resistivity
7	ξ	The rate at which exposed individuals recovered naturally
8	α	Rate of transition to a drug addict
9	ρ	The rate at which illicit drug users are being imprisoned
10	γ	The recovery rate from drug addiction population

Table 2: Variables of the Model and Their Description

S/No	Variables	Descriptions
1	S	Susceptible individuals
2	E	Exposed individuals
3	D	Drug individuals
4	P	Prison individuals
5	R	Recovered individuals

Considering the initial conditions

$$S(0) = S_0 \geq 0, E(0) = E_0 \geq 0, D(0) = D_0 \geq 0, P(0) = P_0 \geq 0, R(0) = R_0 \geq 0$$

All the parameters of the system (1) are assumed to be positive for all time $t > 0$

3 BASIC PROPERTIES OF THE MODEL

This section of the paper focuses on examining and analyzing the key properties and characteristics of the model that has been proposed.

3.1 The Invariant Region

Lemma: The initial conditions of the solutions of system (1) are contained in the region $B \in \mathfrak{R}_+^5$, defined by

$$B = \{(S, E, D, P, R) \in \mathfrak{R}^5: 0 \leq N \leq \frac{\pi}{\mu}\}$$

Proof: Summation of all equations of model system (1) gives

$$N = S + E + D + P + R \quad (2)$$

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dE}{dt} + \frac{dD}{dt} + \frac{dP}{dt} + \frac{dR}{dt} \quad (3)$$

$$\begin{aligned} \frac{dN}{dt} = & \pi - \beta SD - \mu S + \beta SD + (1 - \delta)\beta DR - (\alpha + \mu + \xi)E + \alpha E + \sigma P - (\gamma + \rho + \mu + d)D \\ & + \rho D - (\sigma + \vartheta + \mu + d)P + \xi E + \gamma D + \vartheta P - (\mu + (1 - \delta)\beta D)R \end{aligned} \quad (4)$$

$$\frac{dN}{dt} = \pi - \mu S - \mu E - \mu D - \mu P - \mu R - dD - dP \quad (5)$$

$$\frac{dN}{dt} = \Lambda - \mu(S + E + D + P + R) - dD - dP \quad (6)$$

If there is no drug-related death then the equation becomes

$$\frac{dN}{dt} \leq \pi - \mu N \quad (7)$$

So if we differentiate N with respect to time and substitute the values of $\frac{dS}{dt}, \frac{dE}{dt}, \frac{dD}{dt}, \frac{dP}{dt}$ and $\frac{dR}{dt}$ and simplify, we get

$$\frac{dN}{dt} \leq \pi - \mu N \quad (8)$$

Solve this inequality gives

$$B = \{(S, E, D, P, R) \in \mathfrak{R}^5: 0 \leq N \leq \frac{\pi}{\mu}\} \quad (9)$$

3.2 Positivity of the Solution

Theorem: Let $D = \{(S, E, D, P, R) \in \mathfrak{R}^5: S(0) = S_0 > 0, E(0) = E_0 > 0, D(0) = D_0 > 0, P(0) = P_0 > 0, R(0) = R_0 > 0\}$; then the solutions of $\{S, E, D, P, R\}$ are all positive for future time

Proof: From equation (1) of the system of differential equations (1), we get

$$\frac{dS}{dt} \geq -\mu S \quad (10)$$

$$\frac{dS}{S} \geq -\mu dt \quad (11)$$

Solving this equation and applying the initial condition yields

$$S(t) \geq S_0 e^{-\mu t} \geq 0 \quad (12)$$

Similarly, from the second, third, fourth and five equations we get

$$\left. \begin{aligned} E(t) &\geq E_0 e^{-(\alpha+\mu+\xi)t} \geq 0, \\ D(t) &\geq D_0 e^{-(\gamma+\mu+\rho+d)t} \geq 0 \\ P(t) &\geq P_0 e^{-(\vartheta+\mu+d)t} \geq 0 \\ R(t) &\geq R_0 e^{-(\mu+\delta)t} \geq 0 \end{aligned} \right\} \quad (13)$$

Hence, the solution of the model is positive for all future time ■.

3.3 Equilibrium Point of the Model

3.3.1 Drug-Free Equilibrium Point

The drug-free equilibrium (DFE) state, E_{drug}^0 , of the system model (1) is obtained by setting the right-hand side of the equation equal to zero and letting $E = D = P = 0$, we solve for S and we get:

$$E_{drug}^0 = \left(\frac{\pi}{\mu}, 0, 0, 0, 0 \right) \quad (14)$$

3.3.2 Effective Reproduction Number \mathfrak{R}_{drug}

The next-generation matrix method [22,23,28] was applied to calculate the effective reproduction number of the model. This involved utilizing the matrix F, which represents the terms related to new drug addicts, and the matrix V, which represents the transition terms.

Thus:

$$\left. \begin{aligned} \frac{dE}{dt} &= \beta SD + (1 - \delta)\beta DR - (\alpha + \mu + \xi)E \\ \frac{dD}{dt} &= \alpha E - (\gamma + \rho + \mu + d)D \\ \frac{dP}{dt} &= \rho D - (\vartheta + \mu + d)P \end{aligned} \right\} \quad (15)$$

$$F = \begin{pmatrix} \beta SD + (1 - \delta)\beta DR \\ 0 \\ 0 \end{pmatrix}, \quad V = \begin{pmatrix} (\alpha + \mu + \xi)E \\ -\alpha E + (\gamma + \rho + \mu + d)D \\ -\rho D + (\vartheta + \mu + d)P \end{pmatrix} \quad (16)$$

Therefore, our next-generation matrices are the determinant of the matrix of the new drug addicts' terms and that of the matrix of the transition terms at the drug-free equilibrium.

$$F = \begin{pmatrix} \frac{df_1}{dE} & \frac{df_1}{dD} & \frac{df_1}{dP} \\ \frac{df_2}{dE} & \frac{df_2}{dD} & \frac{df_2}{dP} \\ \frac{df_3}{dE} & \frac{df_3}{dD} & \frac{df_3}{dP} \end{pmatrix}, \quad V = \begin{pmatrix} \frac{dv_1}{dE} & \frac{dv_1}{dD} & \frac{dv_1}{dP} \\ \frac{dv_2}{dE} & \frac{dv_2}{dD} & \frac{dv_2}{dP} \\ \frac{dv_3}{dE} & \frac{dv_3}{dD} & \frac{dv_3}{dP} \end{pmatrix} \quad (17)$$

$$F = \begin{pmatrix} 0 & \frac{\beta\pi}{\mu} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} (\alpha + \mu + \xi) & 0 & 0 \\ -\alpha & (\gamma + \rho + \mu + d) & 0 \\ 0 & -\rho & (\vartheta + \mu + d) \end{pmatrix} \quad (18)$$

$$FV^{-1} = \begin{pmatrix} \frac{\alpha\beta\pi}{\mu(\alpha+\mu+\xi)(d+\gamma+\mu+\rho)} & \frac{\beta\pi}{\mu(d+\gamma+\mu+\rho)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (19)$$

$$\lambda^3 - \frac{\alpha\beta\pi}{\mu(\alpha+\mu+\xi)(\mu+\gamma+\rho+d)} \lambda^2 \quad (20)$$

$$|FV^{-1} - \lambda I| = \frac{\alpha\beta\pi}{\mu(\alpha+\mu+\xi)(\mu+\gamma+\rho+d)} \quad (21)$$

Hence the basic reproduction number is

$$\mathfrak{R}_{drug} = \frac{\alpha\beta\pi}{\mu(\alpha+\mu+\xi)(\mu+\gamma+\rho+d)} \quad (22)$$

3.3.3 Sensitivity Analysis of the \mathfrak{R}_{drug} Parameters

Performing a sensitivity analysis is a valuable approach to assessing the sensitivity of the basic reproductive number to different parameters, particularly when there is uncertainty involved in determining specific parameter values. This analysis allows us to identify constant control measures that can lead to significant reductions in R_{drug} , indicating the most effective strategies for mitigating the spread of drug-induced issues. To accomplish this, we calculate the normalized forward sensitivity index of the reproduction number with respect to these parameters. This index measures the relationship between the relative change in a variable and the corresponding changes in the parameters. The forward normalized sensitivity index, as outlined in [22], can be computed using the following formula: [formula].

$$\Lambda_{\theta}^{\mathfrak{R}_{drug}} = \frac{d\mathfrak{R}_{drug}}{d\theta} X \frac{\theta}{\mathfrak{R}_{drug}} \quad (23)$$

The sensitivity indices for each parameter, computed in relation to the basic reproduction number, are presented in Table 3 below:

Table 3: Sensitivity indices of the parameters of the basic reproduction number

Parameter	Relationship with other Parameters	Index
α	$\Lambda_{\alpha}^{\mathfrak{R}_{drug}} = \frac{(\mu + \xi)}{(\alpha + \mu + \xi)} > 0$	0.45078
β	$\Lambda_{\beta}^{\mathfrak{R}_{drug}} = 1 > 0$	1.0000
π	$\Lambda_{\pi}^{\mathfrak{R}_{drug}} = 1 > 0$	1.0000
ξ	$\Lambda_{\xi}^{\mathfrak{R}_{drug}} = \frac{-\xi}{(\alpha + \mu + \xi)} < 0$	-0.43936
γ	$\Lambda_{\gamma}^{\mathfrak{R}_{drug}} = \frac{-\gamma}{(\mu + \gamma + \rho + d)} < 0$	-0.29557
ρ	$\Lambda_{\rho}^{\mathfrak{R}_{drug}} = \frac{-\rho}{(\mu + \gamma + \rho + d)} < 0$	-0.29557
d	$\Lambda_{d}^{\mathfrak{R}_{drug}} = \frac{-d}{(\mu + \gamma + \rho + d)} < 0$	-0.34483
μ	$\Lambda_{\mu}^{\mathfrak{R}_{drug}} = \frac{-[(\alpha + \mu + \xi)(\mu + \gamma + \rho + d) + \mu(\alpha + \mu + \xi) + \mu(\mu + \gamma + \rho + d)]}{(\alpha + \mu + \xi)(\mu + \gamma + \rho + d)} < 0$	-1.0755

3.3.4 Interpretation of the Sensitivity Indices of the Parameters

The parameters in Table 2 that exhibit positive sensitivity indices, specifically β , π , and α , have a significant impact on the drug addicts spread when their values are raised while keeping other parameters constant. This is because the average number of secondary drug addicts rises with higher values of these parameters. To control the addiction, it is necessary to decrease the values of β , π , and α .

Nevertheless, the results of the sensitivity analysis indicate that it is not physiologically realistic to propose increasing the Drug-induced death rate (d) and the natural death rate (μ) as control measures for the drug's addition, despite their negative sensitivity indices. Conversely, the recovery rate (γ) of a Drug, which exhibits a negative sensitivity index, plays a vital role in effectively controlling the drug's addiction when its value is augmented.

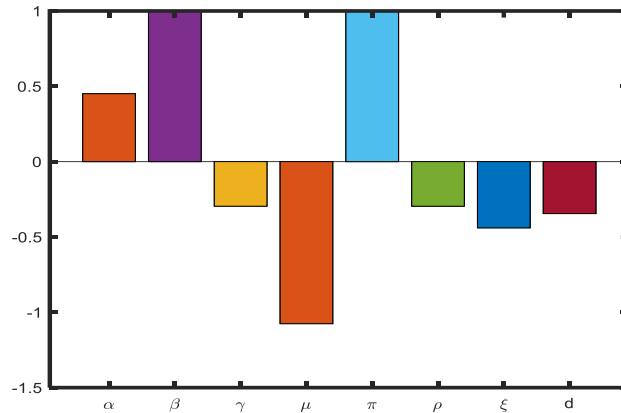


Figure 2: The pictorial representation of the sensitivity indices in a bar chat.

3.3.5 Drug Equilibrium Point

We compute the drug equilibrium points in terms of the drug addicts using the model system equation (1) which gives

From the first equation, we got

$$S^* = \frac{\pi}{\beta D + \mu} \quad (24)$$

From the third equation, we got

$$E^* = \frac{(\gamma + \rho + \mu + d)D}{\alpha} \quad (25)$$

From the fourth equation, we got

$$P = \frac{\rho D}{(\vartheta + \mu + d)} \quad (26)$$

$$D^* = D \quad (27)$$

From the fifth equation in the system, we got

$$R = \frac{\xi(\vartheta + \mu + d)(\gamma + \rho + \mu + d)D + \alpha\gamma(\vartheta + \mu + d)D + \alpha\vartheta\rho D}{\alpha(\vartheta + \mu + d)(\mu + (1 - \delta)\beta D)} \quad (28)$$

Hence the endemic equilibrium point is

$$E_* = \left(\frac{\pi}{\beta D + \mu}, \frac{(\gamma + \rho + \mu + d)D}{\alpha}, D^*, \frac{\rho D}{(\vartheta + \mu + d)}, \frac{\xi(\vartheta + \mu + d)(\gamma + \rho + \mu + d)D + \alpha\gamma(\vartheta + \mu + d)D + \alpha\vartheta\rho D}{\alpha(\vartheta + \mu + d)(\mu + (1 - \delta)\beta D)} \right) \quad (29)$$

When the values of S, E, D, P and R are substituted into equation 2 we get

$$\begin{aligned} & (\beta^2(\delta - 1)(d + \mu + \vartheta)(\alpha + \mu + \xi)(d + \gamma + \mu + \rho) - \beta^2\xi(\delta - 1)(d + \mu + \vartheta)(d + \mu + \eta + \rho) - \\ & \alpha\beta^2\gamma(\delta - 1)(d + \mu + \vartheta) - \alpha\beta^2\rho\vartheta(\delta - 1))D^2 + \left(-(\beta\mu - \beta\mu(\delta - 1))(d + \mu + \vartheta)(\alpha + \mu + \xi)(d + \gamma + \right. \\ & \left. \mu + \rho) - \pi\alpha\beta^2(\delta - 1)(d + \mu + \vartheta) - \beta\mu\xi(\delta - 1)(d + \mu + \vartheta)(d + \mu + \eta + \rho) - \alpha\beta\gamma\mu(\delta - 1)(d + \mu + \right. \end{aligned}$$

$$\vartheta) - \alpha\beta\mu\rho\vartheta(\delta - 1))D + (\pi\alpha\beta\mu(d + \mu + \vartheta) - \mu^2(d + \mu + \vartheta)(\alpha + \mu + \xi)(d + \gamma + \mu + \rho)) = 0 \quad (30)$$

which is of the form

$$M_1D^2 + M_2D + M_3 = 0 \quad (31)$$

where,

$$M_1 = (\beta^2(\delta - 1)(d + \mu + \vartheta)(\alpha + \mu + \xi)(d + \gamma + \mu + \rho) - \beta^2\xi(\delta - 1)(d + \mu + \vartheta)(d + \gamma + \mu + \rho) - \alpha\beta^2\gamma(\delta - 1)(d + \mu + \vartheta) - \alpha\beta^2\rho\vartheta(\delta - 1)) \quad (32)$$

$$M_2 = -(\beta\mu - \beta\mu(\delta - 1))(d + \mu + \vartheta)(\alpha + \mu + \xi)(d + \gamma + \mu + \rho) - \pi\alpha\beta^2(\delta - 1)(d + \mu + \vartheta) - \beta\mu\xi(\delta - 1)(d + \mu + \vartheta)(d + \gamma + \mu + \rho) - \alpha\beta\gamma\mu(\delta - 1)(d + \mu + \vartheta) - \alpha\beta\mu\rho\vartheta(\delta - 1) \quad (33)$$

$$M_3 = -\mu^2(d + \mu + \vartheta)(\alpha + \mu + \xi)(d + \gamma + \mu + \rho)(1 - \mathfrak{R}_{drug}) \quad (34)$$

The plot of the drugs addicted population versus the basic reproduction number is shown in Figure 3 below.

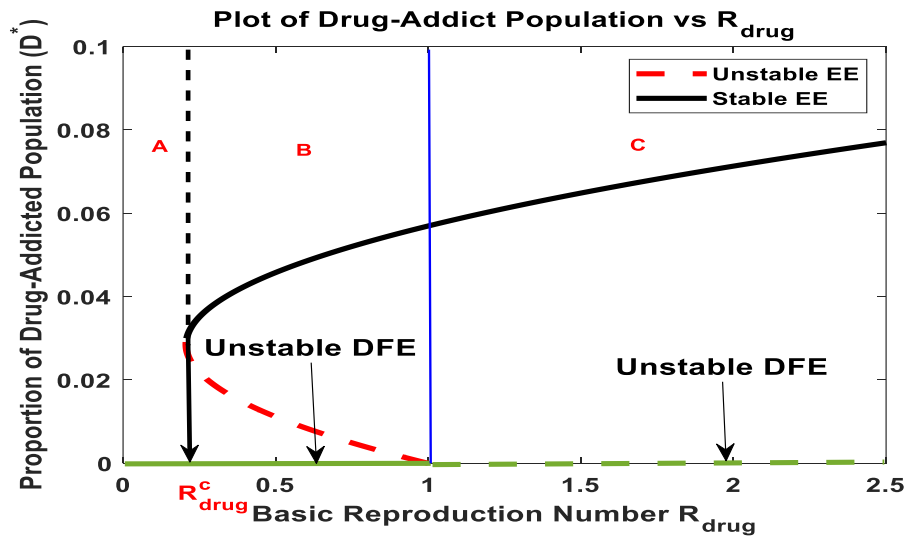


Figure 3: Plot of addicted population vs \mathfrak{R}_{drug} showing backward bifurcation.

The graph illustrates backward bifurcation typically showing the relationship between two variables, often representing drug prevalence and basic reproduction number. In the backward bifurcation graph, there is a distinct point where the system's behaviour changes qualitatively.

Before this point, reducing the basic reproduction number or intervention efforts may not necessarily lead to an immediate decrease in drug prevalence. Instead, the system exhibits a stable equilibrium, indicating that drug addicts persist even with reduced intervention.

After this critical point, reducing the basic reproduction number leads to a significant increase in drug prevalence. This is a crucial concept in epidemiology and public health because it means that even a slight relaxation of efforts to control the drugs can result in a rapid resurgence of drug addicts within the population.

3.4 Local Stability Analysis of the Equilibrium Points

In this section, we analyse the equilibrium of the model to ascertain their stabilities.

3.4.1 Local Stability Analysis of the Drug-Free Equilibrium

Using Routh Hurwitz criteria [15], we established the following theorem.

Theorem: The drug addiction-free equilibrium point is locally asymptotically stable if $\mathfrak{R}_{cor} < 1$ and unstable if $\mathfrak{R}_{drug} > 1$.

Proof: To prove the local stability of drugs-free equilibrium, we obtained the Jacobian matrix of the system (1) at the drugs-free equilibrium E_{drug}^0

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial S} & \frac{\partial f_1}{\partial E} & \frac{\partial f_1}{\partial D} & \frac{\partial f_1}{\partial P} & \frac{\partial f_1}{\partial R} \\ \frac{\partial f_2}{\partial S} & \frac{\partial f_2}{\partial E} & \frac{\partial f_2}{\partial D} & \frac{\partial f_2}{\partial P} & \frac{\partial f_2}{\partial R} \\ \frac{\partial f_3}{\partial S} & \frac{\partial f_3}{\partial E} & \frac{\partial f_3}{\partial D} & \frac{\partial f_3}{\partial P} & \frac{\partial f_3}{\partial R} \\ \frac{\partial f_4}{\partial S} & \frac{\partial f_4}{\partial E} & \frac{\partial f_4}{\partial D} & \frac{\partial f_4}{\partial P} & \frac{\partial f_4}{\partial R} \\ \frac{\partial f_5}{\partial S} & \frac{\partial f_5}{\partial E} & \frac{\partial f_5}{\partial D} & \frac{\partial f_5}{\partial P} & \frac{\partial f_5}{\partial R} \end{pmatrix} \quad (35)$$

The Jacobian matrix of the system of equation (1) at the drugs-free equilibrium point becomes

$$J(E_0) = \begin{pmatrix} -\mu & 0 & -\frac{\beta\pi}{\mu} & 0 & 0 \\ 0 & -(\alpha + \mu + \xi) & \frac{\beta\pi}{\mu} & 0 & 0 \\ 0 & \alpha & -(\gamma + \rho + \mu + d) & 0 & 0 \\ 0 & 0 & \rho & -(\vartheta + \mu + d) & 0 \\ 0 & \xi & \gamma & \vartheta & -\mu \end{pmatrix} \quad (36)$$

with characteristic equation

$$(\lambda + \mu)^2(\lambda + \vartheta + \mu + d)(A\lambda^2 + B\lambda + C) = 0 \quad (37)$$

where,

$$\left. \begin{aligned} A &= \mu \\ B &= \mu(d + 2\mu + \alpha + \gamma + \xi + \rho) \\ C &= \mu(\alpha + \mu + \xi)(\gamma + \rho + \mu + d) - \alpha\beta\pi \end{aligned} \right\} \quad (38)$$

which means

$$(\lambda + \mu) = 0 \quad (39)$$

$$(\lambda + \mu) = 0 \quad (40)$$

$$(\lambda + \vartheta + \mu + d) = 0 \quad (41)$$

$$(A\lambda^2 + B\lambda + C) = 0 \quad (42)$$

or

$$[\mu\lambda^2 + \mu(d + 2\mu + \alpha + \gamma + \xi + \rho)\lambda + \mu(\alpha + \mu + \xi)(\gamma + \rho + \mu + d) - \alpha\beta\pi] = 0 \quad (43)$$

Solving for λ we get

$$\left. \begin{aligned} \lambda_1 &= -\mu \\ \lambda_2 &= -\mu \\ \lambda_3 &= -(\mu + \vartheta + d) \end{aligned} \right\} \quad (44)$$

3.4.2 Global Stability Analysis of the Drug-Free Equilibrium Point

We shall investigate the global stability of the system using the techniques implemented by Castillo-Chavez and Song.[27] Let's re-write system (1) as follows

$$\left. \begin{aligned} \frac{dX}{dt} &= F(X, Z), \\ \frac{dZ}{dt} &= G(X, Z), G(X, 0) = 0 \end{aligned} \right\} \quad (57)$$

Where X represent the non-drug population, that is $X = \{S, R\}$ while Z represents the drug addicts population, that is $Z = \{E, D, P\}$. The drug-free equilibrium point of the model is denoted by $U = (X^*, 0)$. The point $U = (X^*, 0)$ is globally asymptotically stable equilibrium for the model provided that $\mathfrak{R}_{drug} < 1$, which is locally asymptotically stable, and the following conditions must be met:

$$\left. \begin{aligned} (\mathbb{H}_1): & \text{For } \frac{dX}{dt} = F(X, 0), X^* \text{ is globally asymptotically stable.} \\ (\mathbb{H}_2): & G(X, Z) = AZ - \tilde{G}(X, Z), \tilde{G}(X, Z) \geq 0 \text{ for } (X, Z) \in \Omega \end{aligned} \right\} \quad (58)$$

If the model (1) meets the given two criteria, then the following theorem holds

Theorem: The drugs-free equilibrium points $U = (X^*, 0)$ is globally asymptotically stable provided $R_{drug} < 1$ and conditions (\mathbb{H}_1) and (\mathbb{H}_2) are satisfied.

Proof: From the system we can get $F(X, Z)$ and $G(X, Z)$:

$$F(X, Z) = \begin{pmatrix} \pi - \beta SD - \mu S \\ \xi E + \gamma D + \vartheta P - (\mu + (1 - \delta)\beta D)R \end{pmatrix} \quad (59)$$

$$G(X, Z) = \begin{pmatrix} \beta SD - (\alpha + \mu + \xi)E \\ \alpha E + (1 - \delta)\beta DR - (\gamma + \rho + \mu + d)D \\ \rho D - (\vartheta + \mu + d)P \end{pmatrix} \quad (60)$$

$$\text{At } E = D = P = R = 0 \quad (61)$$

$$\frac{dX}{dt} = F(X, 0) = \begin{pmatrix} \pi - \mu S \\ 0 \end{pmatrix} \quad (62)$$

From the above system, we see that $X^* = \left(\frac{\pi}{\mu}, 0\right)$ is a globally asymptotic point. This can be verified from the solutions, namely

$$S(t) = \frac{\pi}{\mu} + \left(S(0) - \frac{\pi}{\mu}\right) e^{-\mu t}. \text{ As } t \rightarrow \infty, \text{ the solution } S(\infty) \rightarrow \frac{\pi}{\mu}$$

This implies the global convergence of (2) in \mathfrak{D} and this conforms with condition \mathbb{H}_1 . Now from \mathbb{H}_2 we have that $G(X, Z) = AZ - \tilde{G}(X, Z), \tilde{G}(X, Z) \geq 0$ for $(X, Z) \in \mathfrak{D}$

Therefore, $\tilde{G}(X, Z) = AZ - G(X, Z)$.

Where A is a $n \times n$ matrix, Z and $G(X, Z)$ are column vectors formed from the drug equations.

The first partial derivative of $G(X, Z)$ with respect to E, D and P computed at the drug-free equilibrium point gives matrix A .

$$A = \begin{pmatrix} -(\alpha + \mu + \xi) & \frac{\beta\pi}{\mu} & 0 \\ \alpha & -(\gamma + \rho + \mu + d) & 0 \\ 0 & \rho & -(\vartheta + \mu + d) \end{pmatrix} \quad (63)$$

$$AZ = \begin{pmatrix} -(\alpha + \mu + \xi)E + \frac{\beta\pi}{\mu}D \\ \alpha E - (\gamma + \rho + \mu + d)D \\ \rho D - (\vartheta + \mu + d)P \end{pmatrix} \quad (64)$$

From the expression $\tilde{G}(X, Z) = AZ - G(X, Z)$, we have

$$\tilde{G}(X, Z) = \begin{pmatrix} -(\alpha + \mu + \xi)E + \frac{\beta\pi}{\mu}D \\ \alpha E - (\gamma + \rho + \mu + d)D \\ \rho D - (\vartheta + \mu + d)P \end{pmatrix} - \begin{pmatrix} \beta SD + (1 - \delta)\beta DR - (\alpha + \mu + \xi)E \\ \alpha E - (\gamma + \rho + \mu + d)D \\ \rho D - (\vartheta + \mu + d)P \end{pmatrix} \quad (65)$$

$$\tilde{G}(X, Z) = \begin{pmatrix} -(1 - \delta)\beta CR \\ 0 \\ 0 \end{pmatrix} \quad (66)$$

From $\tilde{G}(X, Z)$, we can see that $\tilde{G}_1(X, Z) \leq 0, \tilde{G}_2(X, Z) = 0, \tilde{G}_3(X, Z) = 0$ which leads to $\tilde{G}(X, Z) \leq 0$. Which indicates that the second condition (\mathbb{H}_2) is not satisfied. So, the system of equation may not be globally asymptotically stable when $\mathfrak{R}_{drug} < 1$ ■

3.4.3 Local Stability of the Drug Equilibrium Point

In order to investigate whether the model system (1) exhibits backward or forward bifurcation, the Center manifold theory is employed. By studying these bifurcation phenomena, we can ascertain the local stability of the system at the endemic equilibrium point. The analysis begins with the calculation of the basic reproduction number.

$$\mathfrak{R}_{drug} = \frac{\alpha\beta\pi}{\mu(\alpha + \mu + \xi)(\mu + \gamma + \rho + d)} \quad (67)$$

We let $\beta=\beta^*$ be the bifurcation parameter and consider the value of \mathfrak{R}_{drug} at 1, then, we make β the subject of the formula to obtain

$$\beta^* = \frac{\mu(\alpha+\mu+\xi)(\mu+\gamma+\rho+d)}{\alpha\pi} \tag{68}$$

Let's rename the variables as $S=x_1, E=x_2, D=x_3, P=x_4$ and $R=x_5$.

Also let use the vector $X= (x_1, x_2, x_3, x_4, x_5)^T$

formulated as $\frac{dX}{dt} = F(x)$,

Where $F = (f_1, f_2, f_3, f_4, f_5)^T$

Now the drugs addiction-free equilibrium becomes $X_0 = (x_1 = \frac{\pi}{\mu}, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0)$

Substitute these values of $S=x_1, E=x_2, D=x_3, P=x_4$ and $R=x_5$ into our model system of equation we get

$$\left. \begin{aligned} \frac{dx_1}{dt} &= \pi - \beta x_3 x_1 - \mu x_1 && f_1 \\ \frac{dx_2}{dt} &= \beta x_3 x_1 + (1 - \delta)\beta x_3 x_5 - (\alpha + \mu + \xi)x_2 && f_2 \\ \frac{dx_3}{dt} &= \alpha x_2 - (\gamma + \rho + \mu + d)x_3 && f_3 \\ \frac{dx_4}{dt} &= \rho x_3 - (\vartheta + \mu + d)x_4 && f_4 \\ \frac{dx_5}{dt} &= \xi x_2 + \gamma x_3 + \vartheta x_4 - (\mu + (1 - \delta)\beta x_3)x_5 && f_5 \end{aligned} \right\} \tag{69}$$

and its Jacobian at the drugs addiction-free equilibrium point is

$$J(E_{cov}^0) = \begin{pmatrix} -\mu & 0 & -\frac{\beta\pi}{\mu} & 0 & 0 \\ 0 & -(\alpha + \mu + \xi) & \frac{\beta\pi}{\mu} & 0 & 0 \\ 0 & \alpha & -(\gamma + \rho + \mu + d) & 0 & 0 \\ 0 & 0 & \rho & -(\vartheta + \mu + d) & 0 \\ 0 & \xi & \gamma & \vartheta & -\mu \end{pmatrix} \tag{70}$$

With characteristic equation

$$(\lambda + \mu)^2(\lambda + \vartheta + \mu + d)(A\lambda^2 + B\lambda + C) = 0 \tag{71}$$

where,

$$\left. \begin{aligned} A &= \mu \\ B &= \mu(d + 2\mu + \alpha + \gamma + \xi + \rho) \\ C &= \mu(\alpha + \mu + \xi)(\gamma + \rho + \mu + d) - \alpha\beta\pi \end{aligned} \right\} \tag{72}$$

Solving for λ we get

$$\left. \begin{array}{l} \lambda_1 = -\mu \\ \lambda_2 = -\mu \\ \lambda_3 = -(\mu + \vartheta + d) \end{array} \right\} \quad (73)$$

The roots of the quadratic equation are the second other two eigenvalues.

$$[\mu\lambda^2 + \mu(d + 2\mu + \alpha + \gamma + \xi + \rho)\lambda + \mu(\alpha + \mu + \xi)(\gamma + \rho + \mu + d) - \alpha\beta\pi] = 0 \quad (74)$$

Which can be simplified to

$$\mu\lambda^2 + \mu(d + 2\mu + \alpha + \gamma + \xi + \rho)\lambda + \mu(\alpha + \mu + \xi)(\gamma + \rho + \mu + d) \left(1 - \frac{\alpha\beta\pi}{\mu(\alpha + \mu + \xi)(\gamma + \rho + \mu + d)}\right) = 0 \quad (75)$$

When we substitute the values of β and solve for λ , we have

$$\lambda_4 = -(d + 2\mu + \alpha + \gamma + \xi + \rho) \text{ and } \lambda_5 = 0 \quad (76)$$

Now λ_5 is a simple eigenvalue, so that allows us to use the center manifold theorem. So, we proceed with the computation as follows:

$$W = (w_1, w_2, w_3, w_4, w_5)^T \quad (77)$$

$$\begin{pmatrix} -\mu & 0 & -\frac{\beta\pi}{\mu} & 0 & 0 \\ 0 & -(\alpha + \mu + \xi) & \frac{\beta\pi}{\mu} & 0 & 0 \\ 0 & \alpha & -(\gamma + \rho + \mu + d) & 0 & 0 \\ 0 & 0 & \rho & -(\vartheta + \mu + d) & 0 \\ 0 & \xi & \gamma & \vartheta & -\mu \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (78)$$

The system of equation form from the matrix is

$$\left. \begin{array}{l} -\mu w_1 - \frac{\beta\pi}{\mu} w_3 = 0 \\ -(\alpha + \mu + \xi)w_2 + \frac{\beta\pi}{\mu} w_3 = 0 \\ \alpha w_2 - (\gamma + \rho + \mu + d)w_3 = 0 \\ \rho w_3 - (\vartheta + \mu + d)w_4 = 0 \\ \xi w_2 + \gamma w_3 + \vartheta w_4 - \mu w_5 = 0 \end{array} \right\} \quad (79)$$

Which has solutions

$$W = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} = \begin{pmatrix} -\frac{\beta\pi}{\mu^2} w_3 \\ \frac{\beta\pi w_3}{\mu(\alpha + \mu + \xi)} \\ w_3 = w_3 > 0 \\ \frac{\rho w_3}{(\vartheta + \mu + d)} \\ \frac{(\vartheta + \mu + d)\xi\beta\pi w_3 + \mu(\alpha + \mu + \xi)(\vartheta + \mu + d)\gamma w_3 + \mu(\alpha + \mu + \xi)\vartheta\rho w_3}{\mu^2(\alpha + \mu + \xi)(\vartheta + \mu + d)} \end{pmatrix} \quad (80)$$

The transposition of the matrix gives

$$\begin{pmatrix} -\mu & 0 & 0 & 0 & 0 \\ 0 & -(\alpha + \mu + \xi) & \alpha & 0 & \xi \\ -\frac{\beta\pi}{\mu} & \frac{\beta\pi}{\mu} & -(\gamma + \rho + \mu + d) & \rho & \gamma \\ 0 & 0 & 0 & -(\vartheta + \mu + d) & \vartheta \\ 0 & 0 & 0 & 0 & -\mu \end{pmatrix} \quad (81)$$

The left eigenvectors of $J(E_0)$ associated with the zero eigenvalue at $\beta = \beta^*$ are given by

$$V = (v_1, v_2, v_3, v_4, v_5)^T \quad (82)$$

$$\begin{pmatrix} -\mu & 0 & 0 & 0 & 0 \\ 0 & -(\alpha + \mu + \xi) & \alpha & 0 & \xi \\ -\frac{\beta\pi}{\mu} & \frac{\beta\pi}{\mu} & -(\gamma + \rho + \mu + d) & \rho & \gamma \\ 0 & 0 & 0 & -(\vartheta + \mu + d) & \vartheta \\ 0 & 0 & 0 & 0 & -\mu \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (83)$$

Write the above into a system of equation we have

$$\left. \begin{aligned} -\mu v_1 &= 0 \\ -(\alpha + \mu + \xi)v_2 + \alpha v_3 + \xi v_5 &= 0 \\ -\frac{\beta\pi}{\mu} v_1 + \frac{\beta\pi}{\mu} v_2 - (\gamma + \rho + \mu + d)v_3 + \rho v_4 + \gamma v_5 &= 0 \\ -(\vartheta + \mu + d)v_4 + \vartheta v_5 &= 0 \\ -\mu v_5 &= 0 \end{aligned} \right\} \quad (84)$$

Now solve for v_1, v_2, v_3, v_4, v_5 we get

$$V = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\alpha v_3}{(\alpha + \mu + \xi)} \\ v_3 \\ 0 \\ 0 \end{pmatrix} \quad (85)$$

Next, we calculate the second partial derivatives of the f_i with respect to the x_i and obtain.

$$\frac{\partial^2 f_2}{\partial x_1 \partial x_3} = \frac{\partial^2 f_2}{\partial x_3 \partial x_1} = \beta, \quad \frac{\partial^2 f_2}{\partial x_3 \partial x_4} = \frac{\partial^2 f_2}{\partial x_4 \partial x_3} = (1 - \delta)\beta \quad \text{while the rest of the partial derivatives are zero}$$

After some calculation, we got the expression for \mathcal{A} as

$$\mathcal{A} = \frac{2\alpha\beta v_3 w_3^2}{\mu^2(\vartheta + \mu + d)(\alpha + \mu + \xi)} \mathfrak{S} \quad (86)$$

Where

$$\mathfrak{S} = (\mu^2(1 - \delta)\rho) - \beta\pi(\vartheta + \mu + d) \quad (87)$$

Now we compute the value of \mathcal{B} using the formula

Now let's compute the values of the second partial derivatives of the f_i 's with respect to the x_i 's and β 's to get $\frac{\partial^2 f_2}{\partial x_3 \partial \beta} = \beta x_1 = \frac{\beta\pi}{\mu}$, The rest of the partial derivatives are zero.

Substitute the values of $\frac{\partial^2 f_2}{\partial x_3 \partial \beta}$, \mathcal{V}_2 and \mathcal{W}_3 into the expression for \mathcal{B} , we got

$$\mathcal{B} = \frac{\alpha\beta\pi}{\mu(\alpha+\mu+\xi)} \mathcal{V}_3 \mathcal{W}_3 > 0 \quad (88)$$

We now use the centre manifold theorem as described by Castillo-Chavez and Song in [27].

Theorem 6: consider the following general system of ordinary differential equation with parameter φ .

$$\frac{dx}{dt} = f(x, \varphi), f: \mathbb{R}^n \times \mathbb{R} \text{ and } f \in C^2(\mathbb{R}^n \times \mathbb{R}) \quad (89)$$

Where 0 is an equilibrium point of the system is (*that is* $f(0, \varphi) \equiv 0$ for all φ) and assume:

1. $A = D_x f(0, 0) = \left(\frac{\partial f_i}{\partial x_j}(0, 0) \right)$ is the linearization matrix of the system around the equilibrium point 0 with φ evaluated at 0;
2. All other eigenvalues of A have negative real components except for the simple eigenvalue zero.
3. Matrix A has right eigenvector W and a left eigenvector V corresponding to the zero eigenvalue.

Let $f_{\mathcal{K}}$ be the k^{th} component of f and

$$\mathcal{A} = \sum_{\mathcal{K}, i, j=1}^n \mathcal{V}_{\mathcal{K}} \mathcal{W}_i \mathcal{W}_j \frac{\partial^2 f_{\mathcal{K}}}{\partial x_i \partial x_j}(E_0), \quad \mathcal{B} = \sum_{\mathcal{K}, i=1}^n \mathcal{V}_{\mathcal{K}} \mathcal{W}_i \frac{\partial^2 f_{\mathcal{K}}}{\partial x_i \partial a}(E_0, \beta) \quad (90)$$

Then the local dynamics of the system around the $x = 0$ are totally determined by \mathcal{A} and \mathcal{B} . Particularly,

- 1) $\mathcal{A} > 0, \mathcal{B} > 0$, when $\varphi < 0$ with $||\varphi|| \ll 1$, $(0,0)$ is locally asymptotically stable hence there exists a positive unstable equilibrium. When $0 < \varphi \ll 1$, $(0,0)$ is unstable then there exists a negative and locally asymptotically stable equilibrium.
- 2) $\mathcal{A} < 0, \mathcal{B} < 0$, when $\varphi < 0$ with $||\varphi|| \ll 1$, $(0,0)$ is unstable when $0 < \varphi \ll 1$, $(0,0)$ is locally unstable and there exists a positive unstable equilibrium.
- 3) $\mathcal{A} > 0, \mathcal{B} < 0$, when $\varphi < 0$ with $||\varphi|| \ll 1$, $(0,0)$ is unstable, and there exists locally asymptotically stable equilibrium; when $0 < \varphi \ll 1$, $(0,0)$ is stable and positive unstable equilibrium appears.
- 4) $\mathcal{A} < 0, \mathcal{B} > 0$, when φ changes from negative to positive, $x=0$ changes its stability from stable to unstable. Correspondingly, a negative unstable equilibrium becomes locally asymptotically stable.

Now from our values of \mathcal{A} and \mathcal{B} , \mathcal{B} is positive so whether the bifurcation will be forward bifurcation or backwards at $\beta=\beta^*$ depends on the values of \mathcal{A} and in turn depends on the values of \mathfrak{J} . If the value of \mathfrak{J} is negative we will have forward bifurcation while if \mathfrak{J} is positive we will have backward bifurcation.

(a) We have forward bifurcation if

$$(\mu^2(1 - \delta)\rho) - \beta\pi(\vartheta + \mu + d) < 0 \text{ or } \mu^2(1 - \delta)\rho < \beta\pi(\vartheta + \mu + d) \quad (91)$$

(b) We have backward bifurcation if

$$(\mu^2(1 - \delta)\rho) - \beta\pi(\vartheta + \mu + d) > 0 \text{ or } (\mu^2(1 - \delta)\rho) > \beta\pi(\vartheta + \mu + d) \quad (92)$$

3.4.4 Global Stability Analysis of the Drug Equilibrium

Theorem: If $R_{drug} > 1$, the endemic equilibrium E_{drug}^0 of the model is globally asymptotically stable.

Proof: By Lyapunov's direct method and LaSale's Invariant principle, we prove the above theorem by defining a Lyapunov's function

$$L(S^*, E^*, I^*, R^*) = (S - S^* - \ln \frac{S^*}{S}) + (E - E^* - E^* \ln \frac{E^*}{E}) + (D - D^* - D^* \ln \frac{D^*}{D}) + (P - P^* - P^* \ln \frac{P^*}{P}) + (R - R^* - R^* \ln \frac{R^*}{R}) \quad (93)$$

Differentiating L with respect to t produced

$$\frac{dL}{dt} = \frac{(S-S^*)}{S} \frac{dS}{dt} + \frac{(E-E^*)}{E} \frac{dE}{dt} + \frac{(D-D^*)}{D} \frac{dD}{dt} + \frac{(P-P^*)}{P} \frac{dP}{dt} + \frac{(R-R^*)}{R} \frac{dR}{dt} \quad (94)$$

Substitute the values of $\frac{dS}{dt}, \frac{dE}{dt}, \frac{dD}{dt}, \frac{dP}{dt}$ and $\frac{dR}{dt}$ into $\frac{dL}{dt}$ and then simplify to get

$$\begin{aligned} \frac{dL}{dt} = & (\pi + S^* \beta D + S^* \mu + E^* (\alpha + \mu + \xi) + D^* (\gamma + \rho + \mu + d) + P^* (\vartheta + \mu + d) + \vartheta P + R^* (\mu + (1 - \\ & \delta) \beta D)) - \left(\mu S + \frac{S^*}{S} \pi + \mu E + \frac{E^*}{E} \beta S D + \frac{E^*}{E} (1 - \delta) \beta D R + \mu D + d D + \frac{D^*}{D} \alpha E + (\vartheta + \mu + d) P + \right. \\ & \left. \frac{P^*}{P} \rho D + \mu R + \frac{R^*}{R} \xi E + \frac{R^*}{R} \gamma D + \frac{R^*}{R} \vartheta P \right) \end{aligned} \quad (95)$$

Which of the form

$$\frac{dL}{dt} = \mathfrak{S}_1 - \mathfrak{S}_2 \quad (96)$$

Where,

$$\mathfrak{S}_1 = \pi + S^* \beta D + S^* \mu + E^* (\alpha + \mu + \xi) + D^* (\gamma + \rho + \mu + d) + P^* (\vartheta + \mu + d) + \vartheta P + R^* (\mu + (1 - \delta) \beta D) \quad (97)$$

$$\mathfrak{S}_2 = \mu S + \frac{S^*}{S} \pi + \mu E + \frac{E^*}{E} \beta S D + \frac{E^*}{E} (1 - \delta) \beta D R + \mu D + d D + \frac{D^*}{D} \alpha E + (\vartheta + \mu + d) P + \frac{P^*}{P} \rho D + \mu R + \frac{R^*}{R} \xi E + \frac{R^*}{R} \gamma D + \frac{R^*}{R} \vartheta P \quad (98)$$

$$\frac{dL}{dt} \leq 0 \text{ if } \mathfrak{S}_1 \text{ is less than } \mathfrak{S}_2$$

$$\frac{dL}{dt} = 0 \text{ if and only if } S = S^*, E = E^*, D = D^*, P = P^*, R = R^*$$

Thus, the largest invariant impact invariant set in $\{(S^*, E^*, D^*, P^*, R^*) \in \Omega: \frac{dL}{dt} = 0\}$ is the singleton set E_{drug}^* , where E_{drug}^* is the drug equilibrium of the system (1). Therefore, by Lasalle's Invariant principle, implies that E_{drug}^* is globally asymptotically stable in Ω if \mathfrak{S}_1 is less than \mathfrak{S}_2 .

4 NUMERICAL SIMULATION

Table 4: The parameters' values and their sources

S/no	Parameter	Parameter Value	Source
1	π	0.0057	Assumed
2	μ	0.15	Assumed
3	β	0.5	[35]
4	d	0.08	[35]
5	ξ	0.25	Assumed
6	α	0.0012502	[36]
7	ρ	0.07	[35]
8	γ	0.3	[34]

4.1 Effect of varying the recycling parameter on the direction of the bifurcation diagram

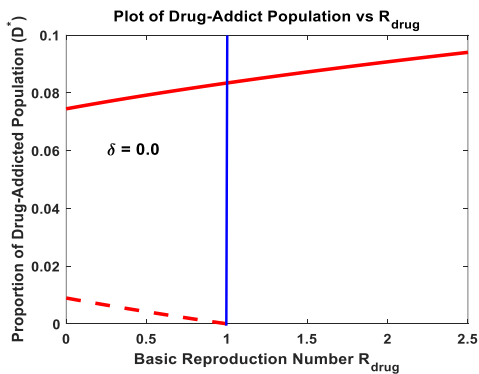


Figure 4a: Proportion of addicted population at $\delta = 0$

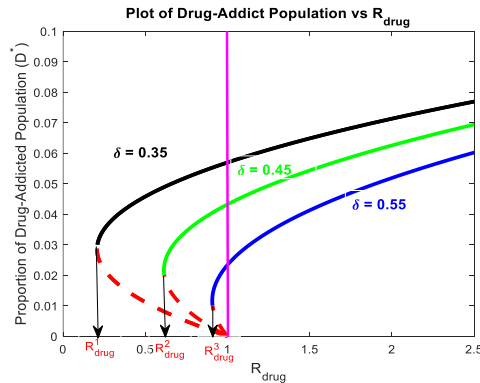


Figure 4b: Proportion of addicted population at $\delta = 0.35 - 0.55$

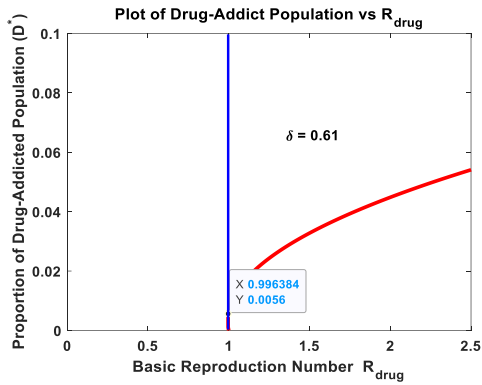


Figure 4c: Proportion of addicted population at $\delta = 0.61$

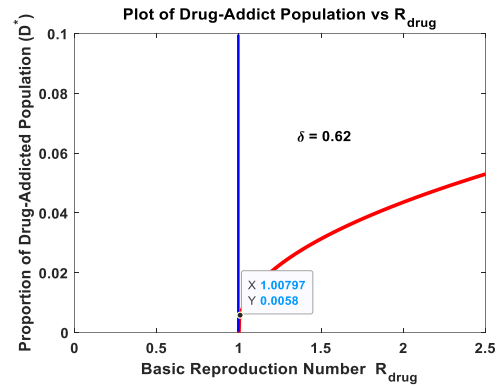


Figure 4d: Proportion of addicted population at $\delta = 0.62$

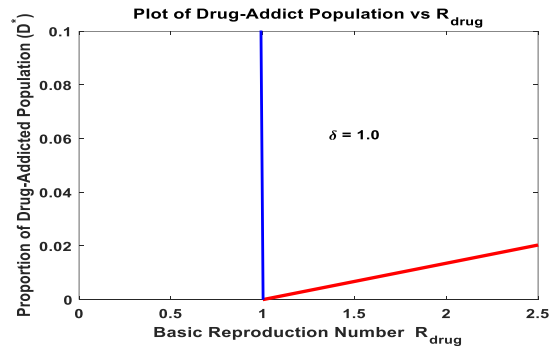


Figure 4d: Proportion of addicted population at $\delta = 1.0$

4.2 Plot of Drug Addicted Population versus Time with varying parameters values of the model

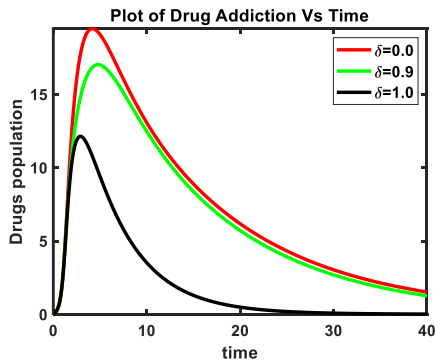


Figure 5a.

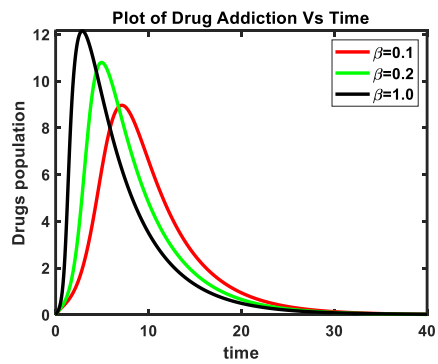


Figure 5b.

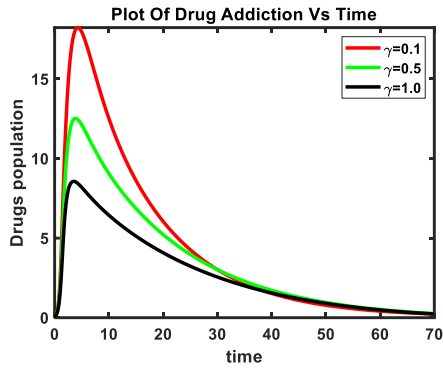


Figure 5c.

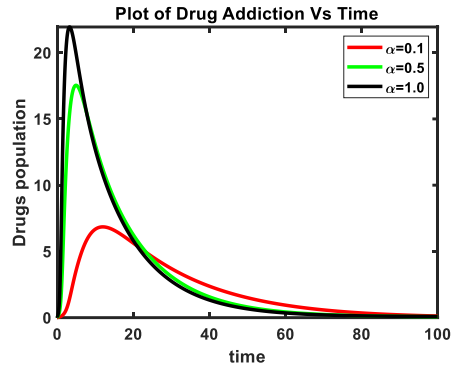


Figure 5d.

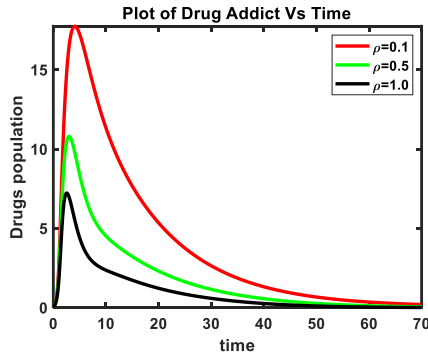


Figure 5e.

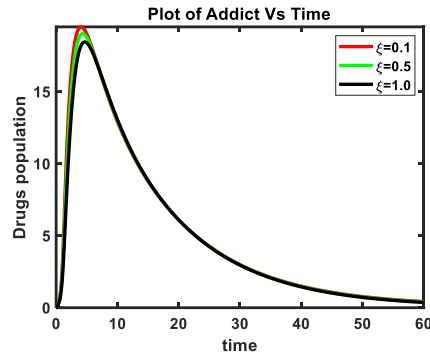


Figure 5f.

5 DISCUSSIONS OF RESULT AND CONCLUSION

At $\delta = 0.0$, the contact rate is equal to the recycling rate of drug addicts into the population, therefore it will be impossible to eradicate the drug addicts from the population as can be seen in Figure 4a. When the value of R_{drug} lies between 0 and 1, there exists a critical value R_{drug}^c such that the value of R_{drug} must be controlled to go below the critical value R_{drug}^c before the drug addicts can be eradicated. For example, when $\delta = 0.35$, there exists a critical value $R_{drug}^1 = 0.208535$ such that the drug addicts' population can be eradicated only if $0 < R_{drug} \leq R_{drug}^1$, if $\delta = 0.45$, the critical point increases to $R_{drug}^2 = 0.613429$. Therefore, the value of R_{drug} must be below 0.613429 before the drug addict population can be eliminated in society. When the value of $\delta = 0.55$ the value of the critical point increases to 0.911628 and the drug addict's population can be removed from the society if $R_{drug} < 0.911628$. The reason for the existence of R_{drug}^c is because there exists a bi-stability. That is the existence of two stable equilibria between 0 and 1, one is a stable endemic equilibrium and a stable drug addicts-free equilibrium. The last value for R_{drug}^c is $R_{drug}^c = 0.996934$ when the value of $\delta = 0.61$. (See Figure 4c). The type of bifurcation direction for $\delta = (0,0.61)$ is a backward bifurcation. When the $\delta = 0.62$, then $R_{drug}^c = R_{drug} = 1$ and the model exhibits a forward

bifurcation for all values of $0.62 \leq \delta \leq 1$ and $R_{drug} > 1$ (see Figure 4d and 4e). In this case, it is easy to eradicate drug usage in the population by controlling the values of R_{drug} below 1. Note that since the recycling rate of the drug addict population is $(1 - \delta)\beta$ and $\delta = [0,1]$, this means that as the values of δ increase from 0 to 1, the recycling rate of the drug addicts' population decreases from 0.9587 to 0. Hence when $\delta = 1$, there is no recycling of drug addicts in the population while if $\delta = 0$, all the drug addicts that have recovered go back into their drug addict's lifestyle. At this point, the rate at which people are exposed to drugs and the rate at which recovered people go back to drug use is the same.

Next, we analyze the effect of the parameters' value of the model on the drug addict's population over time, figure 5a shows the plot of drug addicts' population over time with varying values of δ . The results on the graph show that as the values of δ increase from 0 to 1, the drug addict population decreases over time. Note that an increase in the value of δ signifies a decrease in the population of drug addicts. Therefore, the recycling rate of the addicted population needs to be prevented to reduce the population of drug addicts in the population. Figure 5b and 5c show the plots of contact rate and transition rate from exposure to drug addicts respectively. The results reveal that an increase in contact rate and transition rate increases the population of the drug addicts, therefore, to prevent the growth rate of the drug addict's population we need to prevent the contact rate and treat the exposed population. Figures 5d, 5e and 5e show the effect of recovery rate, rate of rehabilitation and recovery rate of the exposed class respectively, in each case, the graph shows a decrease in the drug addicts' population. Therefore, to reduce the drug addict population, we need to increase the recovery rate of drug addicts, the rate of rehabilitation and the rate of recovery of the exposed class.

Figure 2 above illustrates a bar chart depicting the sensitivity indices of the parameters. Parameters with bars pointing upward, that is α , β and π , need to be reduced to eliminate drug usage in society. This is because these parameters are increasing the function of R_{drug} . That is the rate of change of R_{drug} with respect to these parameters is increasing functions as shown in the equations below.

$$\frac{dR_{drug}}{d\alpha} = \frac{(\mu+\xi)R_{drug}}{\alpha(\alpha+\mu+\xi)}, \frac{dR_{drug}}{d\beta} = \frac{R_{drug}}{\beta} \text{ and } \frac{dR_{drug}}{d\pi} = \frac{R_{drug}}{\pi} \quad (99)$$

while those with bars pointing downward, that is ξ , ρ and γ need to be raised to eliminate the drug abuse habit from the society. This is because the derivative of R_{drug} with respect to these parameters is decreasing functions. This means that increasing these parameters will decrease the value of R_{drug} and therefore eliminate the population of drug addicts in society.

$$\begin{aligned} \frac{dR_{drug}}{d\xi} &= -\frac{R_{drug}}{(\alpha+\mu+\xi)}, \frac{dR_{drug}}{d\xi} = -\frac{R_{drug}}{(\mu+\gamma+\rho+d)}, \frac{dR_{drug}}{d\rho} = -\frac{R_{drug}}{(\mu+\gamma+\rho+d)} \text{ and} \\ \frac{dR_{drug}}{d\mu} &= -\frac{R_{drug}}{\mu(\alpha+\mu+\xi)(\mu+\gamma+\rho+d)} (d\alpha + 2d\mu + d\xi + 3\mu^2 + \alpha\gamma + 2\alpha\mu + 2\gamma\mu + \gamma\xi + 2\mu\xi + \alpha\rho + 2\mu\rho + \xi\rho) \end{aligned} \quad (100)$$

However, even though the bars of μ and d are pointing downward (that they have negative sensitivity indices), it is biologically unreasonable to propose increasing the natural death and Drug-induced death rates to eradicate the habit as our objective is to save lives, not lose them.

Ultimately, this research contributes to the ongoing endeavour to tackle the multifaceted problem of illicit drug use. The insights gained here can inform evidence-based policies and interventions, promoting community well-being and assisting in curbing the spread of drug abuse.

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