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Numerical Technique of Solving Boundary Delay Differential Equations

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ABSTRACT

The two-points direct multistep block approach order four is used to approximate the solutions for second order delay differential equations of the constant type with boundary conditions. Direct integration is suggested since reducing the second order equations to first-order equations will require longer computational time. The two-points approximate solutions for each iteration are calculated simultaneously in the block method can minimize computational time. In contrast to the one-step approach, the multistep method frequently proves to be efficient in decreasing the frequency of function calls. Since the concerns involved boundary values, the shooting method is used to determine the guessing of the initial value. The representation of the stability regions is used to analyze the method's stability. The method's reliability is observed through four numerical problems.

Keywords: Boundary Value Problems, Block Method, Delay Differential Equations, Direct Method, Shooting Method.

1 INTRODUCTION

Numerous engineering and science problems use mathematical models based on the Delay Differential Equations (DDEs). Various aspects of real life situations include delays, for example in signal processing, epidemics of dengue fever and physiological processes [1]. The DDEs of second order can generally be shown as below:

$$y'' = f\big(t, y(t), y'(t), y(t-\tau), y'(t-\tau)\big), t \in [a, b] \tag{1}$$

$$y(t) = \phi(t), \quad t \in [a-\tau, a], \quad \tau \in \Re^{+} \tag{2}$$

with the boundary conditions as the following:

$$y(a) = \alpha, y(b) = \beta \tag{3}$$

where $\alpha = \phi(a)$, τ is the delay term, which is a positive constant and $\phi(t)$ is the continuous function. The dependent variable of Ordinary Differential Equations (ODEs) is being considered in the current time, while the dependent variable of DDEs is considered in both the current and past time. Since the 1970s, several researchers have worked numerically and analytically to solve the constant delay of second order DDEs subject to the boundary conditions. Recently, however, very few studies were made, so this paper became our interests. In 1971, Nevers and Schmitt [2] were the first researchers applied the shooting strategy to tackle this problem and iterated by using the Euler's method. Afterward, Cryer [3] solved the same problem as [2] by using the method of successive approximation. Reddien and Travis [4] then solved the same problem as [2] by using the collocation and Galerkin methods based on the projection type to the Boundary Value Problems (BVPs). Sakai [5] implemented the cubic splines method to solve two DDEs problems while Reddien [6] suggested the midpoint difference method for solving BVPs of the second order DDEs. Other than that, Bellen and Zennaro [7] studied two DDEs problems with the global and piecewise polynomials for the collocation method. Then, by implementing the finite difference approach, Agarwal and Chow [8] addressed the constant and time dependent delay problems while Bakke and Jackiewicz [9] used the central finite difference and Richardson extrapolation method. Besides that, Qu and Agarwal [10] also solved both constant and time dependent delay problems by applying the subdivision technique in the approximation of basis function for the collocation approach.

Several research investigations have been conducted to address the singular perturbation issues related to second order DDEs with constant delay. The following represents the standard form of second order DDEs with singular perturbation:

$$f(t, y, y', y'') = \epsilon y''(t) + a(t)y(t) + b(t)y'(t - \tau), t \in [a, b] \tag{4}$$

with the initial function is given as (2), and the boundary conditions as (3). The functions $a(t)$ and $b(t)$ are continuous functions with ϵ is a small parameter, which is $\epsilon \in (0, 1)$.

The past studies for solving (4) numerically were made by a few researchers. The first and second derivatives were approximated by Kadalbajoo and Sharma [11] by using the forward and central differences respectively to generate a three point scheme and solved by using a discrete invariant embedding algorithm. Andargie and Reddy [12] proposed a fitting parameter to the modified problem's highest order derivative term. Challa and Reddy [13] employing an exponential integrating factor into the first order neutral type DDEs obtained whereas Kanth and Murali [14] implemented the quasilinearization method and exponentially fitted spline method.

This paper presents two points direct multistep block technique of order four for calculating the solutions of second order DDEs with constant delays and singular perturbations. Compared to the other two studies by [15] and [16], where [16] used three points multistep block method of order 5 to resolve the second order DDEs with singular perturbations. This study additionally helps to prove the capability of the multistep method in solving the mentioned problems. On the other hand, [15] is more concerned with the fifth order of two point direct multistep block method for solving second order DDEs with constant and pantograph delay. The simulation of these two different previous methods has shown the ability of the multistep method to handle different types of delays that can model various scientific and engineering fields.

2 DERIVATION OF THE METHOD

A sequence of blocks is obtained after the division of the interval $[a, b]$ as illustrated in Figure 1. The two points in each block, y_{i+1} and y_{i+2}, were solved at the same time at the nodes t_{i+1} and t_{i+2}, re-

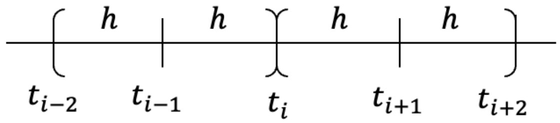

Figure 1 : Two-Point Block Method

spectively. Each block is selected such that it has the interpolation points in diagonally form of order four, which is $\{t_{i+1}, t_i, t_{i-1}, t_{i-2}\}$ and $\{t_{i+2}, t_{i+1}, t_i, t_{i-1}\}$ for the first and second points, respectively. The following second order of general ODEs as shown below:

$$y'' = f(t, y, y').$$

The once and twice integration of y'' are obtained for the first and second points along the interval $[t_i, t_{i+1}]$ and $[t_i, t_{i+2}]$ respectively.

First point:
Integrate once:

$$\int_{t_i}^{t_{i+1}} y'' dt = \int_{t_i}^{t_{i+1}} f(t, y, y') dt$$

$$y'_{i+1} = y'_i + \int_{t_i}^{t_{i+1}} f(t, y, y') dt. \tag{5}$$

Integrate twice:

$$\int_{t_i}^{t_{i+1}} \int_{t_i}^{t} y'' dt dt = \int_{t_i}^{t_{i+1}} \int_{t_i}^{t} f(t, y, y') dt dt.$$

After using integration by parts, then

$$y_{i+1} = y_i + hy'_i + \int_{t_i}^{t_{i+1}} (t_{i+1} - t) f(t, y, y') dt. \tag{6}$$

Second point:
Integrate once:

$$\int_{t_i}^{t_{i+2}} y'' dt = \int_{t_i}^{t_{i+2}} f(t, y, y') dt$$

$$y'_{i+2} = y'_i + \int_{t_i}^{t_{i+2}} f(t, y, y') dt. \tag{7}$$

Integrate twice:

$$\int_{t_i}^{t_{i+2}}\int_{t_i}^{t} y'' dtdt = \int_{t_i}^{t_{i+2}}\int_{t_i}^{t} f(t,y,y')dtdt.$$

After using integration by parts, then

$$y_{i+2} = y_i + 2hy_i' + \int_{t_i}^{t_{i+2}} (t_{i+2} - t) f(t,y,y')dt \tag{8}$$

where $f(t,y,y')$ in (5)-(6) is approximated by the Lagrange polynomial of degree 3 for the first point, $P_{1,3}(t)$ as the following:

$$\begin{aligned} P_{1,3}(t) = & \frac{(t-t_i)(t-t_{i-1})(t-t_{i-2})}{(t_{i+1}-t_i)(t_{i+1}-t_{i-1})(t_{i+1}-t_{i-2})} f_{i+1} \\ & + \frac{(t-t_{i+1})(t-t_{i-1})(t-t_{i-2})}{(t_i-t_{i+1})(t_i-t_{i-1})(t_i-t_{i-2})} f_i \\ & + \frac{(t-t_{i+1})(t-t_i)(t-t_{i-2})}{(t_{i-1}-t_{i+1})(t_{i-1}-t_i)(t_{i-1}-t_{i-2})} f_{i-1} \\ & + \frac{(t-t_{i+1})(t-t_i)(t-t_{i-1})}{(t_{i-2}-t_{i+1})(t_{i-2}-t_i)(t_{i-2}-t_{i-1})} f_{i-2} \end{aligned} \tag{9}$$

while $f(t,y,y')$ in (7)-(8) is approximated by the Lagrange polynomial of degree 3 for the second point, $P_{2,3}(t)$ as the following:

$$\begin{aligned} P_{2,3}(t) = & \frac{(t-t_{i+1})(t-t_i)(t-t_{i-1})}{(t_{i+2}-t_{i+1})(t_{i+2}-t_i)(t_{i+2}-t_{i-1})} f_{i+2} \\ & + \frac{(t-t_{i+2})(t-t_i)(t-t_{i-1})}{(t_{i+1}-t_{i+2})(t_{i+1}-t_i)(t_{i+1}-t_{i-1})} f_{i+1} \\ & + \frac{(t-t_{i+2})(t-t_{i+1})(t-t_{i-1})}{(t_i-t_{i+2})(t_i-t_{i+1})(t_i-t_{i-1})} f_i \\ & + \frac{(t-t_{i+2})(t-t_{i+1})(t-t_i)}{(t_{i-1}-t_{i+2})(t_{i-1}-t_{i+1})(t_{i-1}-t_i)} f_{i-1}. \end{aligned} \tag{10}$$

Substitute $t = t_{i+2} + sh$ in (9) and (10) while $dt = hds$ in (5)-(8). The integration limit at the first point is substituted from -2 to -1 while at the second point; the limit is -2 to 0. Thus, the corrector formula of order four for the two-point block of direct approach (2PBM4) is obtained as below:

$$\begin{aligned} y'_{i+1} &= y_i' + \frac{h}{24}(9f_{i+1} + 19f_i - 5f_{i-1} + f_{i-2}) \\ y_{i+1} &= y_i + hy_i' + \frac{h^2}{360}(38f_{i+1} + 171f_i - 36f_{i-1} + 7f_{i-2}) \\ y'_{i+2} &= y_i' + \frac{h}{3}(f_{i+2} + 4f_{i+1} + f_i) \\ y_{i+2} &= y_i + 2hy_i' + \frac{h^2}{45}(2f_{i+2} + 54f_{i+1} + 36f_i - 2f_{i-1}). \end{aligned} \tag{11}$$

The method follows a predictor-corrector approach with predictor formula is one order less than the corrector formula.

3 ANALYSIS OF METHOD

3.1 Order of Method

Introducing the general linear k-steps method for second order general ODEs may be expressed below:

$$\sum_{j=0}^{k} \alpha_j y_{i+j} = h \sum_{j=0}^{k} \beta_j y'_{i+j} + h^2 \sum_{j=0}^{k} \gamma_j f_{i+j} \tag{12}$$

where the characteristic polynomial, ρ, σ and ω are given as:

$$\rho(R) = \sum_{j=0}^{k} \alpha_j R^j, \quad \sigma(R) = \sum_{j=0}^{k} \beta_j R^j, \quad \omega(R) = \sum_{j=0}^{k} \gamma_j R^j \tag{13}$$

where $R \in \mathbb{C}$. The proposed method, 2PBM4 can also be illustrated by using the matrix difference equation as the following:

$$\alpha_j Y_M = h\beta_j Y'_M + h^2 \gamma_j F_M$$

where

$$Y_M = \begin{bmatrix} y_{i-2} \\ y_{i-1} \\ y_i \\ y_{i+1} \\ y_{i+2} \end{bmatrix}, Y'_M = \begin{bmatrix} y'_{i-2} \\ y'_{i-1} \\ y'_i \\ y'_{i+1} \\ y'_{i+2} \end{bmatrix}, F_M = \begin{bmatrix} f_{i-2} \\ f_{i-1} \\ f_i \\ f_{i+1} \\ f_{i+2} \end{bmatrix}.$$

Linear difference operator L of the method introduced in Lambert [17] is as follows:

$$L[y(t);h] = \sum_{j=0}^{k} \left[\alpha_j y(t+jh) - h\beta_j y'(t+jh) - h^2 \gamma_j y''(t+jh)\right] \tag{14}$$

where $y(t)$ is continuously differentiable function on $[a,b]$. The function $y(t+jh)$ and its derivatives are expanded with Taylor series expansion, then, after the arrangement of terms in (14) eventually gives as below

$$L[y(t);h] = C_0 y(t) + C_1 h y^{(1)}(t) + \dots + C_q h^q y^{(q)}(t) + \dots,$$

with the constants C_q are given as:

$$\begin{aligned}
C_0 &= \alpha_0 + \alpha_1 + \alpha_2 + \dots + \alpha_s, \\
C_1 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + \dots + s\alpha_s - (\beta_0 + \beta_1 + \beta_2 + \dots + \beta_s), \\
C_2 &= \frac{1}{2!}(\alpha_1 + 2^2\alpha_2 + 3^2\alpha_3 + \dots + s^2\alpha_s) - (\beta_1 + 2\beta_2 + 3\beta_3 + \dots + s\beta_s) \\
&\quad - (\gamma_0 + \gamma_1 + \gamma_2 + \dots + \gamma_s), \\
&\ \vdots \\
C_q &= \frac{1}{q!}(\alpha_1 + 2^q\alpha_2 + 3^q\alpha_3 + \dots + s^q\alpha_s) - \frac{1}{(q-1)!}(\beta_1 + 2^{q-1}\beta_2 + 3^{q-1}\beta_3 + \dots + s^{q-1}\beta_s) \\
&\quad - \frac{1}{(q-2)!}(\gamma_1 + 2^{q-2}\gamma_2 + 3^{q-2}\gamma_3 + \dots + s^{q-2}\gamma_s), \\
&\quad q = 3, 4, 5, \dots .
\end{aligned} \tag{15}$$

The corrector formula (11) then portrayed in the form of a matrix below:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{i-2} \\ y_{i-1} \\ y_i \\ y_{i+1} \\ y_{i+2} \end{bmatrix} = h \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} y'_{i-2} \\ y'_{i-1} \\ y'_i \\ y'_{i+1} \\ y'_{i+2} \end{bmatrix}$$

$$+ h^2 \begin{bmatrix} \frac{1}{24} & -\frac{5}{24} & \frac{19}{24} & \frac{3}{8} & 0 \\ \frac{7}{360} & -\frac{1}{10} & \frac{19}{40} & \frac{19}{180} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ 0 & -\frac{2}{45} & \frac{4}{5} & \frac{6}{5} & \frac{2}{45} \end{bmatrix} \begin{bmatrix} f_{i-2} \\ f_{i-1} \\ f_i \\ f_{i+1} \\ f_{i+2} \end{bmatrix}.$$

The order of the method is calculated according to formulation in (15) as follows:

$$\begin{aligned} C_0 &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T \\ C_1 &= 2\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \\ &- \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T \\ C_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T \\ C_3 &= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T \\ C_4 &= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T \\ C_5 &= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T \\ C_6 &= \begin{bmatrix} \frac{73}{144} & \frac{751}{1440} & \frac{47}{90} & \frac{97}{90} \end{bmatrix}^T \neq \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T. \end{aligned} \tag{16}$$

According to Fatunla [18] and Lambert [17], the linear multistep method (12) achieved order p when $C_0 = C_1 = C_2 = \ldots = C_{p+1} = 0$, but $C_{p+2} \neq 0$. The error constant of the method is the first non vanishing coefficient, C_{p+2}. Considering $C_6 \neq 0$ hence $p = 4$ making our method, 2PBM4 have order four.

3.2 Stability of Method

Two types of stability are going to be presented; zero stability, which considers only stability at the limit of $h \to 0$ and the stability theory that concerns only for step size h is a non-zero constant.

The suggested approach, 2PBM4, has zero stability if the first characteristic polynomial is specified as $\rho(R) = det\left[\sum_{i=0}^{k} A^{(i)} R^{k-i}\right] = 0$ then its roots R_j will fulfill the condition $|R_j| \le 1$.

The matrix form of the corrector formula for 2PBM4 in (11) is presented below:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y'_{i+1} \\ y_{i+1} \\ y'_{i+2} \\ y_{i+2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y'_{i-1} \\ y_{i-1} \\ y'_{i} \\ y_{i} \end{bmatrix} + h \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} y'_{i-1} \\ y_{i-1} \\ y'_{i} \\ y_{i} \end{bmatrix}$$

$$+h \begin{bmatrix} -\frac{5}{24} & \frac{19}{24} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{i-1} \\ f_{i} \\ f_{i+1} \\ f_{i+2} \end{bmatrix} + h \begin{bmatrix} 0 & 0 & 0 & \frac{1}{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{i-5} \\ f_{i-4} \\ f_{i-3} \\ f_{i-2} \end{bmatrix}$$

$$+h^2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{10} & \frac{19}{40} & \frac{19}{180} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{2}{45} & \frac{4}{5} & \frac{6}{5} & \frac{2}{45} \end{bmatrix} \begin{bmatrix} f_{i-1} \\ f_{i} \\ f_{i+1} \\ f_{i+2} \end{bmatrix} + h^2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{7}{360} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{i-5} \\ f_{i-4} \\ f_{i-3} \\ f_{i-2} \end{bmatrix}.$$

Thus, the $\rho(R)$ of 2PBM4 for $k = 1$ is given by:

$$\rho(R) = det\left[\sum_{i=0}^{k} A_{(i)} R^{k-i}\right] = det\left[A_0 R^1 + A_1\right] = 0$$

where $A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

$$\rho(R) = det \begin{bmatrix} R & 0 & 1 & 0 \\ 0 & R & 0 & 1 \\ 0 & 0 & R+1 & 0 \\ 0 & 0 & 0 & R+1 \end{bmatrix} = 0$$

$$R^2 (R+1)^2 = 0$$

$$R = 0, 0, -1, -1.$$

Therefore, the suggested approach is zero stable provided that $|R_j| \le 1$.

Now, the stability theory needed to analyze DDEs of the second order (1) apply the linear test equation for stability as shown below:

$$f = ay(t) + by(t-\tau) + cy'(t-\tau). \tag{17}$$

The matrix form of 2PBM4 formula in (11) is as below:

$$A_0Y_{M+1} = A_1Y_M + hB_1Y_M + h\sum_{j=0}^{1} C_{j+1}F_{M+j} + h^2\sum_{j=0}^{1} D_{j+1}F_{M+j} \tag{18}$$
$$A_0Y_{M+1} = A_1Y_M + hB_1Y_M + hC_1F_M + hC_2F_{M+1} + h^2D_1F_M + h^2D_2F_{M+1}$$

where:

$$Y_{M+1} = \begin{bmatrix} y'_{i+1} \\ y_{i+1} \\ y'_{i+2} \\ y_{i+2} \end{bmatrix}, Y_M = \begin{bmatrix} y'_{i-1} \\ y_{i-1} \\ y'_{i} \\ y_{i} \end{bmatrix}, F_{M+1} = \begin{bmatrix} f_{i-1} \\ f_{i} \\ f_{i+1} \\ f_{i+2} \end{bmatrix}, F_M = \begin{bmatrix} f_{i-5} \\ f_{i-4} \\ f_{i-3} \\ f_{i-2} \end{bmatrix},$$

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} -\frac{5}{24} & \frac{19}{24} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}, C_1 = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{10} & \frac{19}{40} & \frac{19}{180} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{2}{45} & \frac{4}{5} & \frac{6}{5} & \frac{2}{45} \end{bmatrix}, D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{7}{360} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Substituting test equation (17) into (18) where:

$$F_M = aY_M + bY_{M-m} + cY'_{M-m},$$
$$F_{M+1} = aY_{M+1} + bY_{M+1-m} + cY'_{M+1-m}.$$

After substituting linear multistep formula as below:

$$\sum_{j=0}^{1} A_{j+1}Y_{M+j} = h\sum_{j=0}^{1} C_{j+1}Y'_{M+j-m} + h^2\sum_{j=0}^{1} D_{j+1}F_{M+j-m}$$

and rearranging the equation while assuming $m = 1$, we finally obtain the following:

$$(A_0 - haC_2)Y_{M+1} - (A_1 + hB_1 + haC_1 + C_2 + D_2 + hbC_2)Y_M - (C_1 + D_1 + hbC_1) = 0. \tag{19}$$

Let $H_1 = ha$ and $H_2 = hb$.
Thus, the characteristic polynomial of equation (19) becomes:

$$(A_0 - H_1C_2)x^2 - (A_1 + hB_1 + H_1C_1 + C_2 + D_2 + H_2C_2)x - (C_1 + D_1 + H_2C_1) = 0. \tag{20}$$

After computing the determinant of equation (20), the stability polynomial obtained is as follows:

$$\begin{aligned}
&-\frac{20071}{97200}x^4H_2H_1-\frac{7}{6480}x^4H_1^2H_2-\frac{52457}{97200}x^5H_1H_2-\frac{7}{12960}x^3H_2^2H_1+\frac{34597}{129600}x^4H_2\\
&-\frac{27203}{129600}x^5H_1+\frac{42193}{48600}x^5H_2-\frac{7}{12960}x^3H_1+\frac{41}{48600}x^3H_2+\frac{11}{48600}x^3H_2^2-\frac{21211}{194400}x^4H_1\\
&-\frac{7}{6480}x^3H_2H_1-\frac{7}{12960}x^5H_1^3-\frac{7}{6480}x^4H_1^2-\frac{9}{8}x^7H_2+\frac{5093}{8640}x^6H_2-\frac{5}{18}x^6H_2^2\\
&-\frac{419}{4320}x^6H_1^3-\frac{5}{18}x^8H_1^2-\frac{9799}{12960}x^7H_1^2-\frac{5}{24}x^7H_1^3+\frac{3161}{12960}x^5H_2^2-\frac{9}{8}x^8H_1+\frac{5093}{8640}x^7H_1\\
&-\frac{6407}{48600}x^6H_1-\frac{20093}{97200}x^5H_1^2-\frac{119057}{194400}x^6H_1^2+\frac{14143}{194400}x^4H_2^2+x^8+\frac{1}{1620}x^3+\frac{81553}{388800}x^4\\
&+\frac{89977}{43200}x^6+\frac{177409}{194400}x^5-\frac{164}{45}x^7-\frac{3319}{6480}x^6H_1H_2-\frac{5}{9}x^7H_1H_2-\frac{419}{4320}x^4H_2^2H_1\\
&-\frac{419}{2160}x^5H_1^2H_2-\frac{5}{12}x^6H_1^2H_2-\frac{5}{24}x^5H_2^2H_1=0.
\end{aligned} \tag{21}$$

Next, the stability regions are established in $(H_1 - H_2)$ plane by replacing the values of $x = 1, -1$ and $x = cos\theta + isin\theta$, $0 \leq \theta \leq 2\pi$ in the stability polynomial (21). To find the points in the region when $x = cos\theta + isin\theta$, the real and imaginary parts are solved separately. The absolute stability is defined as the set of all roots in the stability polynomial (21) that specifying $|x| \leq 1$ and lying inside the region's border. Figure 2 depicts the 2PBM4's stability region.

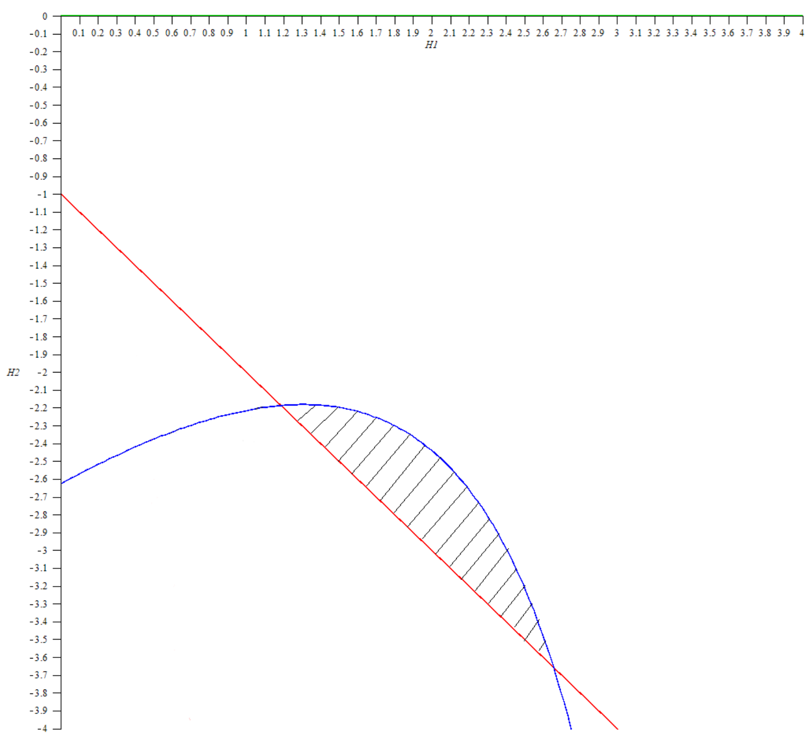

Figure 2 : Stability region for 2PBM4

3.3 Consistency of Method

The general linear multistep method (12) is said to be consistent when the local truncation error (LTE) approaching zero as the step size of the method, h approaching zero [1].

Definition 1. *The LTE at t_{i+k} of the multistep method* (12) *is interpreted to be the operator $L[y(t_i);h]$ in* (14) *when $y(t) \in C^2[a,b]$ is the theoretical solution of the problems for second order ODEs, $y'' = f(t_i, y_i, y_i')$.*

Now, introducing the localizing assumption, which is the truncation error, T_{i+k} is assumed to be local if there is no truncation error obtained previously. Thus, assume that:

$$y_{i+j} = y(t_{i+j}), \quad j = 0, 1, \ldots, k-1$$

$$y'_{i+j} = y'(t_{i+j}), \quad j = 0, 1, \ldots, k-1$$

where y_{i+k} is the approximate solution of the method (12) at t_{i+k}.
From (14):

$$\sum_{j=0}^{k} \alpha_j y(t_i + jh) = h \sum_{j=0}^{k} \beta_j y'(t_i + jh) + h^2 \sum_{j=0}^{k} \gamma_j f\big(t_i + jh, y(t_{i+j}), y'(t_{i+j})\big) + L[y(t_i); h] \tag{22}$$

where $y(t)$ is considered to be the theoretical solution of the problems. The approximate solution, y_{i+k} of the method (12) satisfies:

$$y_{i+k} + \sum_{j=0}^{k-1} \alpha_j y_{i+j} = h \sum_{j=0}^{k-1} \beta_j y'_{i+j} + h^2 \sum_{j=0}^{k-1} \gamma_j f(t_{i+j}, y_{i+j}, y'_{i+j}) + h\beta_k y'_{i+k} + h^2 \gamma_k f(t_{i+k}, y_{i+k}, y'_{i+k}) \tag{23}$$

where $\alpha_k = 1$. Subtracting (23) from (22) under the consideration of the localizing assumption, eventually gives:

$$y(t_{i+k}) - y_{i+k} = h\beta_k[y'(t_{i+k}) - y'_{i+k}] + h^2\gamma_k\big[f\big(t_{i+k}, y(t_{i+k}), y'(t_{i+k})\big) - f(t_{i+k}, y_{i+k}, y'_{i+k})\big] + T_{i+k}.$$

By using mean value theorem,

$$f\big(t_{i+k}, y(t_{i+k}), y'(t_{i+k})\big) - f(t_{i+k}, y_{i+k}, y'_{i+k}) = [y(t_{i+k}) - y_{i+k}]\frac{\partial f}{\partial y}(\eta_{i+k}) + [y'(t_{i+k}) - y'_{i+k}]\frac{\partial f}{\partial y'}(\eta_{i+k})$$

where η_{i+k} is the interior point of interval whose endpoints are $\big(t_{i+k}, y(t_{i+k}), y'(t_{i+k})\big)$ and $(t_{i+k}, y_{i+k}, y'_{i+k})$. Thus,

$$y(t_{i+k}) - y_{i+k} = h\beta_k[y'(t_{i+k}) - y'_{i+k}] + h^2\gamma_k\left[\big[y(t_{i+k}) - y_{i+k}\big]\frac{\partial f}{\partial y}(\eta_{i+k}) + \big[y'(t_{i+k}) - y'_{i+k}\big]\frac{\partial f}{\partial y'}(\eta_{i+k})\right] + T_{i+k}$$

$$T_{i+k} = \big[y(t_{i+k}) - y_{i+k}\big]\Big[1 - h^2\gamma_k\frac{\partial f}{\partial y}(\eta_{i+k})\Big] - \big[y'(t_{i+k}) - y'_{i+k}\big]\Big[h\beta_k + h^2\gamma_k\frac{\partial f}{\partial y'}(\eta_{i+k})\Big].$$

Thus, for an explicit method which is $\beta_k = 0$ and $\gamma_k = 0$, the LTE, T_{i+k} is then:

$$T_{i+k} = y(t_{i+k}) - y_{i+k}.$$

Meanwhile, the LTE of the implicit method is approaching the difference above when h tends to zero. Consider that theoretical solution $y(t)$ is continuously differentiable for the higher order. Thus the LTE for the explicit and implicit methods can be written such that:

$$T_{i+k} = C_{p+2}h^{p+2}y^{(p+2)}(t_i) + O(h^{p+3}). \tag{24}$$

Take $C_{p+2} = C_6 = \begin{bmatrix} \frac{73}{144} & \frac{751}{1440} & \frac{47}{90} & \frac{97}{90} \end{bmatrix}^T$, thus

$$T_{i+k} = h^6 y^{(6)}(t_i)\begin{bmatrix} \frac{73}{144} & \frac{751}{1440} & \frac{47}{90} & \frac{97}{90} \end{bmatrix}^T + O(h^7). \tag{25}$$

The method (11) is said to be consistent if LTE, $T_{i+k} \to 0$ when $h \to 0$.Thus, by (25), it is proved that our proposed method (11) is consistent.

3.4 Convergence of Method

Definition 2. *The linear multistep method* (12) *is convergent if it achieved the consistency and zero stability of the method.*

Considering the proposed method already achieved consistency and zero stable therefore, 2PBM4 is said to be convergent.

4 IMPLEMENTATION OF METHOD

4.1 Delay Differential Equation

Consider a uniform grid, $t_i = ih$ where $i = -m, -m+1, ..., -1, 0, 1, ..., N$. The variable h is the step size such that $h = \dfrac{b-a}{N}$ and $h = \frac{\tau}{m}$ where N is the subinterval while m is the position for initial function $\phi(t)$ in (2) and $m \geq 1$, where m is an integer. The approach of solving DDEs of constant type is by using the previously calculated solutions, $y(t-\tau)$ if the delay falls in interval $[a, b]$ or else use the initial function given if the delay falls in interval $[a-\tau, a]$. The idea is based on the following relations as discussed in [19]:

$$\begin{aligned}
&i = 0, 1, 2, \dots, N && y(t) \to y_i \\
&i = 0, 1, 2, \dots, N && y(t-\tau) \to y_{i-m} \\
&i = -m, -m+1, -m+2, \dots, 0 && y_i = \phi_i.
\end{aligned}$$

4.2 Boundary Value Problems

Considering the problems have boundary conditions; thus, we have to work on the BVPs with the implementation of the shooting approach. The BVPs is transformed into the initial value problem (IVP). However, there is insufficient information for the initial value, which is only $y(a)$ is given. As a result, we have to estimate the most accurate value of the second initial value, $y'(a)$, by using the Newton's like method described in [20]. The purpose of the shooting approach is to predict the desired initial value as accurately as possible while reducing the number of guessing times [21].

Assuming the general second order ODEs may be portrayed in terms of its dependent variabe, y that consider both t and the unknown variable r as shown below:

$$y''(t,r) = f\big(t, y(t,r), y'(t,r)\big) \tag{26}$$

subject to the initial conditions: $y(a,r) = \alpha, y'(a,r) = r_1$
The variable $r = r_k$ is assigned in a way that

$$\lim_{k\to\infty} y(b, r_k) - \beta = 0.$$

The first guess, r_1, is determined by applying the following formula:

$$r_1 = \frac{\beta - \alpha}{b - a}.$$

After that, the partial derivative of (26) towards variable r is obtained and consider that:

$$z(t,r) = \frac{\partial y}{\partial r}(t,r)$$

then,

$$z''(t,r) = \frac{\partial f}{\partial y}(t,y,y')z(t,r) + \frac{\partial f}{\partial y'}(t,y,y')z'(t,r) \tag{27}$$

with initial conditions:

$$z(a,r) = 0, z'(a,r) = 1.$$

The two approximate solutions, $y(t,r)$ and $z(t,r)$ obtained after solving the two IVPs, (26) and (27) simultaneously, will use in the Newton's like formula below to obtain the next guessing of the initial value, r_k.

$$r_k = w_{k-1} - \frac{y(b, w_{k-1}) - \beta}{z(b, w_{k-1})}$$

where:

$$w_{k-1} = r_{k-1} - \frac{y(b, r_{k-1}) - \beta}{z(b, r_{k-1})}.$$

When the procedure reaches the absolute error limit, it will be terminated, which is $|y(b, r_{k-1}) - \beta| \leq TOL$ where TOL is chosen.

Algorithm of 2PBM4:

The algorithm for 2PBM4 used as reference to write in C programming is as follows:

Step 1. Set $h = \dfrac{b-a}{N}$, $r_1 = \frac{(\beta-\alpha)}{(b-a)}$, $y_0 = \alpha, y'_0 = r_1, z_0 = 0, z'_0 = 1$. The tolerence, TOL is set to be $TOL = 10^{-5}$.

Step 2. Set the initial values $t_0, (t_0 - \tau), (y_{0-\tau}), (y'_{0-\tau}), (z_{0-\tau}), (z'_{0-\tau}), f\big(t_0, y_0, y'_0, y(t_0-\tau), y'(t_0-\tau)\big)$, and $f\big(t_0, z_0, z'_0, z(t_0-\tau), z'(t_0-\tau)\big)$.

Step 3. Calculate the value of $(t_i - \tau)$ in (1).

Step 4. Locate the position of $(t_i - \tau)$. If $(t_i - \tau) \leq t_0$ then use the initial function (2) to compute the solution of delay term, $y(t_i - \tau)$ and use the finite difference method to approximate $y'(t_i - \tau)$ or else if $(t_i - \tau)$ falls in the previous points then take the previous approximate solutions. $z(t_i - \tau)$ and $z'(t_i - \tau)$ are always equal to zero because $z = \dfrac{\partial y}{\partial r} = 0$ and $z' = \dfrac{\partial y'}{\partial r} = 0$.

Step 5. Calculate the starting values, which are $y'_1, z'_1, y_1, z_1, y'_2, z'_2, z_2$ and y_2, by using the predictor and corrector technique, which are direct Euler's method and direct modified Euler's method respectively.

Step 6. Calculate the values of f_1 and f_2 from the starting values obtained in ***Step 5*** and the solution of delay terms, $y(t_i - \tau)$ and $y'(t_i - \tau)$ obtained in ***Step 4***.

Step 7. Calculate the predictor and corrector values of the next iteration of the sets $\{y_{i+1}, z_{i+1}\}$ and $\{y_{i+2}, z_{i+2}\}$ simultaneously for $i = 0, 1, 2, ..., N$ by using the same procedures from ***Step 3*** to ***Step 6*** but substitute the direct Euler's method and direct modified Euler's method with 2PBM4 predictor-corrector formula.

Step 8. Check whether $(y_N - \beta) \leq TOL$, if so, calculate the maximum absolute errors, (absolute errors=exact solution-approximate solution) or else set the new r_k by using the Newton's like method.

Step 9. The procedure is complete.

5 RESULTS AND DISCUSSION

Four problems are solved by applying 2PBM4. The exact solution of *Problem 1, Problem 3* and *Problem 4* are not known. Thus, for *Problem 1*, the reference solution for the exact solution is given when $h = \frac{1}{2048}$ and compare with previous methods by using this reference solution. Meanwhile, for *Problem 3* and *Problem 4*, the approximate solutions are illustrated in a graph to compare with previous papers.

Problem 1 (Type: Retarded DDEs of constant delay):
$y''(t) = -\dfrac{1}{16}\sin\big(y(t)\big) - (t+1)y(t-1) + t, 0 \leq t \leq 2$
$y(t) = t - \dfrac{1}{2}, t \leq 0$
$y(2) = -\dfrac{1}{2}$
Source: Nevers and Schmitt [2].

Problem 2 (Type: Retarded DDEs of constant delay):
$y''(t) = -y(t - 0.1) + 10\sin(10t - 1) - 1000\sin(10t), 0 \leq t \leq 1$
$y(t) = 10\sin(10t), t \leq 0$
$y(1) = 10\sin(10)$
Exact solution:
$y(t) = 10\sin(10t)$
Source: Sakai [5].

Problem 3 (Type: Singular perturbation DDEs of constant delay):
$\epsilon y''(t) + e^{-0.5t} y'(t-\tau) - y(t) = 0, 0 < t < 1$
$y(0) = 1, -\tau \leq t \leq 0,$
$y(1) = 1$
where $\tau = 0.1\epsilon$
Source: Challa and Reddy [13].

Problem 4 (Type: Singular perturbation DDEs of constant delay):
$\epsilon y''(t) - e^{t} y'(t-\tau) - t y(t) = 0, 0 \leq t \leq 1$
$y(0) = 1, y(1) = 1$
where $\tau = 0.1\epsilon$
Source: Kadalbajoo and Sharma [11].

Below is the notations used in Tables 1-9:

h	:	Step size.
MAXE	:	Maximum absolute errors.
AVE	:	Average absolute errors.
ITN	:	Number of guessing times.
FCN	:	Total function calls at the last guessing iteration.
TS	:	Total step.
Time(s)	:	Computation time taken in seconds(s).
r_{last}	:	Last guessing t_k at last iteration.
2PBM4	:	Two Point Block Method order 4.
NS	:	The shooting technique using Euler's method in [2].
CRY	:	The finite differences method in [3].
RT	:	The approximation methods of projection type to the BVP in [4].
MS	:	The cubic splines method in [5].

Table 1 : The results of 2PBM4 for Problem 1

h	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{128}$
ITN	3	3	3
FCN	29	33	513
TS	5	9	129
t_{last}	-2.52515	-2.52543	-2.52601
Time(s)	0.000029	0.000034	0.000103

6 DISCUSSIONS

Table 2-4 shows that the absolute errors of 2PBM4 are better than NS, CRY and RT as h decreases for *Problem 1*. Table 6-7 shows that 2PBM4 is more accurate compare to MS as h decrease for *Problem 2*.

Table 2 : The comparison of absolute errors when $h = \frac{1}{4}$ for Problem 1

$h = \frac{1}{4}$	$y(0.0)$	$y(0.5)$	$y(1.0)$	$y(1.5)$	$y(2.0)$
2PBM4	0.00E+00	3.38E-04	5.86E-04	4.69E-05	1.23E-06
NS	0.00E+00	0.22E-00	0.40E-00	0.44E-00	1.48E-04
CRY	0.00E+00	1.87E-02	4.03E-02	2.49E-02	-
RT	0.00E+00	4.01E-03	5.21E-03	5.10E-03	-

Table 3 : The comparison of absolute errors when $h = \frac{1}{8}$ for Problem 1

$h = \frac{1}{8}$	$y(0.0)$	$y(0.5)$	$y(1.0)$	$y(1.5)$	$y(2.0)$
2PBM4	0.00E+00	1.08E-05	7.20E-05	7.36E-05	1.22E-06
NS	0.00E+00	0.12E-00	0.23E-00	0.24E-00	2.59E-05
CRY	0.00E+00	4.73E-03	1.02E-02	6.29E-03	-
RT	0.00E+00	9.83E-04	1.26E-03	1.24E-03	-

Table 4 : The comparison of absolute errors when $h = \frac{1}{128}$ for Problem 1

$h = \frac{1}{128}$	$y(0.0)$	$y(0.5)$	$y(1.0)$	$y(1.5)$	$y(2.0)$
2PBM4	0.00E+00	4.29E-07	8.60E-07	1.32E-06	1.22E-06
NS	0.00E+00	8.76E-03	1.65E-02	1.72E-02	2.89E-06
CRY	0.00E+00	7.43E-05	1.60E-04	9.87E-05	-

Table 5 : The results of 2PBM4 for Problem 2

h	$\frac{1}{20}$	$\frac{1}{40}$
MAXE	5.41E-02	1.85E-03
AVE	2.99E-02	9.40E-04
ITN	1	1
FCN	41	81
TS	11	21
t_{last}	99.09822	99.93944
Time(s)	0.00100	0.00133

Table 6 : The comparison of absolute errors when $h = \frac{1}{20}$ for Problem 2

$h = \frac{1}{20}$	$y(0.0)$	$y(0.2)$	$y(0.4)$	$y(0.6)$	$y(0.8)$	$y(1.0)$
2PBM4	0.00E+00	4.58E-02	3.96E-02	3.24E-02	1.22E-02	2.58E-14
MS	-	2.10E-01	1.00E-01	2.00E-02	2.90E-01	-

Table 7 : The comparison of absolute errors when $h = \frac{1}{40}$ for Problem 2

$h = \frac{1}{40}$	$y(0.0)$	$y(0.2)$	$y(0.4)$	$y(0.6)$	$y(0.8)$	$y(1.0)$
2PBM4	0.00E+00	1.35E-03	1.33E-03	1.03E-03	2.17E-04	5.33E-15
MS	-	5.50E-02	2.70E-02	5.00E-03	8.20E-02	-

Table 8 : The results of 2PBM4 when $h = 0.005$ for Problem 3 when $\epsilon = 0.1$

x	$\tau = 0.00$	$\tau = 0.01$	$\tau = 0.03$	$\tau = 0.05$
0.0	1.000000	1.000000	1.000000	1.000000
0.1	0.606419	0.594046	0.567130	0.536443
0.2	0.494668	0.485154	0.465096	0.443265
0.3	0.482627	0.476519	0.464231	0.451958
0.4	0.510369	0.506368	0.498548	0.491109
0.5	0.557896	0.555067	0.549577	0.544387
0.6	0.618936	0.616799	0.612624	0.608615
0.7	0.692271	0.690630	0.687389	0.684219
0.8	0.778778	0.777598	0.775246	0.772912
0.9	0.880430	0.879776	0.878458	0.877138
1.0	0.999995	1.000000	1.000000	1.000000
ITN	1	1	1	1
FCN	401	401	401	401
TS	101	101	101	101
t_{last}	-6.65524	-6.75234	-6.68491	-6.68138
Time(s)	0.00100	0.00200	0.00200	0.00233

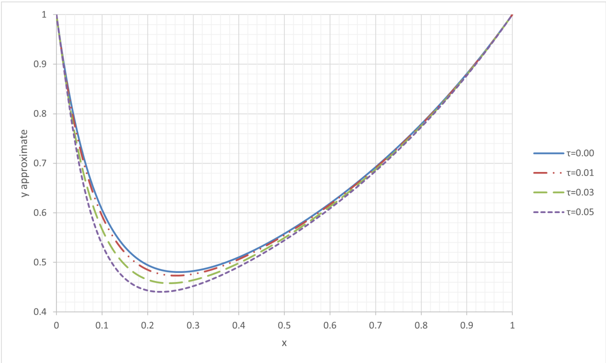

Figure 3 : Approximate value of 2PBM4 to Problem 3.

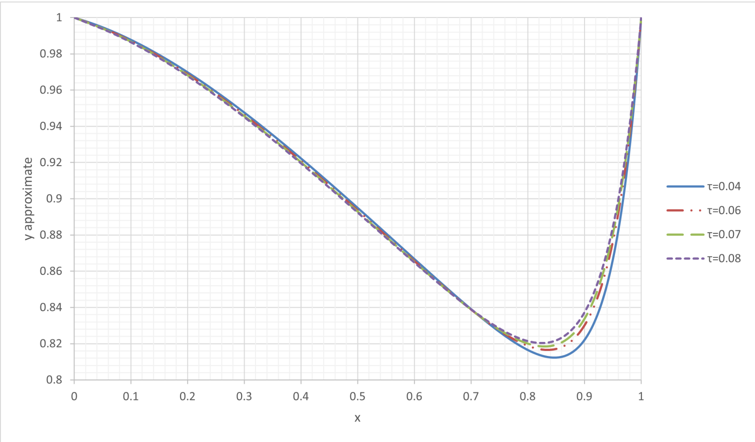

Figure 4 : Approximate value of 2PBM4 to Problem 4.

Table 9 : The results of 2PBM4 when $h = 0.005$ for Problem 4 when $\epsilon = 0.1$

x	$\tau = 0.04$	$\tau = 0.06$	$\tau = 0.07$	$\tau = 0.08$
0.0	1.000000	1.000000	1.000000	1.000000
0.1	0.987697	0.987036	0.986687	0.986337
0.2	0.969786	0.968755	0.968227	0.967703
0.3	0.947555	0.946282	0.945640	0.945006
0.4	0.922251	0.920861	0.920169	0.919491
0.5	0.894967	0.893603	0.892940	0.892302
0.6	0.866750	0.865659	0.865161	0.864705
0.7	0.839100	0.838887	0.838867	0.838905
0.8	0.816573	0.818892	0.820107	0.821348
0.9	0.822151	0.830188	0.833816	0.837224
1.0	1.000000	1.000000	1.000000	1.000000
ITN	1	1	1	1
FCN	401	401	401	401
TS	101	101	101	101
t_{last}	-0.10286	-0.11601	-0.12247	-0.12900
Time(s)	0.00167	0.00100	0.00067	0.00100

In Table 8 and Table 9, we choose $h = 0.005$ instead of $h = 0.01$ in [13] because of the limitation of the multistep method to solve singular perturbation DDEs where we need to use $h \leq \tau$ to give accurate results. Figures 3 and 4 illustrated the graph of approximate solutions of 2PBM4 for *Problem 3* and *Problem 4*, respectively. The graphs obtained are comparable with previous methods in [13] and [11], respectively.

7 CONCLUSION

The proposed method, 2PBM4 is proved to be capable of solving directly both second order DDEs and singular perturbations second order DDEs of constant delay type with boundary conditions.

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