Applied Mathematics and Computational Intelligence
Volume 12, No.1, Apr 2023 [101-124]

Diagonally Implicit Extended 2-Point Super Class of Block Backward Differentiation Formula with Two Off-step Points for Solving First Order Stiff Initial Value Problems

Alhassan Buhari[1], Musa Hamisu[2*]

[1]Department of Mathematics and Statistics, College of Natural and Applied Sciences, Al-Qalam University, Katsina, Nigeria
[2]Department of Mathematics and Statistics, Faculty of Natural and Applied Sciences, Umaru Musa Yar'adua University, Katsina, Nigeria

*Corresponding author: hamisu.musa@umyu.edu.ng

Received: 12 December 2022
Accepted: 31 January 2023

ABSTRACT

A new diagonally implicit extended 2-point super class of block backward differentiation formula with two off–step points is developed for the solution of first order stiff initial value problems. The method computes two solution values with two off–step points concurrently at each integration step. The method is of order five. Sets of different formulae can be generated from the method by varying a free parameter $\rho \in (-1,1)$ *in the formula. A specific choice of the value of the parameter* ρ *within the interval is made and the method is found to be consistent, zero stable and convergent. The region of absolute stability is plotted and it indicated that the method is A-stable. The numerical results obtained demonstrated efficiency of the new method when compared with some existing implicit numerical block methods. The developed method performed better than some existing algorithms in terms of accuracy and competes with others in terms of execution time.*

Keywords: A-stability, block backward differentiation formula, convergence, diagonally implicit, off–step points.

1 INTRODUCTION

In this paper, we shall consider the approximate numerical solution of first order stiff ordinary differential equation in initial value problem (IVP) of the form:

$$y' = f(x, y), \qquad y(x_0) = y_0, \qquad a \le x \le b \tag{1}$$

The syshasof initial value problem (1) is said to be stiff if all its eigenvalues of the coefficient matrix have negative real parts and the stiffness index of the eigenvalues is extremely large [1]. In other words, a system of ordinary differential equations (ODEs) can be regarded as stiff ordinary differential equation if its theoretical solution contains a very fast component as well as a very slow component [2]. These stiff systems are usually found in the fields of physical science, behavioral science, medicinal science, engineering and particularly, in the study of chemical kinetics, vibrations of springs, electrical circuits, statistical thermodynamics, weather predictions, theory of fluid and quantum mechanics and so on [3]. Most of the mathematical modeled relevant real world problems are impossible or difficult to solve analytically. Rather, alternative numerical integration methods are required in obtaining an approximate solution to the problems. However, in dealing with stiff ODEs, stiffness property prevents conventional explicit numerical methods from handling the problem efficiently, thus implicit numerical schemes with infinite region of absolute stability and no restriction on the step length have to be adopted.

Backward differentiation formula (BDF) is one of the most successful classes of implicit linear multistep method for solving stiff initial value problems [4]. Due to the good performance and efficiency of the BDF methods in integrating both linear and nonlinear stiff ODEs, several research efforts have been made by the researchers to develop both fully and diagonally implicit block BDF methods such as those found in [5,6,7,8,9,10,11,12,13,14,15,17,18,19,20] among others to formulate implicit numerical schemes for solving stiff IVPs. In an effort to develop new implicit methods of block type, recent research focuses on computing the solutions of IVPs at off–step points. Examples are [5,10,16,17].

The motivation of this research is to construct a diagonally implicit form of the method developed by [5] by introducing a lower triangular matrix in the formula so as to improve its accuracy and computational time.

2 DERIVATIONS OF THE METHOD

We shall consider the following numerical scheme for the integration of stiff ODEs developed in [5] which is given as:

$$\sum_{j=0}^{1} \alpha_{j,i} y_{n+j-1} + \sum_{j=\frac{3}{2}}^{3} \alpha_{j,i} y_{n+j-1} = h\beta_{k,i}(f_{n+k} - \rho f_{n+k-\frac{3}{2}}), \quad k = i = \frac{1}{2}, 1, \frac{3}{2}, 2. \tag{2}$$

The method (2) is fully implicit, A-stable numerical scheme for solving nonlinear and linear systems of stiff ordinary differential equations which approximates two solution values concurrently at each integration step in block with two off-step points.

In this paper, we focus on the derivation of diagonally implicit form of (2) that also computes two solution values with two off-step points simultaneously in block. We consider the formulation of:

$$\sum_{j=o}^{1} \alpha_{j,i} y_{n+j-1} + \sum_{j=\frac{3}{2}}^{1+k} \alpha_{j,i} y_{n+j-1} = h\beta_{k,i}(f_{n+k} - \rho f_{n+k-\frac{3}{2}}), \qquad k = i = \frac{1}{2}, 1, \frac{3}{2}, 2 \tag{3}$$

with j increment of 1 in the first sum and increment of $\frac{1}{2}$ in the second sum. k and i always assume the same values. Moreover, $k = i = \frac{1}{2}$ represents the first off-step point, $k = i = 1$ represents the first point, $k = i = \frac{3}{2}$ represents the second off-step point and $k = i = 2$ represents the second point. Unlike the extended 2-point super class of block BDF with off-step points method (2), the first off-step, first and second off-step points of (3) have one point less than those in the first off-step, first and second off-step points of (2) respectively. This makes (3) contain a lower triangular matrix in the coefficient matrix when the method is written in matrix form, thereby qualifying it to be diagonally implicit method.

The formula (3) is derived using Taylor's series expansion about x_n as follows

First Off-Step Point: $k = i = \frac{1}{2}$

To derive the first off-step point $y_{n+\frac{1}{2}}$. define the linear operator as

$$L_{\frac{1}{2}}[y(x_n), h]: \alpha_{0,\frac{1}{2}} y_{n-1} + \alpha_{1,\frac{1}{2}} y_n + \alpha_{\frac{3}{2},\frac{1}{2}} y_{n+\frac{1}{2}} - h\beta_{\frac{1}{2},\frac{1}{2}}(f_{n+\frac{1}{2}} - \rho f_{n-1}) = 0, \quad (4)$$

The approximate relationship associated with (4) can be written as

$$\alpha_{0,\frac{1}{2}} y(x_n - h) + \alpha_{1,\frac{1}{2}} y(x_n) + \alpha_{\frac{3}{2},\frac{1}{2}} y(x_n + \frac{1}{2}h) - h\beta_{\frac{1}{2},\frac{1}{2}}(f\left(x_n + \frac{1}{2}h\right) - \rho f(x_n - h)) = 0, \quad (5)$$

The Taylor's series expansion of equation (5) about x_n, after equating and collecting the like terms gives

$$C_{0,\frac{1}{2}} y(x_n) + C_{1,\frac{1}{2}} h y'(x_n) + C_{\frac{3}{2},\frac{1}{2}} h^2 y''(x_n) + \cdots = 0. \quad (6)$$

where,

$$\left.\begin{aligned} C_{0,\frac{1}{2}} &= \alpha_{0,\frac{1}{2}} + \alpha_{1,\frac{1}{2}} + \alpha_{\frac{3}{2},\frac{1}{2}} = 0 \\ C_{1,\frac{1}{2}} &= -\alpha_{0,\frac{1}{2}} + \frac{1}{2}\alpha_{\frac{3}{2},\frac{1}{2}} - \beta_{\frac{1}{2},\frac{1}{2}}(1-\rho) = 0 \\ C_{\frac{3}{2},\frac{1}{2}} &= \frac{1}{2}\alpha_{0,\frac{1}{2}} + \frac{1}{8}\alpha_{\frac{3}{2},\frac{1}{2}} - \beta_{\frac{1}{2},\frac{1}{2}}\left(\frac{1}{2} + p\right) = 0 \end{aligned}\right\}, \quad (7)$$

In deriving the first off-step point $y_{n+\frac{1}{2}}$, the coefficient $\alpha_{\frac{3}{2},\frac{1}{2}}$ is normalized to 1. Solving the set of equation (7) leads to the values of $\alpha_{j,i's}$ and $\beta_{j,i's}$ given as:

$$\alpha_{0,\frac{1}{2}} = \frac{1}{4}\frac{5\rho+1}{\rho+2}, \alpha_{1,\frac{1}{2}} = -\frac{9}{4}\frac{\rho+1}{\rho+2}, \alpha_{\frac{3}{2},\frac{1}{2}} = 1 \text{ and } \beta_{\frac{1}{2},\frac{1}{2}} = \frac{3}{4(\rho+2)}.$$

Substitute these values of $\alpha_{j,i's}$ and $\beta_{j,i's}$ in equation (4) to obtain the following formula of the first off-step point as:

$$y_{n+\frac{1}{2}} = -\frac{1}{4}\frac{5\rho+1}{\rho+2} y_{n-1} + \frac{9}{4}\frac{\rho+1}{\rho+2} y_n + \frac{3}{4(\rho+2)} h f_{n+\frac{1}{2}} - \frac{3}{4(\rho+2)} \rho h f_{n-1} \quad (8)$$

The same procedure is applied for the derivation of first, second off-step and second points respectively. Therefore, the diagonally implicit extended 2-point super class of block BDF with two off-step points (DIE2OSBBDF) is obtained as:

$$\left.\begin{aligned} y_{n+\frac{1}{2}} &= -\frac{1}{4}\frac{5\rho+1}{\rho+2} y_{n-1} + \frac{9}{4}\frac{\rho+1}{\rho+2} y_n + \frac{3}{4(\rho+2)} h f_{n+\frac{1}{2}} - \frac{3}{4(\rho+2)} \rho h f_{n-1} \\ y_{n+1} &= -\frac{1}{3}\frac{11\rho-2}{\rho+14} y_{n-1} + \frac{2(\rho-4)}{\rho+14} y_n + \frac{8}{3}\frac{\rho+8}{\rho+14} y_{n+\frac{1}{2}} + \frac{4}{\rho+14} h f_{n+1} - \frac{4}{\rho+14} \rho h f_{n-\frac{1}{2}} \\ y_{n+\frac{3}{2}} &= \frac{3}{2}\frac{\rho+1}{4\rho-61} y_{n-1} + \frac{5(8\rho-5)}{4\rho-61} y_n - \frac{15(4\rho-5)}{4\rho-61} y_{n+\frac{1}{2}} + \frac{45}{2}\frac{\rho-5}{4\rho-61} y_{n+1} - \frac{15}{4\rho-61} h f_{n+\frac{3}{2}} + \frac{15}{4\rho-61} \rho h f_n \\ y_{n+2} &= -\frac{1}{5}\frac{\rho+4}{\rho-54} y_{n-1} + \frac{9(\rho+2)}{\rho-54} y_n + \frac{4(3\rho-16)}{\rho-54} y_{n+\frac{1}{2}} - \frac{27(\rho-4)}{\rho-54} y_{n+1} + \frac{36}{5}\frac{\rho-16}{\rho-54} y_{n+\frac{3}{2}} - \frac{12}{\rho-54} h f_{n+2} \\ &\quad + \frac{12}{\rho-54} \rho h f_{n+\frac{1}{2}} \end{aligned}\right\} \quad (9)$$

For the attainment of region of absolute stability and zero-stability of the method (9), a value of $\rho = \frac{1}{5}$ is selected within the interval $-1 < \rho < 1$ as in [3].

By substituting $\rho = \frac{1}{5}$ in equation (9), the diagonally implicit extended 2-point super class of block BDF with two off-step points becomes:

$$\left.\begin{aligned} y_{n+\frac{1}{2}} &= -\frac{5}{22}y_{n-1} + \frac{27}{22}y_n + \frac{15}{44}hf_{n+\frac{1}{2}} - \frac{3}{44}hf_{n-1} \\ y_{n+1} &= -\frac{1}{213}y_{n-1} - \frac{38}{71}y_n + \frac{328}{213}y_{n+\frac{1}{2}} + \frac{20}{71}hf_{n+1} - \frac{4}{71}hf_{n-\frac{1}{2}} \\ y_{n+\frac{3}{2}} &= -\frac{9}{301}y_{n-1} + \frac{85}{301}y_n - \frac{45}{43}y_{n+\frac{1}{2}} + \frac{540}{301}y_{n+1} + \frac{75}{301}hf_{n+\frac{3}{2}} - \frac{15}{301}hf_n \\ y_{n+2} &= \frac{21}{1345}y_{n-1} - \frac{99}{269}y_n + \frac{308}{269}y_{n+\frac{1}{2}} - \frac{513}{269}y_{n+1} + \frac{2844}{1345}y_{n+\frac{3}{2}} + \frac{60}{269}hf_{n+2} - \frac{12}{269}hf_{n+\frac{1}{2}} \end{aligned}\right\}, \quad (10)$$

Thus, throughout this paper, we shall be referring equation (10) as diagonally implicit extended 2-point super class of block BDF with two off–step points (DIE2OSBBDF).

3 ORDER AND ERROR CONSTANT OF THE DIE2OSBBDF METHOD

In this section, the derivation of order and error constant of the diagonally implicit extended 2-point super class of block BDF with two off-step points corresponding to the equations in (10) are presented. To derive the order of the method, the formulae (10) can also be expressed as:

$$\left.\begin{aligned} y_{n+\frac{1}{2}} + \frac{5}{22}y_{n-1} - \frac{27}{22}y_n &= \frac{15}{44}hf_{n+\frac{1}{2}} - \frac{3}{44}hf_{n-1} \\ y_{n+1} + \frac{1}{213}y_{n-1} + \frac{38}{71}y_n - \frac{328}{213}y_{n+\frac{1}{2}} &= \frac{20}{71}hf_{n+1} - \frac{4}{71}hf_{n-\frac{1}{2}} \\ y_{n+\frac{3}{2}} + \frac{9}{301}y_{n-1} - \frac{85}{301}y_n + \frac{45}{43}y_{n+\frac{1}{2}} - \frac{540}{301}y_{n+1} &= \frac{75}{301}hf_{n+\frac{3}{2}} - \frac{15}{301}hf_n \\ y_{n+2} - \frac{21}{1345}y_{n-1} + \frac{99}{269}y_n - \frac{308}{269}y_{n+\frac{1}{2}} + \frac{513}{269}y_{n+1} - \frac{2844}{1345}y_{n+\frac{3}{2}} &= \frac{60}{269}hf_{n+2} - \frac{12}{269}hf_{n+\frac{1}{2}} \end{aligned}\right\} \quad (11)$$

The matrix form of equation (11) is given by:

$$\begin{bmatrix} 0 & \frac{5}{22} & 0 & -\frac{27}{22} \\ 0 & \frac{1}{213} & 0 & \frac{38}{71} \\ 0 & \frac{9}{301} & 0 & -\frac{85}{301} \\ 0 & -\frac{21}{1345} & 0 & \frac{99}{269} \end{bmatrix} \begin{bmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{328}{213} & 1 & 0 & 0 \\ \frac{45}{43} & -\frac{540}{301} & 1 & 0 \\ -\frac{308}{269} & \frac{513}{269} & -\frac{2844}{1345} & 1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} =$$

$$h\begin{bmatrix} 0 & -\frac{3}{44} & 0 & 0 \\ 0 & 0 & -\frac{4}{71} & 0 \\ 0 & 0 & 0 & -\frac{15}{301} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{bmatrix} + h\begin{bmatrix} \frac{15}{44} & 0 & 0 & 0 \\ 0 & \frac{20}{71} & 0 & 0 \\ 0 & 0 & \frac{75}{301} & 0 \\ -\frac{12}{269} & 0 & 0 & \frac{60}{269} \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix} \tag{12}$$

Let

$$\gamma_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \gamma_1 = \begin{bmatrix} \frac{5}{22} \\ \frac{1}{213} \\ \frac{9}{301} \\ -\frac{21}{1345} \end{bmatrix}, \gamma_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \gamma_3 = \begin{bmatrix} -\frac{27}{22} \\ \frac{38}{71} \\ -\frac{85}{301} \\ \frac{99}{269} \end{bmatrix}, \gamma_4 = \begin{bmatrix} 1 \\ -\frac{328}{213} \\ \frac{45}{43} \\ -\frac{308}{269} \end{bmatrix}, \gamma_5 = \begin{bmatrix} 0 \\ 1 \\ -\frac{540}{301} \\ \frac{513}{269} \end{bmatrix}, \gamma_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -\frac{2844}{1345} \end{bmatrix},$$

$$\gamma_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \psi_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \psi_1 = \begin{bmatrix} -\frac{3}{44} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \psi_2 = \begin{bmatrix} 0 \\ -\frac{4}{71} \\ 0 \\ 0 \end{bmatrix}, \psi_3 = \begin{bmatrix} 0 \\ 0 \\ -\frac{15}{301} \\ 0 \end{bmatrix}, \psi_4 = \begin{bmatrix} \frac{15}{44} \\ 0 \\ 0 \\ -\frac{12}{269} \end{bmatrix}, \psi_5 = \begin{bmatrix} 0 \\ \frac{20}{71} \\ 0 \\ 0 \end{bmatrix},$$

$$\psi_6 = \begin{bmatrix} 0 \\ 0 \\ \frac{75}{301} \\ 0 \end{bmatrix}, \psi_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{60}{269} \end{bmatrix}$$

Definition 3.1: the order of the block method (10) and its associated linear difference operator given by:

$$L\{y(x),h\} = \sum_{j=0}^{k}\left[\gamma_j y\left(x + j\frac{h}{2}\right) - h\psi_j y'\left(x + j\frac{h}{2}\right)\right] \tag{13}$$

is the unique integer p such that if $A_q = 0$, $q = 0(1)p$ and $A_{P+1} \neq 0$; where A_q are constant (column) matrices defined by:

$$\left.\begin{array}{c} A_0 = \gamma_0 + \gamma_1 + \gamma_2 + \cdots + \gamma_k \\ A_1 = \gamma_1 + 2\gamma_2 + \cdots + k\gamma_k - 2(\psi_0 + \psi_1 + \psi_2 + \cdots + \psi_k) \\ . \\ . \\ . \\ A_q = \frac{1}{q!}(\gamma_1 + 2^q\gamma_2 + \cdots + k^q\gamma_k) - \frac{2}{(q-1)!}(\psi_1 + 2^{q-1}\psi_2 + \cdots + k^{q-1}\psi_k) \end{array}\right\} \tag{14}$$

$$q = 2,3, \ldots \ldots \ldots$$

For $q = 0(1)6$, we have

$$A_0 = \sum_{j=0}^{7}\gamma_j = \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6 + \gamma_7$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{5}{22} \\ \frac{1}{213} \\ \frac{9}{301} \\ -\frac{21}{1345} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{27}{22} \\ \frac{38}{71} \\ -\frac{85}{301} \\ \frac{99}{269} \end{bmatrix} + \begin{bmatrix} 1 \\ -\frac{328}{213} \\ \frac{45}{43} \\ -\frac{308}{269} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -\frac{540}{301} \\ \frac{513}{269} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -\frac{2844}{1345} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A_1 = \sum_{j=0}^{7}(j\gamma_j) - 2\sum_{j=0}^{7}\psi_j = ((0)\gamma_0 + (1)\gamma_1 + (2)\gamma_2 + (3)\gamma_3 + (4)\gamma_4 + (5)\gamma_5 + (6)\gamma_6 + (7)\gamma_7)$$
$$-2(\psi_0 + \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6 + \psi_7)$$

$$= \left[(0)\begin{bmatrix}0\\0\\0\\0\end{bmatrix} + (1)\begin{bmatrix}\frac{5}{22}\\ \frac{1}{213}\\ \frac{9}{301}\\ -\frac{21}{1345}\end{bmatrix} + (2)\begin{bmatrix}0\\0\\0\\0\end{bmatrix} + (3)\begin{bmatrix}-\frac{27}{22}\\ \frac{38}{71}\\ -\frac{85}{301}\\ \frac{99}{269}\end{bmatrix} + (4)\begin{bmatrix}\frac{1}{-\frac{328}{213}}\\ \frac{45}{43}\\ -\frac{308}{269}\end{bmatrix} + (5)\begin{bmatrix}0\\ \frac{1}{-\frac{540}{301}}\\ \frac{513}{269}\end{bmatrix} + (6)\begin{bmatrix}0\\0\\ \frac{1}{-\frac{2844}{1345}}\end{bmatrix} + (7)\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right]$$

$$-2\left[\begin{bmatrix}0\\0\\0\\0\end{bmatrix} + \begin{bmatrix}-\frac{3}{44}\\0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\-\frac{4}{71}\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\-\frac{15}{301}\\0\end{bmatrix} + \begin{bmatrix}\frac{15}{44}\\0\\0\\-\frac{12}{269}\end{bmatrix} + \begin{bmatrix}0\\ \frac{20}{71}\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\ \frac{75}{301}\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\ \frac{60}{269}\end{bmatrix} \right] = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$A_2 = \sum_{j=0}^{7} \frac{(j^2\gamma_j)}{2!} - 2\sum_{j=0}^{7}(j\psi_j) = \frac{1}{2!}\left((0)^2\gamma_0 + (1)^2\gamma_1 + (2)^2\gamma_2 + (3)^2\gamma_3 + (4)^2\gamma_4 + (5)^2\gamma_5 + (6)^2\gamma_6 + (7)^2\gamma_7\right)$$
$$-2\left((0)\psi_0 + (1)\psi_1 + (2)\psi_2 + (3)\psi_3 + (4)\psi_4 + (5)\psi_5 + (6)\psi_6 + (7)\psi_7\right)$$

$$= \frac{1}{2!}\left[(0)^2\begin{bmatrix}0\\0\\0\\0\end{bmatrix} + (1)^2\begin{bmatrix}\frac{5}{22}\\ \frac{1}{213}\\ \frac{9}{301}\\ -\frac{21}{1345}\end{bmatrix} + (2)^2\begin{bmatrix}0\\0\\0\\0\end{bmatrix} + (3)^2\begin{bmatrix}-\frac{27}{22}\\ \frac{38}{71}\\ -\frac{85}{301}\\ \frac{99}{269}\end{bmatrix} + (4)^2\begin{bmatrix}\frac{1}{-\frac{328}{213}}\\ \frac{45}{43}\\ -\frac{308}{269}\end{bmatrix} + (5)^2\begin{bmatrix}0\\ \frac{1}{-\frac{540}{301}}\\ \frac{513}{269}\end{bmatrix} + (6)^2\begin{bmatrix}0\\0\\ \frac{1}{-\frac{2844}{1345}}\end{bmatrix} + (7)^2\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right]$$

$$-2\left[(0)\begin{bmatrix}0\\0\\0\\0\end{bmatrix} + (1)\begin{bmatrix}-\frac{3}{44}\\0\\0\\0\end{bmatrix} + (2)\begin{bmatrix}0\\-\frac{4}{71}\\0\\0\end{bmatrix} + (3)\begin{bmatrix}0\\0\\-\frac{15}{301}\\0\end{bmatrix} + (4)\begin{bmatrix}\frac{15}{44}\\0\\0\\-\frac{12}{269}\end{bmatrix} + (5)\begin{bmatrix}0\\ \frac{20}{71}\\0\\0\end{bmatrix} + (6)\begin{bmatrix}0\\0\\ \frac{75}{301}\\0\end{bmatrix} + (7)\begin{bmatrix}0\\0\\0\\ \frac{60}{269}\end{bmatrix} \right]$$

$$= \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$A_3=\sum_{j=0}^{7}\frac{\left(j^3\gamma_j\right)}{3!}-2\sum_{j=0}^{7}\frac{\left(j^2\psi_j\right)}{2!}=\frac{1}{3!}\left((0)^3\gamma_0+(1)^3\gamma_1+(2)^3\gamma_2+(3)^3\gamma_3+(4)^3\gamma_4+(5)^3\gamma_5+(6)^3\gamma_6+(7)^3\gamma_7\right)$$
$$-\frac{2}{2!}\left((0)^2\psi_0+(1)^2\psi_1+(2)^2\psi_2+(3)^2\psi_3+(4)^2\psi_4+(5)^2\psi_5+(6)^2\psi_6+(7)^2\psi_7\right)$$

$$=\frac{1}{3!}\left[(0)^3\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^3\begin{bmatrix}\frac{5}{22}\\\frac{1}{213}\\\frac{9}{301}\\-\frac{21}{1345}\end{bmatrix}+(2)^3\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^3\begin{bmatrix}-\frac{27}{22}\\\frac{38}{71}\\-\frac{85}{301}\\\frac{99}{269}\end{bmatrix}+(4)^3\begin{bmatrix}\frac{1}{328}\\-\frac{213}{}\\\frac{45}{43}\\-\frac{308}{269}\end{bmatrix}+(5)^3\begin{bmatrix}0\\\frac{1}{540}\\-\frac{}{301}\\\frac{513}{269}\end{bmatrix}+(6)^3\begin{bmatrix}0\\0\\\frac{1}{2844}\\-\frac{}{1345}\end{bmatrix}+(7)^3\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right]$$

$$-\frac{2}{2!}\left[(0)^2\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^2\begin{bmatrix}-\frac{3}{44}\\0\\0\\0\end{bmatrix}+(2)^2\begin{bmatrix}0\\-\frac{4}{71}\\0\\0\end{bmatrix}+(3)^2\begin{bmatrix}0\\0\\-\frac{15}{301}\\0\end{bmatrix}+(4)^2\begin{bmatrix}\frac{15}{44}\\0\\0\\-\frac{12}{269}\end{bmatrix}+(5)^2\begin{bmatrix}0\\\frac{20}{71}\\0\\0\end{bmatrix}+(6)^2\begin{bmatrix}0\\0\\\frac{75}{301}\\0\end{bmatrix}+(7)^2\begin{bmatrix}0\\0\\0\\\frac{60}{269}\end{bmatrix}\right]$$

$$=\begin{bmatrix}-\frac{9}{44}\\0\\0\\0\end{bmatrix}$$

$$A_4=\sum_{j=0}^{7}\frac{\left(j^4\gamma_j\right)}{4!}-2\sum_{j=0}^{7}\frac{\left(j^3\psi_j\right)}{3!}=\frac{1}{4!}\left((0)^4\gamma_0+(1)^4\gamma_1+(2)^4\gamma_2+(3)^4\gamma_3+(4)^4\gamma_4+(5)^4\gamma_5+(6)^4\gamma_6+(7)^4\gamma_7\right)$$
$$-\frac{2}{3!}\left((0)^3\psi_0+(1)^3\psi_1+(2)^3\psi_2+(3)^3\psi_3+(4)^3\psi_4+(5)^3\psi_5+(6)^3\gamma_6+(7)^3\gamma_7\right)$$

$$=\frac{1}{4!}\left[(0)^4\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^4\begin{bmatrix}\frac{5}{22}\\\frac{1}{213}\\\frac{9}{301}\\-\frac{21}{1345}\end{bmatrix}+(2)^4\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^4\begin{bmatrix}-\frac{27}{22}\\\frac{38}{71}\\-\frac{85}{301}\\\frac{99}{269}\end{bmatrix}+(4)^4\begin{bmatrix}\frac{1}{328}\\-\frac{213}{}\\\frac{45}{43}\\-\frac{308}{269}\end{bmatrix}+(5)^4\begin{bmatrix}0\\\frac{1}{540}\\-\frac{}{301}\\\frac{513}{269}\end{bmatrix}+(6)^4\begin{bmatrix}0\\0\\\frac{1}{2844}\\-\frac{}{1345}\end{bmatrix}+(7)^4\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right]$$

$$-\frac{2}{3!}\left[(0)^3\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^3\begin{bmatrix}-\frac{3}{44}\\0\\0\\0\end{bmatrix}+(2)^3\begin{bmatrix}0\\-\frac{4}{71}\\0\\0\end{bmatrix}+(3)^3\begin{bmatrix}0\\0\\-\frac{15}{301}\\0\end{bmatrix}+(4)^3\begin{bmatrix}\frac{15}{44}\\0\\0\\-\frac{12}{269}\end{bmatrix}+(5)^3\begin{bmatrix}0\\\frac{20}{71}\\0\\0\end{bmatrix}+(6)^3\begin{bmatrix}0\\0\\\frac{75}{301}\\0\end{bmatrix}+(7)^3\begin{bmatrix}0\\0\\0\\\frac{60}{269}\end{bmatrix}\right]$$

$$=\begin{bmatrix}-\frac{63}{88}\\-\frac{35}{213}\\0\\0\end{bmatrix}$$

$$A_5=\sum_{j=0}^{7}\frac{\left(j^5\gamma_j\right)}{5!}-2\sum_{j=0}^{7}\frac{\left(j^4\psi_j\right)}{4!}=\frac{1}{5!}\left((0)^5\gamma_0+(1)^5\gamma_1+(2)^5\gamma_2+(3)^5\gamma_3+(4)^5\gamma_4+(5)^5\gamma_5+(6)^5\gamma_6+(7)^5\gamma_7\right)$$
$$-\frac{2}{4!}\left((0)^4\psi_0+(1)^4\psi_1+(2)^4\psi_2+(3)^4\psi_3+(4)^4\psi_4+(5)^4\psi_5+(6)^4\psi_6+(7)^4\psi_7\right)$$

$$=\frac{1}{5!}\left[(0)^5\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^5\begin{bmatrix}\frac{5}{22}\\\frac{1}{213}\\\frac{9}{301}\\-\frac{21}{1345}\end{bmatrix}+(2)^5\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^5\begin{bmatrix}-\frac{27}{22}\\\frac{38}{71}\\-\frac{85}{301}\\\frac{99}{269}\end{bmatrix}+(4)^5\begin{bmatrix}1\\-\frac{328}{213}\\\frac{45}{43}\\-\frac{308}{269}\end{bmatrix}+(5)^5\begin{bmatrix}0\\1\\-\frac{540}{301}\\\frac{513}{269}\end{bmatrix}+(6)^5\begin{bmatrix}0\\0\\1\\-\frac{2844}{1345}\end{bmatrix}+(7)^5\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right]$$

$$-\frac{2}{4!}\left[(0)^4\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^4\begin{bmatrix}-\frac{3}{44}\\0\\0\\0\end{bmatrix}+(2)^4\begin{bmatrix}0\\-\frac{4}{71}\\0\\0\end{bmatrix}+(3)^4\begin{bmatrix}0\\0\\-\frac{15}{301}\\0\end{bmatrix}+(4)^4\begin{bmatrix}\frac{15}{44}\\0\\0\\-\frac{12}{269}\end{bmatrix}+(5)^4\begin{bmatrix}0\\\frac{20}{71}\\0\\0\end{bmatrix}+(6)^4\begin{bmatrix}0\\0\\\frac{75}{301}\\0\end{bmatrix}+(7)^4\begin{bmatrix}0\\0\\0\\\frac{60}{269}\end{bmatrix}\right]$$

$$=\begin{bmatrix}-\frac{1071}{880}\\-\frac{217}{355}\\-\frac{81}{602}\\0\end{bmatrix}$$

$$A_6 = \sum_{j=0}^{7} \frac{\left(j^6 \gamma_j\right)}{6!} - 2\sum_{j=0}^{7} \frac{\left(j^5 \psi_j\right)}{5!} = \frac{1}{6!}\left((0)^6 \gamma_0 + (1)^6 \gamma_1 + (2)^6 \gamma_2 + (3)^6 \gamma_3 + (4)^6 \gamma_4 + (5)^6 \gamma_5 + (6)^6 \gamma_6 + (7)^6 \gamma_7\right)$$
$$- \frac{2}{5!}\left((0)^5 \psi_0 + (1)^5 \psi_1 + (2)^5 \psi_2 + (3)^5 \psi_3 + (4)^5 \psi_4 + (5)^5 \psi_5 + (6)^5 \psi_6 + (7)^5 \psi_7\right)$$

$$= \frac{1}{6!}\left[(0)^6\begin{bmatrix}0\\0\\0\\0\end{bmatrix} + (1)^6\begin{bmatrix}\frac{5}{22}\\ \frac{1}{213}\\ \frac{9}{301}\\ -\frac{21}{1345}\end{bmatrix} + (2)^6\begin{bmatrix}0\\0\\0\\0\end{bmatrix} + (3)^6\begin{bmatrix}-\frac{27}{22}\\ \frac{38}{71}\\ -\frac{85}{301}\\ \frac{99}{269}\end{bmatrix} + (4)^6\begin{bmatrix}-\frac{1}{328}\\ \frac{213}{45}\\ \frac{45}{43}\\ -\frac{308}{269}\end{bmatrix} + (5)^6\begin{bmatrix}0\\ -\frac{1}{540}\\ \frac{301}{513}\\ 269\end{bmatrix} + (6)^6\begin{bmatrix}0\\0\\ -\frac{1}{2844}\\ 1345\end{bmatrix} + (7)^6\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right]$$

$$- \frac{2}{5!}\left[(0)^5\begin{bmatrix}0\\0\\0\\0\end{bmatrix} + (1)^5\begin{bmatrix}-\frac{3}{44}\\0\\0\\0\end{bmatrix} + (2)^5\begin{bmatrix}0\\ -\frac{4}{71}\\0\\0\end{bmatrix} + (3)^5\begin{bmatrix}0\\0\\ -\frac{15}{301}\\0\end{bmatrix} + (4)^5\begin{bmatrix}\frac{15}{44}\\0\\0\\ -\frac{12}{269}\end{bmatrix} + (5)^5\begin{bmatrix}0\\ \frac{20}{71}\\0\\0\end{bmatrix} + (6)^5\begin{bmatrix}0\\0\\ \frac{75}{301}\\0\end{bmatrix} + (7)^5\begin{bmatrix}0\\0\\0\\ \frac{60}{269}\end{bmatrix}\right]$$

$$= \begin{bmatrix}-\frac{603}{440}\\ -\frac{3701}{3195}\\ -\frac{669}{1204}\\ -\frac{447}{1345}\end{bmatrix}$$

By the above definition 3.1, we conclude that the diagonally implicit extended 2-point super class of block BDF with two off-step points is found to be of order 5 with error constant given by:

$$E_6 = \begin{bmatrix}-\frac{603}{440}\\ -\frac{3701}{3195}\\ -\frac{669}{1204}\\ -\frac{447}{1345}\end{bmatrix} \neq \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

4 STABILITY OF THE METHOD

This section presents the stability analysis of the method (10) in terms of zero and absolute stability. The formulae (10) are represented as:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+\frac{1}{2}}\\y_{n+1}\\y_{n+\frac{3}{2}}\\y_{n+2}\end{bmatrix}=\begin{bmatrix}0&-\frac{5}{22}&0&\frac{27}{22}\\0&-\frac{1}{213}&0&-\frac{38}{71}\\0&-\frac{9}{301}&0&\frac{85}{301}\\0&\frac{21}{1345}&0&-\frac{99}{269}\end{bmatrix}\begin{bmatrix}y_{n-\frac{3}{2}}\\y_{n-1}\\y_{n-\frac{1}{2}}\\y_{n}\end{bmatrix}+\begin{bmatrix}0&0&0&0\\\frac{328}{213}&0&0&0\\-\frac{45}{43}&\frac{540}{301}&0&0\\\frac{308}{269}&-\frac{513}{269}&\frac{2844}{1345}&0\end{bmatrix}\begin{bmatrix}y_{n+\frac{1}{2}}\\y_{n+1}\\y_{n+\frac{3}{2}}\\y_{n+2}\end{bmatrix}$$

$$+h\begin{bmatrix}0&-\frac{3}{44}&0&0\\0&0&-\frac{4}{71}&0\\0&0&0&-\frac{15}{301}\\0&0&0&0\end{bmatrix}\begin{bmatrix}f_{n-\frac{3}{2}}\\f_{n-1}\\f_{n-\frac{1}{2}}\\f_{n}\end{bmatrix}+h\begin{bmatrix}\frac{15}{44}&0&0&0\\0&\frac{20}{71}&0&0\\0&0&\frac{75}{301}&0\\-\frac{12}{269}&0&0&\frac{60}{269}\end{bmatrix}\begin{bmatrix}f_{n+\frac{1}{2}}\\f_{n+1}\\f_{n+\frac{3}{2}}\\f_{n+2}\end{bmatrix} \tag{15}$$

The absolute stability region of the DIE2OSBBDF is determined by applying linear test differential equation of the form $y' = \lambda y$ (where $\lambda < 0$ is complex) into (15). This leads to:

$$\begin{bmatrix}1-\frac{15}{44}\lambda h&0&0&0\\-\frac{328}{213}&1-\frac{20}{71}\lambda h&0&0\\\frac{45}{43}&-\frac{540}{301}&1-\frac{75}{301}\lambda h&0\\-\frac{308}{269}+\frac{12}{269}\lambda h&\frac{513}{269}&-\frac{2844}{1345}&1-\frac{60}{269}\lambda h\end{bmatrix}\begin{bmatrix}y_{n+\frac{1}{2}}\\y_{n+1}\\y_{n+\frac{3}{2}}\\y_{n+2}\end{bmatrix}=$$

$$\begin{bmatrix}0&-\frac{5}{22}-\frac{3}{44}\lambda h&0&\frac{27}{22}\\0&-\frac{1}{213}&-\frac{4}{71}\lambda h&-\frac{38}{71}\\0&-\frac{9}{301}&0&\frac{85}{301}-\frac{15}{301}\lambda h\\0&\frac{21}{1345}&0&-\frac{99}{269}\end{bmatrix}\begin{bmatrix}y_{n-\frac{3}{2}}\\y_{n-1}\\y_{n-\frac{1}{2}}\\y_{n}\end{bmatrix} \tag{16}$$

Let $\bar{h} = \lambda h$ in (16) to obtain

$$\begin{bmatrix} 1-\frac{15}{44}\bar{h} & 0 & 0 & 0 \\ -\frac{328}{213} & 1-\frac{20}{71}\bar{h} & 0 & 0 \\ \frac{45}{43} & -\frac{540}{301} & 1-\frac{75}{301}\bar{h} & 0 \\ -\frac{308}{269}+\frac{12}{269}\bar{h} & \frac{513}{269} & -\frac{2844}{1345} & 1-\frac{60}{269}\bar{h} \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{5}{22}-\frac{3}{44}\bar{h} & 0 & \frac{27}{22} \\ 0 & -\frac{1}{213} & -\frac{4}{71}\bar{h} & -\frac{38}{71} \\ 0 & -\frac{9}{301} & 0 & \frac{85}{301}-\frac{15}{301}\bar{h} \\ 0 & \frac{21}{1345} & 0 & -\frac{99}{269} \end{bmatrix} \begin{bmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_{n} \end{bmatrix} \quad (17)$$

Let the number of block be r and the number of points in the block be m, then $n = rm$. where $m = 2$ and $n = 2r$. By [16], we let

$$Y_m = \begin{bmatrix} y_{2m+\frac{1}{2}} \\ y_{2m+1} \\ y_{2m+\frac{3}{2}} \\ y_{2m+2} \end{bmatrix} = \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} \text{and}\, Y_{m-1} = \begin{bmatrix} y_{2(m-1)+\frac{1}{2}} \\ y_{2(m-1)+1} \\ y_{2(m-1)+\frac{3}{2}} \\ y_{2(m-1)+2} \end{bmatrix} = \begin{bmatrix} y_{2m-\frac{3}{2}} \\ y_{2m-1} \\ y_{2m-\frac{1}{2}} \\ y_{2m} \end{bmatrix} = \begin{bmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_{n} \end{bmatrix}$$

Equation (17) is equivalent to

$$AY_m = BY_{m-1} \quad (18)$$

where,

$$A = \begin{bmatrix} 1-\frac{15}{44}\bar{h} & 0 & 0 & 0 \\ -\frac{328}{213} & 1-\frac{20}{71}\bar{h} & 0 & 0 \\ \frac{45}{43} & -\frac{540}{301} & 1-\frac{75}{301}\bar{h} & 0 \\ -\frac{308}{269}+\frac{12}{269}\bar{h} & \frac{513}{269} & -\frac{2844}{1345} & 1-\frac{60}{269}\bar{h} \end{bmatrix}, \; B = \begin{bmatrix} 0 & -\frac{5}{22}-\frac{3}{44}\bar{h} & 0 & \frac{27}{22} \\ 0 & -\frac{1}{213} & -\frac{4}{71}\bar{h} & -\frac{38}{71} \\ 0 & -\frac{9}{301} & 0 & \frac{85}{301}-\frac{15}{301}\bar{h} \\ 0 & \frac{21}{1345} & 0 & -\frac{99}{269} \end{bmatrix}$$

The evaluation of $R(t,\bar{h}) = det(At - B)$ leads to the stability polynomial of the DIE2OSBBDF as:

$$R(t,\bar{h}) = \frac{7543685}{63236789}t^2 + \frac{22931115}{252947156}t^2\bar{h} + \frac{5887899}{252947156}t^2\bar{h}^2 - \frac{95982151}{126473578}t^3\bar{h} - \frac{973605}{2941246}t^3\bar{h}^2 + \frac{1175400}{63236789}t^3\bar{h}^3 - \frac{276930845}{252947156}t^4\bar{h} + \frac{112715775}{252947156}t^4\bar{h}^2 - \frac{722250}{9033827}t^4\bar{h}^3 + \frac{2295}{9033827}t\bar{h}^3 - \frac{540}{63236789}th^4 + \frac{104800}{63236789}t\bar{h} + \frac{1647}{1470623}t\bar{h}^2 - \frac{205875}{63236789}t^2\bar{h}^3 + \frac{337500}{63236789}t^4\bar{h}^4 + t^4 - \frac{70780474}{63236789}t^3 \quad (19)$$

The region of absolute stability of the method (10) is drawn by plotting the graph of

$$R(t,\bar{h}) = 0. \quad (20)$$

$$R(t,\bar{h}) = \frac{7543685}{63236789}t^2 + \frac{22931115}{252947156}t^2\bar{h} + \frac{5887899}{252947156}t^2\bar{h}^2 - \frac{95982151}{126473578}t^3\bar{h} - \frac{973605}{2941246}t^3\bar{h}^2 + \frac{1175400}{63236789}t^3\bar{h}^3 - \frac{276930845}{252947156}t^4\bar{h} + \frac{112715775}{252947156}t^4\bar{h}^2 - \frac{722250}{9033827}t^4\bar{h}^3 + \frac{2295}{9033827}t\bar{h}^3 - \frac{540}{63236789}th^4 +$$

$\frac{104800}{63236789}t\bar{h}+\frac{1647}{1470623}t\bar{h}^2-\frac{205875}{63236789}t^2\bar{h}^3+\frac{337500}{63236789}t^4\bar{h}^4+t^4-\frac{70780474}{63236789}t^3$=0.
(21)

Letting $\bar{h}=0$ in equation (21) to obtain the first characteristics polynomial given by:

$$t^4+\frac{7543685}{63236789}t^2-\frac{70780474}{63236789}t^3=0. \quad (22)$$

The following roots of the first characteristic polynomial (22) are obtained after solving (22) as:

$$t=0, t=0, t=1, t=\frac{7543685}{63236789}$$

Definition 4.1: a block method (10) is said to be zero stable if no root of the first characteristics polynomial has modulus greater than one and any root with modulus one is simple [3].

Thus, by the definition 4.1, the DIE2OSBBDF is zero-stable since no modulus of any of the root is $>$ 1and the root $t=1$ is simple.

Definition 4.2: a block method (10) is said to be A-stable if its region of absolute stability covers the whole of the left-hand half-plane $Re(h\lambda)<0$ [20].

The stability region of the method is plotted using a boundary locus by setting$t=e^{i\theta}$in (21). Therefore, the graph of absolute stability region for the DIE2OSBBDF method is plotted using Maple 18 software as given below.

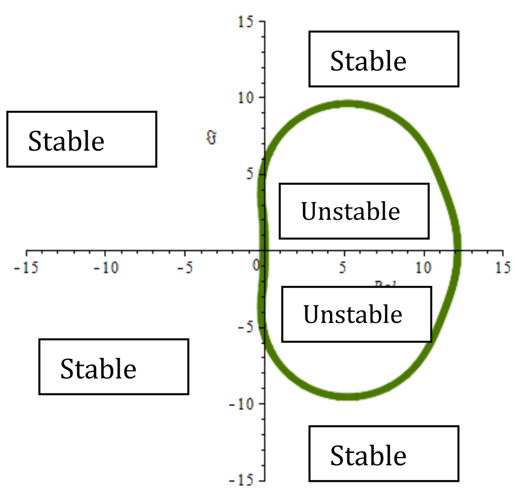

Figure 1: Stability region of the DIE2OSBBDF method

The stability region covers the whole left half plane. Therefore, by the definition 4.2, the method is A-stable and hence it is suitable for the numerical integration of first order stiff IVPs.

5 CONVERGENCE OF THE METHOD

This section presents the convergence of the DIE2OSBBDF method. According to theorem stated by [1], the necessary and sufficient conditions for a numerical integration method for solving initial value problems to be convergent is that it has to be both consistent and zero-stable.

Definition 5.1: the DIE2OSBBDF method is said to be consistent if it has order $p \geq 1$. It follows that the method is consistent if the following conditions are satisfied:

i. $\sum_{j=0}^{7} \gamma_j = 0$
ii. $\sum_{j=0}^{7} j\gamma_j = \sum_{j=0}^{7} \Psi_j$

Based on this definition, we have already shown in section 2 that the order of DIE2OSBBDF is five which is clearly greater than one. Thus by the definition of consistency, we deduced that the method (10) is consistent.

Let $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7$ and $\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5, \Psi_6, \Psi_7$ be as previously defined. Then

i. $\sum_{j=0}^{7} \gamma_j = \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6 + \gamma_7$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{5}{22} \\ \frac{1}{213} \\ \frac{9}{301} \\ -\frac{21}{1345} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{27}{22} \\ \frac{38}{71} \\ -\frac{85}{301} \\ \frac{99}{269} \end{bmatrix} + \begin{bmatrix} 1 \\ -\frac{328}{213} \\ \frac{45}{43} \\ -\frac{308}{269} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -\frac{540}{301} \\ \frac{513}{269} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -\frac{2844}{1345} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

ii. $\sum_{j=0}^{7} j\gamma_j = (0)\gamma_0 + (1)\gamma_1 + (2)\gamma_2 + (3)\gamma_3 + (4)\gamma_4 + (5)\gamma_5 + (6)\gamma_6 + (7)\gamma_7$

$$= (0)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (1)\begin{bmatrix} \frac{5}{22} \\ \frac{1}{213} \\ \frac{9}{301} \\ -\frac{21}{1345} \end{bmatrix} + (2)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (3)\begin{bmatrix} -\frac{27}{22} \\ \frac{38}{71} \\ -\frac{85}{301} \\ \frac{99}{269} \end{bmatrix} + (4)\begin{bmatrix} 1 \\ -\frac{328}{213} \\ \frac{45}{43} \\ -\frac{308}{269} \end{bmatrix} + (5)\begin{bmatrix} 0 \\ 1 \\ -\frac{540}{301} \\ \frac{513}{269} \end{bmatrix} + (6)\begin{bmatrix} 0 \\ 0 \\ 1 \\ -\frac{2844}{1345} \end{bmatrix} + (7)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{6}{11} \\ \frac{32}{71} \\ \frac{120}{301} \\ \frac{96}{269} \end{bmatrix} = 2\sum_{j=0}^{7} \psi_j$$

Since the first and second conditions of consistency are satisfied, therefore, we conclude that the proposed method is consistent. However, having satisfied both the conditions of consistent and that of zero-stability, then we also conclude that the diagonally implicit extended 2-point super class of block BDF with two off-step points converges and is suitable for the numerical integration of first order stiff initial value problems.

6 IMPLEMENTATION OF THE METHOD

Newton's iteration is used to implement the diagonally implicit extended 2-point super class of block BDF with two off-step points. We consider the implementation when $\rho = \frac{1}{5}$. The iteration is given below.

Definition 6.1:

Let y_i and $y(x_i)$ be the theoretical and approximate solutions of (1). Then the absolute error is defined by

$$(error_i)_t = |(y_i)_t - (y(x_i))_t| \tag{23}$$

The maximum error is defined by:

$$MAXE = \underbrace{max}_{1\le i\le T}\,(\underbrace{max(error_i)_t}_{1\le i\le T}), \tag{24}$$

where, T is the total number of steps and N is the number of equations.

Define

$$\left.\begin{aligned} F_{\frac{1}{2}} &= y_{n+\frac{1}{2}} - \frac{15}{44}hf_{n+\frac{1}{2}} + \frac{3}{44}hf_{n-1} - \varepsilon_{\frac{1}{2}} \\ F_1 &= y_{n+1} - \frac{328}{213}y_{n+\frac{1}{2}} - \frac{20}{71}hf_{n+1} + \frac{4}{71}hf_{n-\frac{1}{2}} - \varepsilon_1 \\ F_{\frac{3}{2}} &= y_{n+\frac{3}{2}} + \frac{45}{43}y_{n+\frac{1}{2}} - \frac{540}{301}y_{n+1} - \frac{75}{301}hf_{n+\frac{3}{2}} + \frac{15}{301}hf_n - \varepsilon_{\frac{3}{2}} \\ F_2 &= y_{n+2} - \frac{308}{269}y_{n+\frac{1}{2}} + \frac{513}{269}y_{n+1} - \frac{2844}{1345}y_{n+\frac{3}{2}} - \frac{60}{269}hf_{n+2} + \frac{12}{269}hf_{n+\frac{1}{2}} - \varepsilon_2 \end{aligned}\right\} \tag{25}$$

where,

$$\left.\begin{aligned} \varepsilon_{\frac{1}{2}} &= -\frac{5}{22}y_{n-1} + \frac{27}{22}y_n \\ \varepsilon_1 &= -\frac{1}{213}y_{n-1} - \frac{38}{71}y_n \\ \varepsilon_{\frac{3}{2}} &= -\frac{9}{301}y_{n-1} + \frac{85}{301}y_n \\ \varepsilon_2 &= \frac{21}{1345}y_{n-1} - \frac{99}{269}y_n \end{aligned}\right\} \tag{26}$$

are the back values.

Let $y_{n+j}^{(i+1)}$, $\quad j = \frac{1}{2}, 1, \frac{3}{2}, 2$, denote the $(i+1)^{th}$ iterative values of y_{n+j} and define

$$e_{n+j}^{(i+1)} = y_{n+j}^{(i+1)} - y_{n+j}^{(i)}, \qquad j = \frac{1}{2}, 1, \frac{3}{2}, 2 \tag{27}$$

Newton's iteration for the DIE2OSBBDF method takes the form:

$$y_{n+j}^{(i+1)} = y_{n+j}^{(i)} - \left(F'_j\left(y_{n+j}^{(i)}\right)\right)^{-1}\left(F_j\left(y_{n+j}^{(i)}\right)\right), \quad j = \frac{1}{2}, 1, \frac{3}{2}, 2 \tag{28}$$

$\Rightarrow$

$$\left(F'_j\left(y_{n+j}^{(i)}\right)\right)e_{n+j}^{(i+1)} = -\left(F_j\left(y_{n+j}^{(i)}\right)\right), \qquad j = \tfrac{1}{2}, 1, \tfrac{3}{2}, 2 \tag{29}$$

Equation (29) is equivalently written in matrix form as:

$$\underbrace{\begin{bmatrix} \left(1-\frac{15}{44}\frac{\partial F_{n+\frac{1}{2}}^{(i)}}{\partial y_{n+\frac{1}{2}}^{(i)}}\right) & 0 & 0 & 0 \\ -\frac{328}{213} & \left(1-\frac{20}{71}\frac{\partial F_{n+1}^{(i)}}{\partial y_{n+1}^{(i)}}\right) & 0 & 0 \\ \frac{45}{43} & -\frac{540}{301} & \left(1-\frac{75}{301}\frac{\partial F_{n+\frac{3}{2}}^{(i)}}{\partial y_{n+\frac{3}{2}}^{(i)}}\right) & 0 \\ \left(-\frac{308}{269}+\frac{12}{269}h\frac{\partial F_{n+2}^{(i)}}{\partial y_{n+\frac{1}{2}}^{(i)}}\right) & \frac{513}{269} & -\frac{2844}{1345} & \left(1-\frac{60}{269}h\frac{\partial F_{n+2}^{(i)}}{\partial y_{n+2}^{(i)}}\right) \end{bmatrix}}_{JacobianMatrix} \begin{bmatrix} e_{n+\frac{1}{2}}^{(i+1)} \\ e_{n+1}^{(i+1)} \\ e_{n+\frac{3}{2}}^{(i+1)} \\ e_{n+2}^{(i+1)} \end{bmatrix} =$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ \frac{328}{213} & -1 & 0 & 0 \\ -\frac{45}{43} & \frac{540}{301} & -1 & 0 \\ \frac{308}{269} & -\frac{513}{269} & \frac{2844}{1345} & -1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} + h\begin{bmatrix} 0 & -\frac{3}{44} & 0 & 0 \\ 0 & 0 & -\frac{4}{71} & 0 \\ 0 & 0 & 0 & -\frac{15}{301} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{bmatrix} +$$

$$h\begin{bmatrix} \frac{15}{44} & 0 & 0 & 0 \\ 0 & \frac{20}{71} & 0 & 0 \\ 0 & 0 & \frac{75}{301} & 0 \\ -\frac{12}{269} & 0 & 0 & \frac{60}{269} \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix} + \begin{bmatrix} \varepsilon_{\frac{1}{2}} \\ \varepsilon_1 \\ \varepsilon_{\frac{3}{2}} \\ \varepsilon_2 \end{bmatrix} \tag{30}$$

A programme in C language is written to implement the equation (30).

7. PROBLEMS TESTED

To demonstrate the effectiveness and efficiency of the method, the following stiff IVPs in ordinary differential equations are solved:

Problem 1

$$y' = -10xy, y(0) = 1{,}0, \qquad 0 \leq x \leq 10.$$

Exact solution: $y(x) = e^{-5x^2}$.

Source: [13]

Problem 2

$$y'_1 = -100y_1 + 9.901y_2, y_1(0) = 1{,}0, \qquad 0 \leq x \leq 10.$$
$$y'_2 = -0.1y_1 - y_2, y_2(0) = 10,$$

Exact solution:

$$y_1(x) = e^{-0.99x},$$
$$y_2(x) = 10e^{-0.99x}.$$

Eigenvalues: $\lambda = -0.99$ and $\lambda = -100.01$.

Source: [3].

Problem 3

$$y'_1 = -y_1 + 95y_2, \qquad y_1(0) = 1{,}0, \qquad 0 \leq x \leq 10.$$
$$y'_2 = -y_1 - 97y_2, \qquad y_2(0) = 1,$$

Exact solution:

$$y_1(x) = \frac{1}{47}(95e^{-2x} - 48e^{-2x}),$$
$$y_2(x) = \frac{1}{47}(48e^{-96x} - e^{-2x}).$$

Eigenvalues: $\lambda = -2$ and $\lambda = -96$.

Source: [3].

Problem 4

$$y'_1 = 198y_1 + 199y_2 \qquad y_1(0) = 1{,}0, \qquad 0 \leq x \leq 10.$$
$$y'_2 = -398y_1 - 399y_2, \qquad y_2(0) = -1,$$

Exact solution:

$$y_1(x) = e^{-x},$$

$$y_2(x) = -e^{-x}.$$

Eigenvalues: $\lambda = -1$ and $\lambda = -200$.

Source: [13]

8 NUMERICAL RESULTS

The numerical results for the problems tested are presented in Table 1-4. The tested problems are solved with the developed DIE2OSBBDF method and compared with the existing DI2BBDF and E2OSBBDF methods in terms of total number of steps taken to complete the integrations, maximum error and execution time. The notations used in the tables are described below:

H: Step Size.

NS: Total Number of Steps.

$MAXE$: Maximum Error.

$TIME$: Computation time (seconds).

$DI2BBDF$: Diagonally implicit 2-point block BDF method of order 2.

$E2OSBBDF$: Extended 2-point super class of block BDF with off-step points method of order 5.

$DIE2OSBBDF$: Diagonally implicit Extended 2-point super class of block BDF with two off-step points method.

Table 1: Comparison of Maximum error and Computation Time for Problem 1

H	$METHOD$	NS	$MAXE$	$TIME$
10^{-2}	$DI2BBDF$	500	$2.46470e-002$	$2.27900e-001$
	$E2OSBBDF$	500	$9.99654e-004$	$2.57400e-002$
	$DIE2OSBBDF$	500	$8.63160e-004$	$1.48900e-002$
10^{-3}	$DI2BBDF$	5000	$2.86495e-003$	$1.41800e-001$
	$E2OSBBDF$	5000	$1.04543e-005$	$3.56900e-002$
	$DIE2OSBBDF$	5000	$8.84045e-006$	$3.47100e-002$
10^{-4}	$DI2BBDF$	50000	$2.90509e-004$	$2.44600e-001$
	$E2OSBBDF$	50000	$1.04676e-007$	$2.76000e-001$
	$DIE2OSBBDF$	50000	$8.84532e-008$	$2.74500e-001$
10^{-5}	$DI2BBDF$	500000	$2.90910e-005$	$5.33100e+000$
	$E2OSBBDF$	500000	$1.04682e-009$	$2.49800e+000$
	$DIE2OSBBDF$	500000	$8.84539e-010$	$2.48900e+000$
10^{-6}	$DI2BBDF$	5000000	$2.90950e-006$	$3.35500e+000$
	$E2OSBBDF$	5000000	$4.98775e-011$	$2.44500e+001$
	$DIE2OSBBDF$	5000000	$5.11539e-011$	$2.42800e+001$

Table 2: Comparison of Maximum error and Computation Time for Problem 2

H	$METHOD$	NS	$MAXE$	$TIME$
10^{-2}	$DI2BBDF$	500	$7.10925e-002$	$1.72000e-001$
	$E2OSBBDF$	500	$9.43228e-004$	$2.29400e-002$
	$DIE2OSBBDF$	500	$8.17317e-004$	$2.39200e-001$
10^{-3}	$DI2BBDF$	5000	$7.26651e-003$	$1.76300e-001$
	$E2OSBBDF$	5000	$1.01369e-005$	$1.38500e-001$
	$DIE2OSBBDF$	5000	$8.60081e-006$	$1.62400e-001$
10^{-4}	$DI2BBDF$	50000	$7.28226e-004$	$1.82900e-001$
	$E2OSBBDF$	50000	$1.02434e-007$	$1.18200e+000$
	$DIE2OSBBDF$	50000	$8.66072e-008$	$1.18800e+000$
10^{-5}	$DI2BBDF$	500000	$7.28384e-005$	$5.62800e-001$
	$E2OSBBDF$	500000	$1.02573e-009$	$1.14600e+001$
	$DIE2OSBBDF$	500000	$8.66864e-010$	$1.15400e+001$
10^{-6}	$DI2BBDF$	5000000	$7.28409e-006$	$4.28800e+000$
	$E2OSBBDF$	5000000	$7.86802e-010$	$1.15500e+002$
	$DIE2OSBBDF$	5000000	$1.14690e-009$	$1.16300e+002$

Table 3: Comparison of Maximum error and Computation Time for Problem 3

H	$METHOD$	NS	$MAXE$	$TIME$
10^{-2}	$DI2BBDF$	500	$9.37034e+005$	$2.51600e-001$
	$E2OSBBDF$	500	$2.64316e-002$	$2.66900e-002$
	$DIE2OSBBDF$	500	$2.59017e-002$	$3.44700e-002$
10^{-3}	$DI2BBDF$	5000	$5.58180e-002$	$2.43700e-001$
	$E2OSBBDF$	5000	$5.96783e-003$	$1.50300e-001$
	$DIE2OSBBDF$	5000	$5.63595e-003$	$1.56200e-001$
10^{-4}	$DI2BBDF$	50000	$7.04562e-003$	$2.61300e-001$
	$E2OSBBDF$	50000	$9.07731e-005$	$1.23500e+000$
	$DIE2OSBBDF$	50000	$7.86030e-005$	$1.22400e+000$
10^{-5}	$DI2BBDF$	500000	$7.19659e-004$	$7.92400e-001$
	$E2OSBBDF$	500000	$9.73764e-007$	$1.18300e+001$
	$DIE2OSBBDF$	500000	$8.26124e-007$	$1.18500e+001$
10^{-6}	$DI2BBDF$	5000000	$7.21171e-005$	$5.05000e+000$
	$E2OSBBDF$	5000000	$9.83728e-009$	$1.19900e+002$
	$DIE2OSBBDF$	5000000	$8.31721e-011$	$1.20000e+002$

Table 4: Comparison of Maximum error and Computation Time for Problem 4

H	$METHOD$	NS	$MAXE$	$TIME$
10^{-2}	$DI2BBDF$	500	$1.26860e+021$	$1.69600e-001$
	$E2OSBBDF$	500	$9.61694e-005$	$3.51200e-002$
	$DIE2OSBBDF$	500	$8.33504e-005$	$3.50000e-002$
10^{-3}	$DI2BBDF$	5000	$7.33973e-004$	$1.39200e-001$
	$E2OSBBDF$	5000	$1.03416e-006$	$1.44900e-001$
	$DIE2OSBBDF$	5000	$8.77480e-007$	$1.41500e-001$
10^{-4}	$DI2BBDF$	50000	$7.35580e-005$	$2.64300e-001$
	$E2OSBBDF$	50000	$1.04513e-008$	$1.20600e+001$
	$DIE2OSBBDF$	50000	$8.83649e-009$	$1.21800e+001$
10^{-5}	$DI2BBDF$	500000	$7.35741e-006$	$5.57000e-001$
	$E2OSBBDF$	500000	$1.04655e-010$	$1.15700e+001$
	$DIE2OSBBDF$	500000	$8.84469e-011$	$1.16900e+001$
10^{-6}	$DI2BBDF$	5000000	$7.35765e-007$	$4.18100e+000$
	$E2OSBBDF$	5000000	$7.72708e-011$	$1.16300e+002$
	$DIE2OSBBDF$	5000000	$1.14009e-010$	$1.18500e+002$

In order to give visual impact on the reliability and efficiency of the method, the efficiency curves of $Log_{10}(MAXE)$ against H for the tested problems are plotted. The graphs of the scaled maximum error for different problems tested are given below.

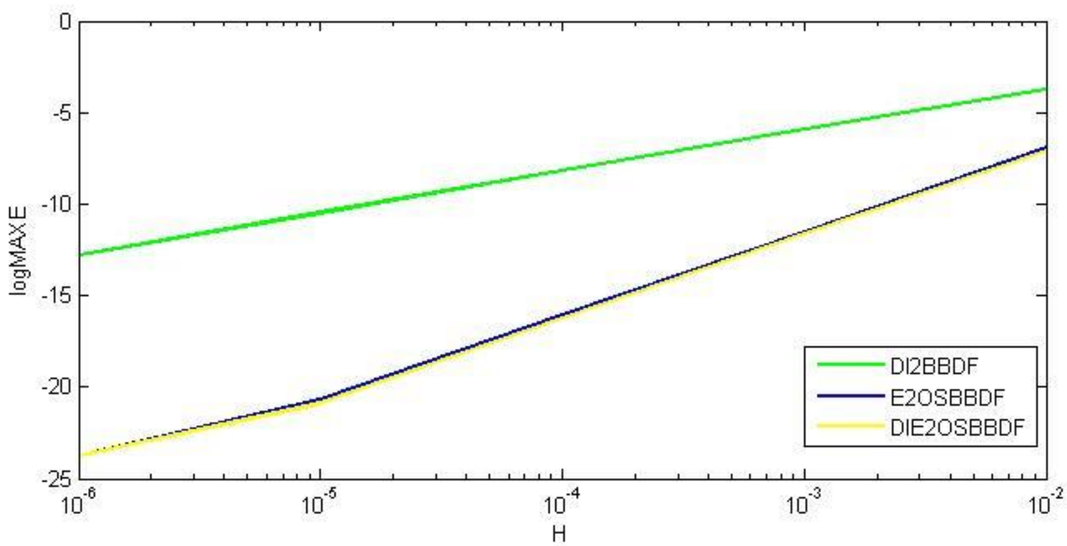

Figure 2: Efficiency Curves for Problem 1

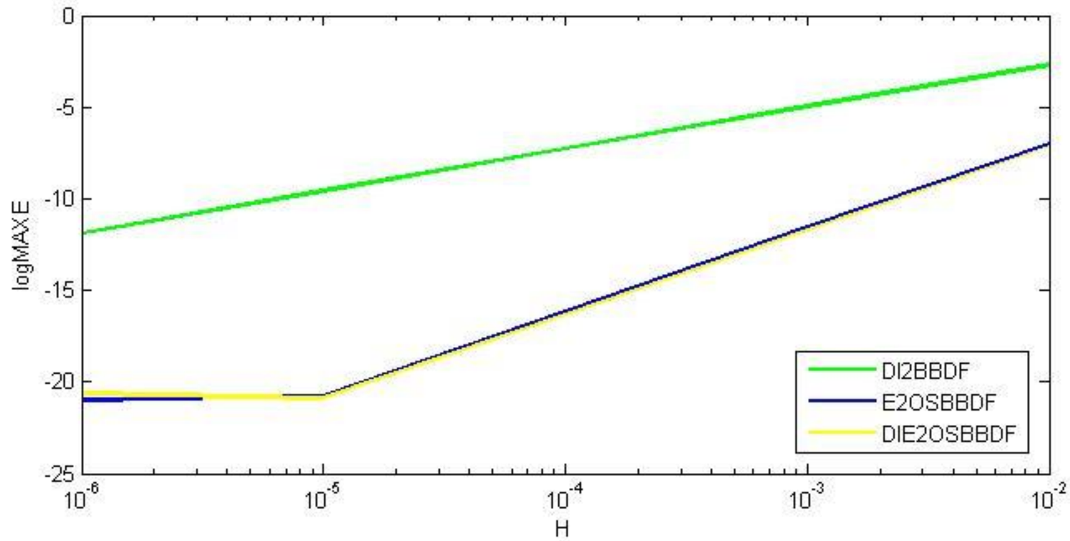

Figure 3: Efficiency Curves for Problem 2

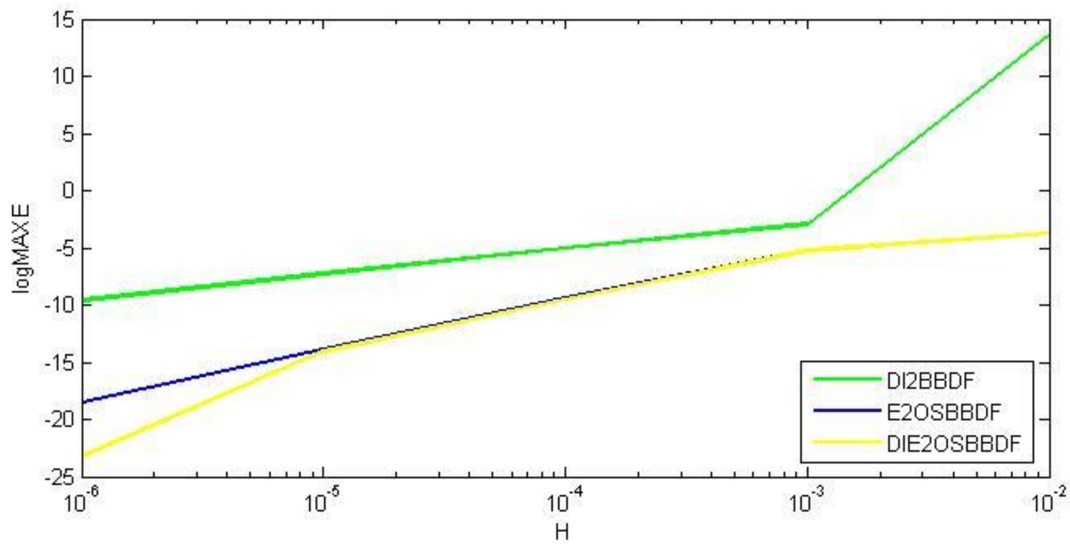

Figure 4: Efficiency Curves for Problem 3

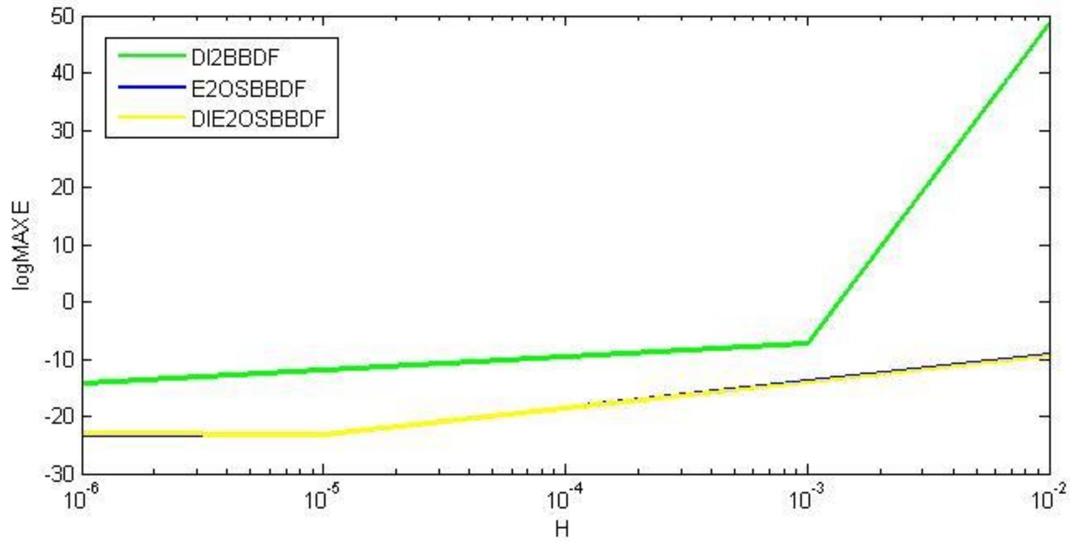

Figure 5: Efficiency Curves for Problem 4

9 DISCUSSION OF RESULT

From the Tables 1– 4 prsented, it is observed that the DIE2OSBBDF gives a better accuracy than the existing DI2BBDF and E2OSBBDF methods. Convergence is noticeable in the developed new method by the decrease in maximum error as the mesh size H approaches zero. The accuracy of the developed method at any fixed point keeps improving as the step size keeps reducing. Therefore, the numerically computed approximation approaches the theoretical solution as H tends to zero. Thus, the execution time of the developed method is competing with those computational times in DI2BBDF and E2OSBBDF methods. The efficiency curves in Figure 2 – 5 also show that the scaled maximum error for the DIE2OSBBDF is less when compared with that in the existing DI2BBDFand E2OSBBDF. This is the reason why the efficiency curves of the DIE2OSBBDF are below that of the DI2BBDF and E2OSBBDF methods.

10 CONCLUSION

A new diagonally implicit extended 2-point super class of block BDF with off-step points is developed. The method is of order 5. The stability properties of the method have shown that the method is consistent, zero-stable and convergent. The region of absolute stability is plotted and it indicates that the proposed method is A-stable. When the DIE2OSBBDF method is compared with the existing DI2BBDF and E2OSBBDF methods, the results obtained show that the new method is better than DI2BBDF and E2OSBBDF in terms of accuracy. However, the new method is competing with existing methods in terms of computational time. Therefore, we conclude that the method is a good alternative solver for stiff initial value problems.

ACKNOWLEDGEMENT

The authors are grateful to Umaru Musa Yar'adua University, Katsina and Al-Qalam University, Katsina, Nigeria for the support and assistance they received during the conduct of the research.

REFERENCES

[1] J. D. Lambert, *Computaional Methods in Ordinary Differential Equations*, New York: John Wiley & Sons, 1973.

[2] G. G. Dahlquist, "Problems related to the numerical treatment of stiff differential equations,", *Proc. Int. Computing Sym*. pp. 307-314, 1974.

[3] M. B. Sulaiman, H. Musa, F. Ismail, N. Senu, and Z. B. Ibrahim, "A new super class of block backward differentiation formula for stiff ODEs," *Asian European Journal of Mathematics*, vol. 7, no. 1, 2014.

[4] C.F. Curtiss and J.O. Hirschfelder, "Integration of stiff equations," in *Proceedings - 1952 of the National Academy of Sciences*, vol. 38, pp. 235-243, 1952.

[5] H. Musa and A. A. Umar, "An extended 2-point super class of block backward differentiation formula with off-step points for solving stiff initial value problems," *Nigerian Journal of Mathematics and Applications*, vol. 32, no. 2, pp. 1-14, 2022.

[6] Z. B. Ibrahim, N. M. Noor, and K. I. Othman," Fixed coefficient $A(\alpha)$ stable block backward differentiation formulas for stiff ordinary differential equations", MDPI, Symmetry, vol. 11, no. 846, pp. 1-12, 2019.

[7] N. Bala, B. Alhassan, and H. Musa," A new 3-point implicit block numerical method for solving first order stiff initial value problems," *International Journal of Science Research and Technology*, vol. 9, no. 9, pp. 219-231, 2022.

[8] S. J. Aksah, Z. B. Ibrahim, and I. S. M. Zawawi, "Stability analysis of singly diagonally implicit block backward differentiation formulas for stiff ordinary differential equations," *Mathematics*, vol. 7, pp. 211, 2019.

[9] Z. B. Ibrahim and A. A. Nasarudin, "A class of hybrid multistep block methods with A-stability for the numerical solution of stiff ordinary differential equations," *Mathematics*, vol. 8, pp. 914, 2020.

[10] A. Buhari, M. Hamisu, and N. Abasi," Convergence and order of the 2-point diagonally implicit block backward differentiation formula with two off-step points," *UMYU Scientifica,* vol. 1, no. 1, pp. 30-38, 2022.

[11] H. Musa, M. B. Suleiman, and N. Senu, "Fully implicit 3-point block extended backward differentiation formula for stiff initial value problems," *Applied Mathematical Sciences*, vol. 6, pp. 4211-4228, 2012.

[12] N. I. N. Ismail, Z. A. Majid, and N.Senu, "Hybrid multistep block method for solving neutral delay differential equations," *Sains Malaysiana*, vol. 49, pp. 929-940, 2020.

[13] Z. B. Ibrahim, K. Othman, and M.B. Suleiman, "Implicit r-point block backward differentiation formula for solving first- order stiff ODEs," *Applied Mathematics and Computation*, vol. 186, pp. 558-565, 2007.

[14] H. Musa, M. B. Suleiman, F. Ismail, N. Senu, Z. A. Majid, and Z. B. Ibrahim, "A new fifth order implicit block method for solving first order stiff ordinary differential equations," *Malaysian Journal of Mathematical Sciences*, vol. 8, pp. 45-59, 2014.

[15] Z. A. Majid, N. M. Nasir, F. Ismail, and N. Bachok, "Two point diagonally block method for solving boundary value problems with Robin boundary conditions," *Malaysian Journal of Mathematical Sciences*, vol. 13, pp. 1-14, 2019.

[16] N. Abasi, M. Suleiman, and H. Musa, "2-point block BDF method with off-step points for solving stiff ODEs," *Journal of Soft Computing and Applications*, vol. 2014, pp.1-15, 2014.

[17] I. S. M. Zawawi, Z. B. Ibrahim, F. Ismail, and Z. A. Majid, "Diagonally implicit block backward differentiation formula for solving ODEs," *International Journal of Mathematics and Mathematical Sciences*, pp. 1-8, 2012.

[18] Z. B. Ibrahim, N. Zainuddin, K. I. Othman, M. Suleiman, and I. S. M. Zawawi, "Variable order block method for solving second order ordinary differential equations," *Sains Malays*., vol. 48, pp. 1761–1769, 2019.

[19] M. B. Suleiman, H. Musa, F. Ismail, and N. Senu, "A new variable step size block backward differentiation formula for solving stiff initial value problems," *International Journal of Computer Mathematics*, vol. 90, pp. 2391-2408, 2013.

[20] I. S. M. Zawawi, Z. B. Ibrahim, and K. I. Othman, "Derivation of Diagonally Implicit Block Backward Differentiation Formulas for solving Stiff Initial Value Problems," *Hindawi Publishing Coperation, Mathematical Problems in Engineering*, pp. 1-13, 2015.