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Unique Solution of an Infinite 2-System Model of First Order Ordinary Differential Equation

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ABSTRACT

This work is to solve an infinite 2-system model of first order ordinary differential equations. The system is in Hilbert space l_2 *with the coefficients are any positive real numbers. The system is rewritten as a system in the form of matrix equations and it is first studied in* $\mathbb{R}^2$ *where its solution is obtained and a fundamental matrix is constructed. The results are carried out to solve the infinite 2-system in Hilbert space* l_2*. The control functions satisfy integral constraint and are elements of the space of square integrable function in* l_2*. The existence and uniqueness of the solution of the system in Hilbert space* l_2 *on an interval time* $[0, T]$ *for a sufficiently large* T *is then proven.*

Keywords: Infinite 2-system, Hilbert space, matrix, differential equation.

1 INTRODUCTION

Parabolic and hyperbolic partial differential equations are used to describe control problems that are related to some problems in economy, engineering, defence industry etc. In mathematics, some control problems described by partial differential equations could be reduced to an infinite system of ordinary differential equation by using decomposition method (see for instance, [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11] and [12]).

In [10] for example, the following parabolic equation was considered:

$$z_t = Az - u + v, z|_{t=0} = z_0(x), z|_{S_T} = 0,$$

where

$$Az = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x) z_{x_j}\right),$$

satisfies the following inequality:

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \gamma \sum_{i=1}^{n} \xi_i^2.$$

The game system was proven to have a unique solution $z = z(x,t)$ in the form of

$$z(x,t) = \sum_{i=1}^{\infty} z_i(t)\psi_i(x),$$

which is also the solution for the following infinite system of differential equations:

$$\dot{z}_i = \lambda_i z_i - u_i(t) + v_i(t), \quad z_i(0) = z_{i0}, \quad i = 1, 2, \ldots . \tag{1}$$

The work by [10] above, illustrates the importance of studying infinite system of ordinary differential equation, as it has a strong relationship with parabolic or hyperbolic differential equation. However, the infinite system can be studied independently from the problem of partial differential equations as shown in [13], [14], [15], [16] and [17].

In [13] for example, an infinite system of ordinary differential equations was described as follows,

$$\begin{aligned} \dot{x}_k &= -\alpha_k x_k - \beta_k y_k + w_{1k}, x_k(0) = x_{k0} \\ \dot{y}_k &= \beta_k x_k - \alpha_k y_k + w_{2k}, y_k(0) = y_{k0} \end{aligned} \tag{2}$$

where α_k is a positive real number and β_k is any real number for $k = 1, 2, \ldots$.. The system is in Hilbert space l_2 and it was shown that there exists a unique solution for the system in the space.

The problem of countable number of first-order differential equations with function coefficients was later studied in [14], in which, the existence-uniqueness theorem of solution to the model in Hilbert space l_2 was proved.

The study of such infinite system was then extended to an infinite first order 2-systems of differential equations as can be seen in [18]. In the article, the system studied was described as follows:

$$\begin{aligned} \ddot{x}_k &= -\alpha_k x_k - \beta_k y_k + w_{1k}, x_k(0) = x_{k0}, \dot{x}_k(0) = x_{k1} \\ \ddot{y}_k &= \beta_k x_k - \alpha_k y_k + w_{2k}, \dot{y}_k(0) = y_{k1} \end{aligned} \tag{3}$$

where α_k, β_k are real numbers for $k = 1, 2, \ldots$.. The system is in Hilbert space ${l_{r+1}}^2$ and it was shown that the solution of the system exists and is unique in the space.

The results obtained in studying the existence and uniqueness of solution in such system were used in the study of optimal control and differential games problems described by the systems in the space considered.

For example, in [16], a pursuit differential game for an infinite first order 2-systems of differential equations in Hilbert space l_2 was studied. The result from this work was a formula for guaranteed pursuit time, which occur when the state of the system coincides with the origin. An explicit pursuer strategy was constructed, where the control of players were constrained by geometric constraints.

Further work of infinite system can also be found in [17]. The work was about a linear pursuit differential game of geometric constraint described by an infinite system of first-order differential equations. The state of the system was to be brought by the pursuer, from a given initial state, to the origin in a finite time. However, the evader tried to avoid this to happen. A strategy for the pursuer was constructed where the guaranteed pursuit time was obtained. On the other hand, a formula of guaranteed evasion time was also obtained in the evasion part of the game.

This independent study continues to be carried out in the present work where the space of the game is Hilbert space l_2, which is a complete linear vector space of any sequences of real numbers as stated below:

$$l_2 = \left\{ \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n, \dots) | \alpha_n \in \mathbb{R}, \sum_{n=1}^{\infty} \alpha_n^2 < \infty \right\}$$

with the following inner product and norm defined by :

$$\langle \alpha, \beta \rangle = \sum_{n=1}^{\infty} \alpha_n \beta_n \text{ and } \|\alpha\| = \sqrt{(\alpha, \alpha)} \text{ for } \alpha, \beta \in l_2.$$

In addition, the control function w is an element of $L(0, T; l_2)$, which is the space of square integrable function in l_2 on the time interval $[0, T]$ for a sufficiently large T.

In this work, an infinite first order 2-systems of differential equations is studied by proving the existence and uniqueness of the solution of the system. The process is carried out by using an obtained fundamental matrix.

2 PROBLEM STATEMENT

The game system of this project is described by the following infinite 2-system of first order ordinary differential equation:

$$\begin{aligned}
&\dot{x}_i = -\lambda_i x_i + y_i + w_{i1},\\
&\dot{y}_i = -\lambda_i y_i + w_{i2}, \qquad\qquad i = 1, 2, ...\\
&x_i(0) = x_{i0},\ y_i(0) = y_{i0},
\end{aligned} \tag{4}$$

where λ_i is a positive scalar function and $x_0 = (x_{10}, x_{20}, ...), y_0 = (y_{10}, y_{20}, ...), x_i = (x_{1i}, x_{2i}, ...), y_i = (y_{1i}, y_{2i}, ...), \dot{x}_i = (\dot{x}_{1i}, \dot{x}_{2i}, ...), \dot{y}_i = (\dot{y}_{1i}, \dot{y}_{2i}, ...)$ are all elements in l_2. In addition, the control function is the function $w : [0, T] \to l_2$ such that $w(t) = (w_1(t), w_2(t), ...)$, with measurable coordinates $w_i(t) = (w_{i1}(t), w_{i2}(t)), 0 \le t \le T$, satisfying the condition

$$\sum_{i=1}^{\infty} \int_0^T \left(w_{i1}^2(t) + w_{i2}^2(t)\right) dt \le \rho_0^2 \tag{5}$$

for a given positive number ρ_0. The purpose is to prove the existence and the uniqueness of the solution of the system (4) in Hilbert space l_2 . In other words, we prove that the solution exists and belongs to the space $C(0, T; l_2)$, which is the space of continous function in l_2 on time interval $[0, T]$ for a sufficiently large T.

3 PRELIMINARY RESULTS

3.1 Solution for a 2-system of differential equation in $\mathbb{R}^2$

A general solution of a 2-system differential equations is first to be obtained in the space of $\mathbb{R}^2$. The method is later extended to the system in Hilbert space l_2. The scalar function in the system could be any positive real number and the system is as follows:

$$\begin{aligned}
&\dot{x} = -\lambda x + y + w_1,\\
&\dot{y} = -\lambda y + w_2,\\
&x(0) = x_0,\ y(0) = y_0,
\end{aligned} \tag{6}$$

where λ is a positive scalar function and $x_0, y_0, x, y, \dot{x}, \dot{y}, w_1, w_2 \in \mathbb{R}$.

To obtain the solution $z = (x, y) \in \mathbb{R}^2$ of the system, (6) is rewritten into a matrix equation as follows:

$$\dot{\begin{bmatrix} x \\ y \end{bmatrix}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}. \tag{7}$$

From (7), let $z = \begin{bmatrix} x \\ y \end{bmatrix}, C = \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$. Then (6) becomes the following ordinary differential equation (ODE) in $\mathbb{R}^2$:

$$\begin{aligned} \dot{z} &= Cz + w \\ z(0) &= z_0 \end{aligned} \tag{8}$$

where C is a scalar function and $z_0, z, \dot{z}, w \in \mathbb{R}^2$. By straight forward calculation, the solution of (8) is

$$z(t) = e^{Ct}\left(z_0 + \int_0^t e^{-sC} w(s)ds\right). \tag{9}$$

3.2 Fundamental Matrix

We are now obtaining our fundamental matrix e^{Ct} which appear in equation (9). The fundamental matrix will later be used in the infinite system in Hilbert space l_2.

Lemma 3.2.1. *Let λ be a positive scalar function, $C = \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}$ and $t \in [0, T]$. Then $e^{Ct} = e^{-\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$.*

Proof.

$$\begin{aligned} e^{Ct} &= I + \frac{Ct}{1!} + \frac{C^2t^2}{2!} + \cdots + \frac{C^nt^n}{n!} + \ldots \\ &= I + \frac{\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix} t}{1!} + \frac{\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}^2 t^2}{2!} + \cdots + \frac{\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}^n t^n}{n!} + \ldots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -\lambda t & t \\ 0 & -\lambda t \end{bmatrix} + \begin{bmatrix} \frac{\lambda^2t^2}{2!} & \frac{-\lambda t^2}{1!} \\ 0 & \frac{\lambda^2t^2}{2!} \end{bmatrix} + \cdots + \begin{bmatrix} \frac{-\lambda^nt^n}{n!} & \frac{\lambda^{(n-1)}t^n}{(n-1)!} \\ 0 & \frac{-\lambda^nt^n}{n!} \end{bmatrix} + \ldots \\ &= \begin{bmatrix} \sum_{n=0}^{\infty} \frac{(-\lambda t)^n}{n!} & t\sum_{n=0}^{\infty} \frac{(-\lambda t)^n}{n!} \\ 0 & \sum_{n=0}^{\infty} \frac{(-\lambda t)^n}{n!} \end{bmatrix} \\ &= \begin{bmatrix} e^{-\lambda t} & te^{-\lambda t} \\ 0 & e^{-\lambda t} \end{bmatrix} \\ &= e^{-\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

□

The following are a few properties of the fundamental matrix.

Lemma 3.2.2. Some Properties of the Fundamental Matrix

Let $A(t) = e^{Ct} = e^{-\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ for $\lambda > 0$ and $h, t \in [0, T]$. Then the matrix A has the following properties:

1. $A(t+h) = A(t)A(h) = A(h)A(t) = A(h+t)$
2. $|A(t)z| = |A^T(t)z| < e^{-\lambda t}\sqrt{t^2+2}|z|$
3. $||A(t) - I_2|| < T + 3 \, for \, t \in [0,T]$

Proof.

1.

$$\begin{aligned} A(t+h) &= e^{-\lambda(t+h)} \begin{bmatrix} 1 & t+h \\ 0 & 1 \end{bmatrix} \\ &= e^{-\lambda t} e^{-\lambda h} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} &= e^{-\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} e^{-\lambda h} \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \\ &= A(t)A(h) \\ &= A(h)A(t) \\ &= e^{-\lambda h} \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} e^{-\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \\ &= e^{-\lambda h} e^{-\lambda t} \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \\ &= e^{-\lambda(h+t)} \begin{bmatrix} 1 & h+t \\ 0 & 1 \end{bmatrix} \\ &= A(h+t). \end{aligned}$$

2.

$$\begin{aligned} A(t)z &= e^{-\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= e^{-\lambda t} \begin{bmatrix} x + ty \\ y \end{bmatrix}. \end{aligned}$$

This implies,

$$
\begin{aligned}
|A(t)z| &= e^{-\lambda t}\sqrt{(x+ty)^2+y^2} \\
&= e^{-\lambda t}\sqrt{x^2+2txy+(t^2+1)y^2} \\
&\le e^{-\lambda t}\sqrt{x^2+(t^2x^2+y^2)+(t^2+1)y^2} \\
&= e^{-\lambda t}\sqrt{x^2(1+t^2)+y^2(t^2+2)} \\
&< e^{-\lambda t}\sqrt{(t^2+2)(x^2+y^2)} \\
&= e^{-\lambda t}\sqrt{t^2+2}|z|.
\end{aligned}
$$

3.

$$
\begin{aligned}
\|A(t)-I_2\| &\le \|A(t)\| + \|I_2\| \\
&= \max_{|d|=1}|A(t)d| + 1 \text{ for any vector } d \text{ of which its norm vector } |d| = 1 \\
&< \max_{|d|=1}\left(e^{-\lambda t}\sqrt{t^2+2}|d|\right) + 1 \\
&\le e^{-\lambda t}\sqrt{t^2+2} + 1 \\
&< 1 + \sqrt{t^2+2} \\
&\le 1 + \sqrt{t^2} + \sqrt{2} \\
&= 1 + t + \sqrt{2} \\
&< t + 3 \\
&\le T + 3
\end{aligned}
$$

□

4 MAIN RESULT

As described in the previous section, the game system (4) is rewritten in a similar fashion as follows:

$$
\begin{aligned}
&\dot{z}_i = C_i z_i + w_i, \quad i = 1, 2, \ldots \\
&z_i(0) = z_{i0},
\end{aligned}
\tag{10}
$$

where $z_i(t) = (x_i(t), y_i(t))$ and $C_i = \begin{bmatrix} -\lambda_i & 1 \\ 0 & -\lambda_i \end{bmatrix}$, $\lambda_i > 0$ with the function $w(t) = (w_1(t), w_2(t), \ldots)$ such that $w_i(t) = (w_{i1}(t), w_{i2}(t))$ be an admissible control function, that is, it satisfies (5).

The idea is that, the existence and uniqueness of solution of (10) in l_2 will imply the existence and uniqueness of solution of (4) in l_2, both on the time interval $[0, T]$.

Definition 4.1. *A function* $z(t) = (z_1(t), z_2(t), ...), 0 \leq t \leq T$ *where* T *is a given positive number, is called the solution of the system* (10) *if each coordinate* z_i *of* z*,*

1. *is continuous and differentiable on* $(0, T)$ *and satisfies the initial condition* $z_i(0) = z_i^0$*,*
2. *has the first derivative* $\dot{z}_i(t)$ *on* $(0, T)$ *and satisfies the system* (10) *almost everywhere on* $(0, T)$*, where* $C_i = \begin{bmatrix} -\lambda_i & 1 \\ 0 & -\lambda_i \end{bmatrix}$, $z_i = (x_i, y_i)$ *and* $w_i = (w_{i1}, w_{i2})$ *be the control parameter.*

Theorem 4.1. *If* $z = (z_1, z_2, ...), z_0 = (z_{10}, z_{20}, ...), \dot{z} = (\dot{z}_1, \dot{z}_2, ...) \in l_2$ *and* $w(t) = (w_1(t), w_2(t), ...) \in L_2(0, T; l_2)$ *for* $0 \leq t \leq T$ *where* $w_i(t) = (w_{i1}(t), w_{i2}(t), ...)$ *be an admissible control function, then the game system* (10) *has a unique solution* $z(t) = (z_1(t), z_2(t), ...)$ *in Hilbert space* l_2 *where* $0 \leq t \leq T$ *for any given* $T > 0$*.*

Proof.
The proof begin by proving the existence of solution in l_2, followed by proving the solution function is continuous on the time interval $[0, T]$, which implies its uniqueness (Theorem 2.2.1 : [19]).

The Existence

Note that our fundamental matrix is $A_i(t) = e^{C_i t}$. By referring to the solution for the system (6) which is (9), we derive that:

$$\begin{aligned} z_i(t) &= A_i(t)\Big(z_{i0} + \int_0^t A_i(-s)w_i(s)ds\Big) \\ &= A_i(t)z_{i0} + \int_0^t A_i(t-s)w_i(s)ds. \end{aligned}$$

Now, by using Lemma 3.2.2, Cauchy Schwartz Inequality, $\lambda_i > 0$ for each $i = 1, 2, \ldots, k$, and the inequality $(a + b)^2 \leq 2(a^2 + b^2)$ for any real number a and b, we have

$$\begin{aligned} |z_i(t)|^2 &\leq 2|A_i(t)z_{i0}|^2 + 2\left|\int_0^t A_i(t-s)w_i(s)ds\right|^2 \\ &< 2\big(e^{-\lambda_i t}\sqrt{t^2+2}|z_{i0}|\big)^2 + 2\Big(\int_0^t e^{-\lambda_i (t-s)}\sqrt{(t-s)^2+2}|w_i(s)|ds\Big)^2 \end{aligned}$$

$$\begin{aligned}
&\leq 2e^{-2\lambda_i t}(t^2+2)|z_{i0}|^2 + 2\int_0^t e^{-2\lambda_i(t-s)}((t-s)^2+2)ds\int_0^t |w_i(s)|^2 ds\\
&\leq 2(t^2+2)|z_{i0}|^2 + 2\int_0^t ((t-s)^2+2)ds\int_0^t |w_i(s)|^2 ds\\
&\leq 2(t^2+2)|z_{i0}|^2 + 2\int_0^t (t^2+2)ds\int_0^t |w_i(s)|^2 ds\\
&= 2(t^2+2)|z_{i0}|^2 + 2t(t^2+2)\int_0^t |w_i(s)|^2 ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{i=1}^{\infty}|z_i(t)|^2 &\leq \sum_{i=1}^{\infty} 2(t^2+2)|z_{i0}|^2 + \sum_{i=1}^{\infty} 2t(t^2+2)\int_0^t |w_i(s)|^2 ds\\
&= 2(t^2+2)\left(\sum_{i=1}^{\infty}|z_{i0}|^2 + t\sum_{i=1}^{\infty}\int_0^t |w_i(s)|^2 ds\right)\\
&\leq 2(t^2+2)\left(\sum_{i=1}^{\infty}|z_{i0}|^2 + T\sum_{i=1}^{\infty}\int_0^T |w_i(s)|^2 ds\right)\\
&= 2(t^2+2)\left(\|z_0\|_{l_2}^2 + T\|w(\cdot)\|_{L_2(0,T;l_2)}^2\right)\\
&< \infty
\end{aligned}$$

since $z_0 \in l_2$ and $w(\cdot) \in L_2(0,T;l_2)$. We conclude that $z(t) \in l_2$ for $t \in [0,T]$.

The continuity on $[0,T]$

Next, we need to show that $z(t)$ is continuous, that is, for any $\varepsilon > 0$, $\exists\delta > 0$ such that $\|z(t+h) - z(t)\|_{l_2}^2 < \varepsilon$ whenever $|h| < \delta$.

Proof.
For $h > 0$:

$$\begin{aligned}
z_i(t+h) - z_i(t) &= A_i(t+h)z_{i0} + \int_0^{t+h} A_i(t+h-s)w_i(s)ds - \left(A_i(t)z_{i0} + \int_0^t A_i(t-s)w_i(s)ds\right)\\
&= \left(A_i(t)A_i(h)\right)z_{i0} + \int_0^t A_i(t+h-s)w_i(s)ds + \int_t^{t+h} A_i(t+h-s)w_i(s)ds
\end{aligned}$$

$$
\begin{aligned}
&- A_i(t)z_{i0} - \int_0^t A_i(t-s)w_i(s)ds \\
&= \big(A_i(t)A_i(h) - A_i(t)\big)z_{i0} + \int_0^t A_i(t+h-s)w_i(s)ds - \int_0^t A_i(t-s)w_i(s)ds + \int_t^{t+h} A_i(t+h-s)w_i(s)ds \\
&= \big(A_i(h) - I_2\big)A_i(t)z_{i0} + \int_0^t A_i(t+h-s)w_i(s)ds - \int_0^t A_i(t-s)w_i(s)ds + \int_t^{t+h} A_i(t+h-s)w_i(s)ds \\
&= \big(A_i(h) - I_2\big)A_i(t)z_{i0} + \int_0^t A_i(h)A_i(t-s)w_i(s)ds - \int_0^t A_i(t-s)w_i(s)ds + \int_t^{t+h} A_i(t+h-s)w_i(s)ds \\
&= \big(A_i(h) - I_2\big)A_i(t)z_{i0} + \int_0^t (A_i(h) - I_2)A_i(t-s)w_i(s)ds + \int_t^{t+h} A_i(t+h-s)w_i(s)ds.
\end{aligned}
$$

Now,

$$
\begin{aligned}
|z_i(t+h) - z_i(t)|^2 &\le 3\big|\big(A_i(h) - I_2\big)A_i(t)z_{i0}\big|^2 + 3\left|\int_0^t (A_i(h) - I_2)A_i(t-s)w_i(s)ds\right|^2 \\
&\quad + 3\left|\int_t^{t+h} A_i(t+h-s)w_i(s)ds\right|^2 \\
&= 3\|A_i(h) - I_2\|^2|A_i(t)z_{i0}|^2 + 3\|A_i(h) - I_2\|^2\left|\int_0^t A_i(t-s)w_i(s)ds\right|^2 + 3\left|\int_t^{t+h} A_i(t+h-s)w_i(s)ds\right|^2 \\
&\le 3\|A_i(h) - I_2\|^2|A_i(t)z_{i0}|^2 + 3\|A_i(h) - I_2\|^2\int_0^t 1ds\int_0^t |A_i(t-s)w_i(s)|^2ds \\
&\quad + 3\int_t^{t+h} 1ds\int_t^{t+h} |A_i(t+h-s)w_i(s)|^2ds \\
&\le 3\|A_i(h) - I_2\|^2 e^{-2\lambda_i t}(t^2+2)|z_{i0}|^2 + 3\|A_i(h) - I_2\|^2 t\int_0^t \left(e^{-\lambda_i(t-s)}\sqrt{t^2+2}|w_i(s)|\right)^2 ds \\
&\quad + 3h\int_t^{t+h} \left(e^{-\lambda_i(t+h-s)}\sqrt{(t+h-s)^2+2}|w_i(s)|\right)^2 ds
\end{aligned}
$$

$$
\begin{aligned}
&\le 3\|A_i(h) - I_2\|^2(t^2+2)|z_{i0}|^2 + 3\|A_i(h) - I_2\|^2 t\int_0^t \left(\sqrt{t^2+2}|w_i(s)|\right)^2 ds \\
&\quad + 3h\int_t^{t+h} \left(\sqrt{(t+h-s)^2+2}|w_i(s)|\right)^2 ds \\
&\le 3\|A_i(h) - I_2\|^2(t^2+2)\left[|z_{i0}|^2 + t\int_0^t |w_i(s)|^2ds\right] + 3h((t+h)^2+2)\int_t^{t+h} |w_i(s)|^2ds.
\end{aligned}
$$

Then,

$$\sum_{i=1}^{\infty}|z_i(t+h)-z_i(t)|^2 \leq \sum_{i=1}^{\infty}\left(3\|A_i(h)-I_2\|^2(t^2+2)\left[|z_{i0}|^2+t\int_0^t|w_i(s)|^2ds\right]+3h((t+h)^2+2)\int_t^{t+h}|w_i(s)|^2ds\right).$$

Now let

$$P_1=\sum_{i=1}^{N}3\|A_i(h)-I_2\|^2(t^2+2)\left[|z_{i0}|^2+t\int_0^t|w_i(s)|^2ds\right],$$

$$P_2=\sum_{i=N+1}^{\infty}3\|A_i(h)-I_2\|^2(t^2+2)\left[|z_{i0}|^2+t\int_0^t|w_i(s)|^2ds\right] \text{ and}$$

$$P_3=\sum_{i=1}^{\infty}3h((t+h)^2+2)\int_t^{t+h}|w_i(s)|^2ds.$$

Hence

$$\begin{aligned}\|z(t+h)-z(t)\|^2&=\sum_{i=1}^{\infty}|z_i(t+h)-z_i(t)|^2\\&\leq P_1+P_2+P_3.\end{aligned}$$

Now for

$$P_1=\sum_{i=1}^{N}3\|A_i(h)-I_2\|^2(t^2+2)\left[|z_{i0}|^2+t\int_0^t|w_i(s)|^2ds\right],$$

as $h\to 0$, $A_i(h)\to A_i(0)=I_2$. Thus, as $h\to 0$ we have $\|A_i(h)-I_2\|\to 0$ for each i. Hence for any $\varepsilon>0$, choose δ_1 such that $P_1<\frac{\varepsilon}{3}$ whenever $|h-0|=|h|<\delta_1$ as the summation in P_1 consists of finite number of summands. Also, for

$$P_2=\sum_{i=N+1}^{\infty}3\|A_i(h)-I_2\|^2(t^2+2)\left[|z_{i0}|^2+t\int_0^t|w_i(s)|^2ds\right],$$

both $\sum_{i=1}^{\infty}|z_{i0}|^2$ and $\sum_{i=N+1}^{\infty}|z_{i0}|^2\to 0$ as $N\to\infty$, since $z_0\in l_2$. Furthermore, $w\in L(0,T;l_2)$ implies that $\sum_{i=1}^{\infty}\int_0^t|w_i(s)|^2$ is convergent. Thus, for any $\varepsilon>0$, choose N such that $P_2<\frac{\varepsilon}{3}$. Now,

$$
\begin{aligned}
P_3 &= \sum_{i=1}^{\infty} 3h((t+h)^2+2)\int_{t}^{t+h} |w_i(s)|^2 ds\\
&\leq 3h((t+h)^2+2)\sum_{i=1}^{\infty}\int_{0}^{T} |w_i(s)|^2 ds\\
&\leq 3h((t+h)^2+2)\rho_0^2
\end{aligned}
$$

where ρ_0^2 is the initial energy.

As $i \to \infty$, for any $\varepsilon > 0$, choose δ_2 such that $P_3 < \frac{\varepsilon}{3}$ whenever $|h-0| = |h| < \delta_2$.

Finally, for each $\varepsilon > 0$, suppose $0 < h < \delta = \min\{\delta_1, \delta_2\}$ and $i = N$ such that $P_1 < \frac{\varepsilon}{3}, P_2 < \frac{\varepsilon}{3}$ and $P_3 < \frac{\varepsilon}{3}$, Then,

$$
\begin{aligned}
\|z(t+h) - z(t)\|_{l_2}^2 &= \sum_{i=1}^{\infty} |z_i(t+h) - z_i(t)|^2\\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}\\
&= \varepsilon.
\end{aligned}
$$

□

Similarly, for any $\varepsilon > 0$, there exists $\delta > 0 \ni \|z(t) - z(t-h)\|_{l_2}^2 < \varepsilon$ whenever $h = |h| < \delta$, where $h > 0$. Thus, we conclude that the function $z(t)$ is continuous on $[0, T]$ in Hilbert space l_2.

□

5 CONCLUSION

An infinite 2-system model of first order ordinary differential equation in Hilbert space l_2 is solved by proving that the solution exist in l_2 and continous on the time interval $[0, T]$, where T is a given positive number. The built model is based on a matrix equation to simplify the problem, and has coefficients of any real number. The work could be used to describe a control or differential game problem.

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