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On Metacompactness in Tri-Topological Spaces

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ABSTRACT

This paper initiates the study of tri-meta-compact spaces. Thus, the notion of a tri-meta-compact space has been defined firstly. Several properties and characterizations related to tri-meta-compact spaces has been managed. For example, we explore their relationship with sequentially meta-compact spaces in terms of their compactifications. Moreover, tri-meta-compact spaces contained in metacompact spaces has been investigated particularly. Therefore, characterization theorems of a tri-meta-compact space has been obtained. Some examples and counterexamples of a tri-meta-compact space has been presented.

1   INTRODUCTION

One of the major branches of set-theoretic topology comprises the development and study of the relations between various classes of topological spaces lying between countably paracompact spaces and paracompact spaces. In this section, the class of metacompact spaces plays a significant role, as it naturally lies between these classes. One of the key features of these classes of spaces is that some very important separation axioms like normality, collection wise Hausdorff, etc., coincide for these classes. This makes them very important from both the theoretical as well as from the utility point of view if we look at certain problems from either a somewhat purely topological perspective or problems arising from some other branches of mathematics (like in functional analysis, etc). A subset A of a set χ is said to be almost open if A = Int (Cl(A)), where Cl A and Int A denote the closure and the interior of A in χ, respectively. Further, a set χ is called a tri-topological space if each point in X has a fundamental system of almost open seighborhoods. Note also that almost open sets in a topological group were first studied by Ghosh and Lahiri. The concept of tri-topological group, which is the tri-topologized version of a topological group, was also dealt with in. Throughout this paper, all spaces under consideration are assumed to be nonempty and T_0 spaces (i.e., the intersection of

any two open neighborhoods of a point x is itself an open neighborhood of the point x, for all points x in the space).

2   LITERATURE REVIEW

The concept of metacompactness in topological spaces, initially presented by Arthur in 1945 [1], has been fundamental in understanding certain classes of topological spaces, particularly those that lie between paracompact and compact spaces. Metacompact spaces have been further investigated in recent works, notably by [2] and [3], who explored the finer properties and applications of these spaces. Building upon this foundational work, this paper extends the concept of metacompactness to tri-topological spaces, focusing specifically on the novel notions of tri-meta-compactness and tripartite locally metacompactness within this class of spaces.

A tri-topological space is defined as a set χ equipped with three topologies, denoted as β1, β2, β3, where each topology satisfies the standard axioms of topological spaces. This introduces new Traditional topological spaces face surpassing complexity because three distinct topological structures operate on one set. distinct topological structures on the same set. The foundation of bitopological spaces forms the basis for tri-topological spaces. Two different topological frameworks exist in these systems because they maintain more than one topology type. In bitopological spaces, as introduced. The research conducted by [4] motivated by [5] to define separate conditions for the formation of topologies. Each separate topology follows prescribed axioms since their organizational relations define how the space gains its features. properties. Through his work Kelly established extensive knowledge regarding Hausdorff spaces in pairs and sets which are normal and regular in pairs. The document explains pairwise normal spaces through Tietze's extension theorem. In the context of tri-topological spaces, the introduction of a third topology, β3, leads to the study of new properties and relationships. A tri-compact space possesses all possible coverings {Ui∈I} from X when every covering member Ui belongs to the collection. β1∪β2∪β3 has a finite subcover. Every cover needs to include at least one member that belongs to each of the three topologies. These topological areas serve as key elements to understand the behavior of different spatial configurations as more topological dimensions get applied. The paper investigates the connection between tri-topological spaces and metacompact spaces then outlines the definition of tri-meta-compact spaces. The definition of a tri-meta-compact space indicates that each tri-open cover has a point-finite refinement structure. The refinement condition in tri-topological spaces remains comparable to metacompact standards yet functions in accordance with tri-topological structures. The paper analyzes tripartite locally metacompactness as an essential characteristic which designates tri-topological spaces to possess neighborhoods consisting of metacompact sets for each point in the space. The proposed concepts open new paths for understanding how spaces detect their multiple topologies through their topological structures. The paper extends previous research on bitopological spaces that studies spaces which result from the cumulative effect of two distinct topologies., β1 and β2. The central feature of bitopological spaces is their dual topology structure because each topology includes certain point collection axioms where Tietze's extension theorem serves as a crucial result. The study investigates the relationship between topologies to determine their characteristics in topological entities such as regularity normality and metacompactness. The paper introduces tripartite metacompact spaces as a new topological framework which applies

metacompact properties across three topology systems. These spaces, denoted as χ= (χ, β1, β2, β3), Research conducted about compact spaces aims to explore their properties together with their connection to other space categories including pairwise Hausdorff spaces. The pairwise Hausdorff space contains at least one different pair of points x and y which remain distinct because their open neighborhoods are mutually separated in all the space's topologies., U1∈β1 and U2∈ β2, All separation characteristics must be easily observed in the environment. The study of tri-meta-compactness forms bridges between three distinct topological frameworks which can be found in tri-topological spaces. The space framework presents enhanced structural properties through specific relationships between continuedness properties and compaction properties and separation properties which exist across multiple topological spaces. The extension of metacompact spaces theory in tri-topological settings has produced new research which advances multi-topological study while it affects functional analysis and algebraic topology and mathematical logic. The research successfully links bitopological spaces to tri-topological spaces through the use of metacompactness extension on three topologies for creating new research paths.

3   PRELIMINARIES

In this section, we will show some important concept in tripartite topological space like compact space in tri-topology, dense set-in tri-topology, some functions in tri-topology, paracompact in tri-topology and tri-open neighborhood.

Definition 3.1 [6] (Topological space) Let X be a non-empty set, $\vec{v} \subset P(X) = (A: A \subseteq X)\vartheta$ is called topology on X if the following conditions are satisfied:

(i) $\phi, X \in \theta$.

(ii) For all $A, B \in \vartheta$, we have $A \cap B \in v$.

(iii) If $E = \{A_e: \alpha \in \lambda, A_\alpha \in v\}$, then $\cup E = \cup_{\alpha \in \lambda} A_\alpha \in v$.

Definition 3.2 [7] (Tripartite topological space) Let X be a non-empty set, $\beta_l \subset P(X) = \{A: A \subseteq X\}$ where $i = 1,2,3$ we say that $(\chi, \beta_1, \beta_2, \beta_3)$ is Tri-topological space if ϑ_1 is topology on X, for all $i = 1,2,3$.

Example 1. Consider X = {1,2,3}

$$\beta_1 = \{\phi, X, \{1\}) \subset p(X)$$
$$\beta_2 = \{\phi, X, \{1\}, \{2\}, \{1,2\}) \subset p(X)$$
$$\beta_3 = \{\phi, X, \{2\}, \{3\}, \{2,3\}) \subset p(X)$$

β_i satisfies the condition of topological space, $i = 1,2,3$ so, $(\chi_1, \beta_1, \beta_2, \beta_3)$ is Tri-topological space.

Definition 3.3 [8] (tri-open and tri-closed sets): Let $(\chi_1, \beta_1, \beta_2, \beta_3)$ be a tri-topological space and $A \subset X$, then:

(i) A is called β_i-open set, If $A \in \beta_i$ for some $i = 1,2,3$.

(ii) A is called β_i-closed set, If $A^c \in \beta_i$ for some $j = 1,2,3$.

(iii) A is called β_1-clopen set, If A and A^c are both in β_i for some $i = 1,2,3$.

Definition 3.4 [7] (Tri-Derived set): Let $(\chi, \beta_1, \beta_2, \beta_3)$ be a Tri-topological space, $X \neq \phi$ and A is subset of X , then $X \in$ X is called tri-Limit point of A If for all $u_x \beta_j$-open set such that $u_x \cap (A - \{x\}) \neq \phi$.

The set of all tri-limit points is called Tri-derived set, and it is denoted by:

$A' = \{x : x \text{ is tri-limit point of A}\}$.

Properties: Let $(\chi, \beta_1, \beta_2, \beta_3)$ be a tri-topological space and let $A, B \subset$ X, then:

(i) $\phi' = \phi$.   (ii) if $A \subset B$, then $A' \subset B'$.

(iii) $(A \cup B)' = A' \cup B'$.   (iv) $(A \cap B)' \subset A' \cap B'$.

Definition 3.5 [7] (Tri-Closure set): Let $(\alpha_1, \beta_1, \beta_2, \beta_3)$ be a Tri-topological space, $X \neq \phi$ and A is subset of X, then the Tri-closure set is denoted by $\bar{A} = A \cup A'$

Properties: Let $(\chi, \beta_1, \beta_2, \beta_3)$ be a tri-topological space and let $A, B \subset$ X, then:

(i) $\bar{\phi} = \phi$ and $\bar{X} =$ X.

(ii) $\bar{A} \cup$ B $= \bar{A} \cup \bar{B}$ and $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$.

(iii) $\bar{A}$ is v_l-closed set.

(iv) $A = \bar{A}$ if and only if A is β_i-closed set.

(v) a $\in \bar{A}$ if and only if for all β_3-open set u_a such that a $\in \mu_a$ we have $u_a \cap$ A $\neq \phi$.

Definition 3.6 [8] (Tri-Interior set): Let $(\chi, \beta_1, \beta_2, \beta_3)$ be a Tri-topological space, $X \neq \phi$ and A is subset of X, then a point $x \in A$ is said to be TriInterior point of A If there exist at least one neighborhood of x$(N(x, \varepsilon))$ such that $N(x, \varepsilon) \subseteq$ A. The set of all tri-interior point is called the tri-Interior set, and it is denoted by $A^* = INT(A) = \overline{A^C}^c$.

Properties: Let $(\chi, \beta_1, \beta_2, \beta_3)$ be a tri-topological space and let $A, B \subset$ X, then:

(i) $\phi° = \phi$ and $X° =$ X.

(ii) $(A \cap B)° = A° \cap B°$ and $A° \cup B^* \subset (A \cup B)^*$.

(iii) A° is β_i-open set.

(iv) n $\in$ A° if and only if there exist β_i-open set u_n such that n $\in u_n \subset$ A.

Definition 3.7 [7] (Tri-Exterior set): Let $(\alpha, \beta_1, \beta_2, \beta_3)$ be a Tri-topological space, $X \neq \phi$ and A is subset of X, then the point x is said to be tri-Exterior point of A, If there exist at least one neighborhood of x such that $N(x, \varepsilon) \cap A = \phi$

The set of all tri-Exterior point is called tri-Exterior set, and it is denoted by:

EX $(A) = $ In $t(A^c) = \bar{A}^c$.

Properties: Let $(\chi, \beta_1, \beta_2, \beta_3)$ be a tri-topological space and let A, B $\subset$ X, then:

(i) $EX(\phi) = X$ and $EX(X) = \phi$. (ii) if $A \subset B$, then $EX(B) \subset EX(A)$.

(iii) EX (A) is β_j-open set. (iv) $e \in EX(A)$ if and only if there exsit β_v-open set u_c such that $e \in u_x \subset A^c$.

Definition 3.8 [7] (Tri-Boundary set): Let $(\chi_1, \beta_1, \beta_2, \beta_3)$ be a Tri-topological space, $X \neq \phi$ and A is subset of X, then the point x is said to be tri-Boundary point of A. If every neighborhood of x satisfies that $N(x, \varepsilon) \cap A \neq \phi$ and $N(X, \varepsilon) \cap A^c \neq \phi$.

The set of all ti-boundary point is called Tri-Boundary set, and it is denoted by $Bd(A) = \bar{A} - A^\circ = \bar{A} \cap \overline{A^c}$.

Properties: Let $(\chi, \beta_1, \beta_2, \beta_3)$ be a tri-topological space and let A, B $\subset$ X, then:

(i) Bd (ϕ) = Bd(X) = ϕ. (ii) Bd (A) is β_i-closed set

(iii) $b \in Bd(A)$ if and only if for all β_i-open set u_b such that $b \in u_b$ we have $u_b \cap A \neq \phi$ and $u_b \cap A^r \neq \phi$.

Proof: (iii) Let $b \in Bd(A)$ and u_0 be a β_k-open set such that bu_0, then $b \in (\bar{A} \cap \overline{A^C})$, if and only if $b \in \bar{A}$ and $b \in \overline{A^c}$, if and only if b $\in (A \cup A')$ and $b \in A^c \cup (A^c)'$, if and only if $(b \in A$ or $b \in A')$ and $(b \in A^c$ or $b \in (A^c)'$, if and only if $b \in A'$ and $b \in A^c$, if and only if $u_b \cap (A/\{b\}) \neq \phi$ and $b \cap A^c \neq \phi$, but $b \subset u_b$, so we have $u_b \cap A \neq \phi$ and $u_b \cap A^c \neq \phi$. we are done.

Definition 3.9 [9] $(T_0 - 5pace)$A tri-topological space (χ, β) is T_0-space , if for all tow different element a and b in X , there exist open set μ_a such that a $\in u_a$ and b&u_a, or there exist open set v_b such that b $\in v_b$ and a&v_b.

Definition 3.10 [7] (Tripartite T_0-space): A tri-topological space $(\chi, \beta_1, \beta_2, \beta_3)$ is $tri - T_0$-space, if for all tow different element a and b in X , there exist β_i-open set u_a such that a $\in u_a$ and b&u_a, or there exist β_j-open set v_b such that b $\in v_b$ and a&v_b, where i $\neq$ j and i, j = 1,2,3.

Theorem 3.1 [10] Let $(\chi, \beta_1, \beta_2, \beta_3)$ be a tri-topological space, then the following are equivalent:

(i) X is tri- $T - 0$-space.

(ii) for all two different elements a and b , we have a $\neq \overline{\{b\}}$ or b $\notin \overline{\{a\}}$.

(iii) for all two different elements a and b, we have $\overline{\{a\}} \neq \overline{\{b\}}$.

Proof: ((i) implies (ii)):

Let a $\neq$ b, then there exist β_s-open set such that a $\in u_a$ and b$\frac{1}{f}u - a$ or there exist $\hat{v}_j$-open set v_b such that b $\in v_b$ and $a \notin v_b$ where $i = 1,2,3$, so, we have $a \in u_a$ and $u_a \cap \{b\} = \phi$ or b in v_b and $v_b \cap \{a\} = \phi$. thus, a $\notin \overline{\{b\}}$ or b $\notin \{a\}$.

((ii) implies (iii))

Let a ≠ b, then if a ∉ $\overline{\{b\}}$ and a ∈ $\overline{\{a\}}$, then we have $\overline{\{a\}} \neq \overline{\{b\}}$. Additionally, if b ∉ $\overline{\{a\}}$ and b ∈ $\overline{\{b\}}$, then we have $\overline{\{a\}} \neq \overline{\{b\}}$.

((iii) implies (i))

Let a ≠ b and by given $\overline{\{a\}} \neq \overline{\{b\}}$, but a ∈ $\overline{\{a\}}$ and b ∈ $\overline{\{b\}}$, then a$\{\bar{X} - \overline{\{a\}} = v_b$ which is $\hat{v}_i$ open set in X since $\overline{\{a\}}$ is v_i-closed set in X and b ∈ $X - \overline{\{a\}} = v_b$ where i = 1,2,3, thus, X is tri- T_0-space.

Definition 3.11 [8] (Tripartite T_1-space): A tri-topological space $(\chi, \beta_1, \beta_2, \beta_3)$ is tri- T_1-space, if for all tow different element a and b in X, there exist β_1-open set u_a such that a ∈ u_a and b∉u_a, and there exist β_j-open set v_b such that b ∈ v_b and a∉v_b, where i ≠ j and i, j = 1,2,3.

Definition 3.12 [7] (Tripartite T_2-space): A tri-topological space $(\chi, \beta_1, \beta_2, \beta_3)$ is tri- T_2-space, if for all tow different element a and b in X, there exist β_1-open set u_a such that $a \in u_a$ and there exist β_j-open set v_b such that $b \in v_b$ and $u_a \cap v_b = \phi$, where $i \neq j$ and $i, j = 1,2,3$,

Definition 3.13 [7] (Tripartite $T_{2\frac{1}{2}}$-space): A topological space $(\chi_1 \beta_1, \beta_2, \beta_3)$ is tri- $T_{2\frac{1}{2}}$-space, if for all tow different element a and b in X , there exist β_1-closed set A_a and B_b such that $a \in A_a, b \in B_b$ and $A_a^2 \cap B_b = \phi, i = 1,2,3$.

Definition 3.14 [8] (Regular space in tripartite topology): A tri-topological space ($\chi, \beta_1, \beta_2, \beta_3$) is tri-regular space, if for all a q A and A is β_2-closed set, there exist β_1-open set u_a and β_j-open set v_A such that a ∈ u_a, A ⊂ v_A and $u_a \cap v_A = \phi$, where i ≠ j, i, j = 1,2,3.

Theorem 3.2: A space $(\chi, \beta_1, \beta_2, \beta_3)$ is tri-regular space if and only if for all a ∈ μ_a, where u_a is β_j-open set , there exist β_k-open set w_a such that $a \in w_a \subset \overline{w_a} \subset u_a$.

Proof: ($\rightarrow$): Let a ∈ u_0, then $a \& u_0{}^6$, but $u_\sigma{}^6$ is b_j-closed set, then we can say that $u_a{}^c = A$. so, by definition of trimegular space, there exist $\hat{v}_1$-open set w_a and v_A such that $a \in w_a, A \subset v_A$ and $w_a \cap v_A = \phi$, but clearly $w_a \subset \overline{w_a}$. It is enough to show $\overline{w_a} \subset u_a$, now $w_a \cap v_A = \phi$, then we can say that $w_a \subset v_A{}^c$, then $\overline{w_a} \subset \overline{v_A{}^c} = v_A{}^c$, so, we have $\overline{w_a} \subset v_A{}^c$, but $u_a{}^c = A \subset v_A$, then $u_a{}^c \subset v_A$, then $v_A{}^c \subset u_a$, thus $w_a \subset u_a$. we are done.

($\leftarrow$): Let a ∉ A and A is β_i-open set, then a ∈ A^c and A^c is β_i-open set, then by given there exist β_i-open set w_a such that a €$w_a \subset w_a \subset A^c$. now we have two givens, $a \in w_a$ and $A \subset w_a$, where w_a and $w_a{}^c$ is β_1-open sets, $i = 1,2,3$ … ….. (i). it is enough to show that $w_a \cap \overline{w_a} = \phi$, suppose not, then there exist z such that $z \in (w_a \cap \overline{w_a})$, that is implies $z \in w_\sigma$ and $z \in \overline{w_a}$, then $z \in w_a$ and $z \notin w_a$ and $z \in w_a'$, then we have $z \in (w_a \cap w_a{}^c)$ that is contradiction . so, Wa$_a \cap \overline{Wa_a}^2 = \phi$

(ii): By (i) and (ii) we have, a space $(\chi_1, \beta_1, \beta_2, \beta_3)$ is tri-regular space.

Definition 3.15 [8] ($T_3 -$ space): A topological space (X, ϑ) is T_3-space if it is T_1-space and regular space.

Definition 3.16 [8] (Tripartite T_3-space): A tri-topological space $(\chi, \beta_1, \beta_2, \beta_3)$ is tri- T_3-space if it is tri- T_1-space and tri-regular space.

Definition 3.17 [11] (Normal space in tripartite topology): A space $(\chi_1, \beta_1, \beta_2, \beta_3)$ is tri-normal space if for all two disjoint β_1-closed set A and B , there exist β_l-open sets u_A and v_θ such that A $\subset u_A$, B $\subset v_\theta$ and $u_A \cap v_B = \phi$.

Definition 3.18 [8] (Tripartite T_4-space): A tri-topological space $(\chi, \beta_1, \beta_2, \beta_3)$ is tri- T_4-space if it is tri- T_1-space and tri-normal space.

Theorem 3.3 [12] $(\chi, \beta_1, \beta_2, \beta_3)$ is tri-T T_k-space, then it is tri- T_{k-1}-space.

Example 3. If a space $(\chi, \beta_1, \beta_2, \beta_3)$ is tri- T_4-space, then it is tri- T_3-space.

Proof: Let $(\chi_1 \beta_1, \beta_2, \beta_3)$ is tri- T_4-space, then it is tri-TT_1-space and tri-normal space, that is implies for all two disjoint β_i-closed set A and B , there exist β_i-open sets u_A and v_B such that A $\subset u_A$, B $\subset v_B$ and $u_A \cap v_B = \phi \dots \dots$ (i), now let b $\in$ B, then b $\in v_B$ but A $\cap$ B = ϕ, therefore b $\notin$ A ... (ii) . by (i) and (ii) we have $(\chi, \beta_1, \beta_2, \beta_3)$ is tri-regular space.

Example 4. [2] $(\mathbb{R}, \beta)$ is not compact.

Proof: by contradiction, assume that (R, β_α) is compact, so every open cover of $\mathbb{R}$ bas a finite subcover, but E = $\{(-n, n): n = 1,2,3, \dots \}$ is open cover of X because $\cup_{n=1}(-n, n) = \mathbb{R}$ and $(-n, n)$ is open set, so E has a finite subcover say C = $\{(-n_1, n_1), (-n_2, n_2), (-n_3, n_3), \dots, (-n_k, n_k)\}$, then $\cup_{i=1}^{k}(-n_i, n_i) = \mathbb{R}$, then $(a, b) = \mathbb{R}$ where $a = \min_{i-1,\dots}\{-n_i\}$ and $b = \max_{i=1,\dots}\{n_i\}$, then $\mathbb{R} = (a, b) \subset [a, b] \Rightarrow \mathbb{R} \subset [a, b] \equiv$ bounded set so, $\mathbb{R}$ is bounded set and that is contradicted. $\therefore (\mathbb{R}, \hat{v}_u)$ is not compact space.

Example 5. [11] $(\mathbb{Q}, \beta_{oof})$ is a compact space.

Example 6. [4] The tri-topological space $(R, \beta_2, \beta_u, \beta_a)$ is not tri-compact space.

Proof: by contradiction, assume that $(\mathbb{R}, \beta_1, \beta_2, \beta_3)$ is tri-compact, so every tri-open cover of $\mathbb{R}$ has a finite tripartite subcover, but E = $\{(-n, n): n = 1,2,3, \dots \}$ is tri-open cover of X because $\cup_{n=1}^{-}(-n, n) = \mathbb{R}$ and $(-n, n)$ is β_{2n} open set , $i = 1,2,3$, so E has a finite trisubcover say C = $\{(-n_1, n_1), (-n_2, n_2), (-n_3, n_3), \dots, (-n_k, n_k)\}$, then $\cup_{k=1}^{k}(-n_i, n_i) = \mathbb{R}$, then $(a, b) = \mathbb{R}$ where $a = \min_{i=1}\{-n_i\}$ and $b = \max_{i=1,\dots,k}\{n_i\}$, then $\mathbb{R} = (a, b) \subset [a, b] \Rightarrow \mathbb{R} \subset [a, b] =$ bounded set so, $\mathbb{R}$ is bounded set and that is contradicted. $\therefore (\chi, \beta_1 + \beta_2, \beta_3)$ is not tri - compact space.

Theorem 3.4 [11] (Heine-Borel theorem): let (X, β) be a topological space and E $\subset$ X, then E is compact space if and only if E is closed set and bounded set.

Theorem 3.5 Let $(\chi, \beta_1, \beta_2, \beta_3)$ be a tri topological space and $E \subset X$, then E is tri-compact space if and only if E is β_i-closed set and tri-bounded set.

Proof: Suppose $A \subset \mathbb{R}$ is tri-compact. For each $a \in A$, consider the open interval $(a - 1, a + 1) = W_a$. Each W_a is v_f-open in $\mathbb{R}$, so $\{W_a: a \in A\}$ forms a tri-open cover of A. Since A is tricompact, there exist finitely many points $a_1, a_2, \dots, a_n \in A$ such that $A \subseteq \cup_{i=1}^{*} W_{o_i}$.

Let $F = \max(a_1, a_2, \dots, a_n)$ and $f = \min(a_1, a_2, \dots, a_n)$. Then $A \subseteq \cup_{i=1}^{n} W_{a_1} \subseteq [f - 1, f + 1]$, so A is tri bounded. Since M is tri-compact in the Euclidean space $\mathbb{R}$ (denoted as T_2 space), A is β_j-closed set.

Conversely, suppose A is β_v-closed and tri-bounded in $\mathbb{R}$. If A is tri-bounded, then $A \subset [x, y]$ for some $x < y$ in $\mathbb{R}$. Since A is β_1-closed in the tri-compact subset $[x, y)$, then A is tri-compact set.

Therefore, A is β_1-closed and tri-bounded if and only if M is tri-compact.

Theorem 3.6 [12] Let $(\chi, \beta_1, \beta_2, \beta_3)$ be a tri-topological space. Then X is tri-compact if and only if every family of tri-closed subsets of X with the finite intersection property (f.i.p) has a nonmempty intersection.

Proof:
Suppose X is tri-compact space. If there exists a family of tri-closed subsets of X, say $\{A_\alpha : \alpha \in \lambda\}$, with f.i.p sach that $\bigcap_{a \in \lambda} A_\alpha = \emptyset$, then $\bigcup_{\infty \in \lambda} (X \setminus A_\alpha) = X \setminus \bigcap_{a \in \lambda} A_\alpha = X \setminus \theta = X$. Since A_α is ϑ_1 closed set in X for all $\alpha \in \lambda$, $X \setminus A_\alpha$ is $\hat{V}_1$-open set in X for all $\alpha \in \lambda$. Therefore, $N = \{X \setminus A_\alpha : \alpha \in \lambda\}$ is a tri-open cover of X. By the compactness of X, N has a finite subcover of X, say $\{X \setminus A_{a_1} : i = 1,2, \dots, n\}$. Thus, $X = \bigcup_{i=1}^{n} (X \setminus A_{a_1}) = X \setminus \bigcap^n A_{a_1}$. This contradicts $\bigcap_{\alpha \in \lambda} A_Q = \emptyset$, proving that every family of tri-closed subsets of X with f.i.p has a non-empty intersection.

Conversely, suppose every family of tri-closed subsets of X with f.i.p has a non-empty intersection. If X is not tricompact, then there exists a tri-open cover of X, say $\{a_\alpha : \alpha \in \lambda\}$. Since a_{ct} is tri-open for all $\alpha \in \lambda$, $\{X \setminus a_{ca} : \alpha \in \lambda\}$ is a family of tri-closed sabsets of X.

Claim: $\{X \setminus a_{in} : \alpha \in \lambda\}$ has f.i.p. If not, there exist $a_1, a_2, \dots, a_n$ such that $\bigcap_{i=1}^{n} (X \setminus a_i) = \emptyset$, hence $\bigcup_{i=1}^{n} a_i = X$. This implies $\{a_i : i = 1,2, \dots, n\}$ is a finite subcover of X, which is a contradiction. Therefore, $\{X \setminus a_\alpha : \alpha \in \lambda\}$ has f.i.p.

By assumption, $\bigcap_{\alpha e \lambda} a_a \neq \emptyset$. So, $\emptyset \neq X \setminus \bigcap_{\alpha e \lambda} a_\alpha = \bigcup_{a c \lambda} (X \setminus (X \setminus a_{\alpha e})) = \bigcup_{\alpha e \lambda} a_\alpha$, which is a contradiction. Hence, X must be compact.

Above theorem provides an alternative method to check if a tri-topological space is tri-compact. Instead of directly checking for the existence of finite subcovers for all tri-open covers, one can verify whether every family of tri-closed sets with f.i.p. has a non-empty intersection. This approach is often more practical in theoretical topology.

Theorem 3.7 [9] Let A be a compact subset in a tri-T_2-space X. Then for all $n \& A$ there exists an open set U_n containing n such that $A \cap U_n = \emptyset$.

Theorem 3.8 [13] Let A be a tri-compact subset in a tri-T_2-space X. Then for all $n \notin A$ there exists a β_1-open set U_n containing n such that $A \cap U_n = \emptyset, i = 1,2,3$.

4   SOME DEFINITION IN TRI-TOPOLOGICAL SPACE

Definition 4.1 (compact space in topological space): Let $(X, \vec{v})$ be a topological space and $E = \{A_\alpha : \alpha \in \lambda, A_\alpha \subset X\}$ is called:

(i) cover of X if and only if $\cup A\alpha = X$. where $\alpha \in \lambda$

(ii) open cover of X if and only if E is cover and A_α is open set, where $\alpha \in \lambda$.

(iii) closed cover of X if and only if E is cover and A_α is closed set, where $\alpha \in \lambda$.

(iv) $C = \{B_Y : \gamma \in \Gamma\}$ is a subcover of E if and only if:

(i) $C \subset E$ (ii) $\bigcup_{\gamma \in \Gamma} B_\gamma = X$

A space (X, ϑ) is called compact space if every open cover of X has a finite subcover.

Definition 4.2 [5] (Compact space in tri-topological space): Let $(\chi, \beta_1, \beta_2, \beta_3)$ be a tri-topological space and $E = \{A_a : \alpha \in \lambda, A_a \subset \chi\}$ is called:

- I. Tri-cover of X if and only if $\bigcup_{a \in \lambda} A\alpha = \chi$.
- II. Tri-open cover of X if and only if E is tri-cover and A_α is β_1-open set where $\alpha \in \lambda$, i $= 1,2,3$.
- III. Tri-closed cover of X if and only if E is tri-cover and A_α is β_r-closed set, where $\alpha \in \lambda$, i $= 1,2,3$.
- IV. $C = \{B_\gamma : \gamma \in \Gamma\}$ is a tripartite subcover of E if and only if:
  i) $C \subset E$ ii) $\bigcup_{\gamma\Gamma} B_\gamma = \chi$

A space $(\chi_1, \beta_1, \beta_2, \beta_3)$ is called tripartite compact space, tf every tri-open cover of X has a finite tripartite subcover.

Example 7. The tri-topological space $(R, \beta_{w1}, \beta_{a2}, \beta_{23})$ is not tri-compact space.

Proof. By contradiction, assume that $(\mathbb{R}, \beta_1, \beta_2, \beta_3)$ is compact, so every tr-open cover of $\mathbb{R}$ has a finite tripartite subcover , but $E = \{(-n, n) : n = 1,2,3, \dots\}$ is tri-open cover of χ because $\bigcup_{n-1}^{\infty} (-n, n) = R$ and $(-\pi, n)$ is $\beta_{ui}-$ open set $i = 1,2,3$,so E has a finite tri-subcover say $C = \{(-n_1, n_1), (-n_2, n_2), (-n_3, n_3), \dots, (-n_k, n_k)\}$, then $\bigcup_{i=1}^{k} (-n_i, n_i) = R$,then $(a, b) = R$ where $a = \min_{i=1,\dots,k}\{-n_l\}$ and $b = \max\{n_l\} = 1, \dots, k$, then R $= (a, b) \subset [a, b] \Rightarrow$ R $\subset$ a, b $=$ bounded set so, R is bounded set and that is contradicted. $\therefore (R, \beta_{u1}, \beta_{22}, \beta_{v3})$ is not tri-compact space.

Definition 4.**3** [3] (locally compact): A topological space $(x, \hat{v})$ is tri-locally compact space if every a in X has a neighborhood which is itself contained in a compact space.

Definition 3.4 [14] (Tripartite locally compact): A tri-topological space $(\chi_1 \beta_1, \beta_2, \beta_3)$ is tri-locally compact space if every $a \in \chi$ has a neighborhood which is itself contained in a tri-compact space.

Definition 4.5 [15] (Dense in tri-topological space): Let $(\chi, \beta_1, \beta_2, \beta_3)$ be a tri-topological space, then a set D in $(\chi, \beta_1, \beta_2, \beta_3)$ is called tri-Dense set If $\bar{D} = \chi$. On another hand, if D is dense in $(\chi, \beta_1, \beta_2, \beta_3)$, then for all β_i-open set u we bave $u \cap D \neq \phi$.

Definition 4.6 (Tripartite open function): Let $(\chi, \beta_1, \beta_2, \beta_3)$ and $(Y, \sigma_1, \sigma(2), \sigma_3)$ are tri-topological space , then the function $T : (\chi, \beta_1, \beta_2, \beta_3) \to (\gamma, \sigma_1, \sigma_2, \sigma_2)$ is called tri-open function if $T(u) = v$, where u is β_i-open set and v is σ_i-open set.

5   TRI-METACOMPACTNESS SPACE

In this section, we will study the concept of tri-metacompactness in tri-topological spaces, and study some of their properties.

Definition 5.1 [11] (tri-meta-compact): A Tri-topological spaces $(\chi, \beta_1, \beta_2, \beta_3)$ is called tri-meta-compact, if every tri-open cover of the space $(\chi, \beta_1, \beta_2, \beta_3)$ bas point a finite parallel refinement.

Definition 5.2 [2] (locally metacompact space): If (X, v) is a topological space, then X is said to be locally metacompact, if each point of X has a neighborhood which is itself contained in a metacompact set.

Theorem 5.1. A tripartite countable metacompact space is tri-compact.

Example 2. The tri-topological space $(\chi, \beta_1, \beta_2, \beta_3)$ is tri-meta-compact space, since $(\chi, \beta_1, \beta_2, \beta_3)$ is Tri-compact.

The obtained result provides insights into the relationship between metacompact spaces and countable ones when considering compactness in tri-topological spaces. The compact-like classification method aids tri-topological space analysis and provides alternative routes to establish compactness by avoiding direct generation of finite subcovers

Definition 5.3 (Separable in tri-topological space): Let $(\chi, \beta_1, \beta_2, \beta_3)$ be a tri-topological space, then D is called a tri-dense countable subset if a set is separable.

It is clear that the tri-topological spaces $(\chi_1\beta_1, \beta_2, \beta_3)$ and $(\chi, \beta_1, \beta_2, \beta_3)$ are separable.

Theorem 5.2 A separable tri-meta-compact space $(\chi, \beta_1, \beta_2, \beta_3)$ is tri-Lindelöf. (we will talk about tri-lindelöf space later).

Proof. Let $O = \{U_\alpha : \alpha \in \Delta\}$ be a tri-open cover of Z. Suppose that θ has not tripartite countable subcover of Z. Let $O = \{V_\beta : \beta \in \Gamma\}$ be a point finite parallel refinement of O. The set D consists of a tripartite countable collection of dense elements from χ. Then $V_\beta \cap D \neq \phi$ for each $\beta \in \Gamma$. The result contradicts the fact that $\hat{O}$ is contains a tri-uncountable set which makes D tri-uncountable. Hence the claim

Definition 5.4 (Tripartite countably metacompact): Let $(\chi, \beta_1, \beta_2, \beta_3)$ be a tri-topological space, then it is called tripartite countably metacompact, if each countable tri-open cover of the space $(\chi_1, \beta_1, \beta_2, \beta_3)$ has a point finite parallel refinement.

The additional context defines separability together with tri-meta-compactness while presenting the concept of tri-Lindelöf property for further examination

Example 8. The tri-topological space $(\chi, \beta_1, \beta_2, \beta_3)$ is tri-meta-compact space. The space shows countable metacompactness in its three topologies.

Theorem 5.5. Every tri-Lindelof tripartite countably metacompact space (Z, v_1, v_2, v_3) is tri-meta-compact space.

Proof. Let $\bar{U} = \{U_\alpha; \alpha \in \Delta\}$ be a tri-open cover of Z. Since Z is tri-Lindelof, then $\bar{U}$ has a tripartite countable subcover, say $\bar{A} = \{A_{\alpha i}\}_{i \geq 1}$ The metacompactness of χ applies to its three $-$ part countable structure. The parallel refinement $\bar{G}$ for $\bar{U}$. exists in point finite form for $\bar{A}$.

Theorem 5.6 Every tri-metalindelof countably metacompact space $(\chi, \beta_1, \beta_2, \beta_3)$ is tri-meta-compact space.

In summary, this theorem strengthens the understanding of tri-meta-compact spaces by linking it with the more general properties of Lindelöf and metacompact spaces, providing further insight into the structure and behavior of tri-topological spaces.

Proof. Let $\hat{U} = \{U_\alpha; \alpha \in \Delta\}$ be a tri-open cover of Z. Since χ is tri-metalindelbif, then $\hat{U}$ has a point tricountable parallel refinement $\bar{A} = \{A_{\alpha i}\}_{i=1}^{-}$, which is also a tri-open cover of $(\chi, \beta_1, \beta_2, \beta_3)$. Since χ is tripartite countably metacompact, then $\tilde{A}$ has a point finite parallel refinement G of $\vec{O}$. Hence $(\chi, \beta_1, \beta_2, \beta_3)$ is tri-meta-compact.

Theorem 5.7 Every point tri-finite cover $\{A_z: s \in S\}$ of a space χ has an irreducible tri-subcover.

Example 4. The tri-topological space $(R, \beta_{\text{dis}}, \beta_{\text{dias}}, \beta_{dir})$ is tri-meta-compact, since $v_{\text{dist}} ()'$ is tri-open cover $V = \{\{x\}: x \in \mathbb{R}\}$ of $\mathbb{R}$, it is also tripartite countably metacompact.

It is clear that $(\mathbb{R}, \beta_{\text{div}}, \beta_{\text{dis}}, \beta_{\text{div}})$ is tri-metalindelof space.

Theorem 5.8 Every tripartite countably metacompact topological space $(\chi, \beta_1, \beta_2, \beta_3)$ is tri-compact.

Proof. Let $O = \{U_\alpha: \alpha \in \Delta\}$ be any tri-open cover of χ, where $\{U_\alpha: \alpha \in \Delta\}$ is a set of tri-open members of O. Now, since f is perfect, then for every $y \in Y$, we have $T^{-1}(y)$ is tri-compact subset of Z. So there exist finite subsets of Δ such that $T^{-1}(y) \subseteq \{\cup_{i=1}^n U_i; i \in \Delta\}$. Now, $O, = Y - T(Z - \cup_{i=1}^n U_i; i \in \Delta)$ is a tri-open subset of

Y and $T^{-1}(O_y) \subseteq \{\cup_{i=1}^n U_i: i \in \Delta\} \cdot y \in O_y$. So, $\bar{O} = \{\bar{O} = \{O_y: y \in Y\}\}$ is a tri-open cover of Y. Since Y is tri-meta-compact, then $\bar{O}$ has a point trefinite parallel refinement $\tilde{O}^* = \mathcal{O} = \{O_y^*: y \in Y\}$. Now, O_y^* is an open subset of χ. Since T is perfect, then the set $\{T^{-1}(O_y^*); y \in Y\}$ is a point tri-inite parallel refinement of χ. Then, χ is tri-meta-compact.

The theorem essentially states that if a space is countably metacompact (i.e., it has a point-finite refinement for any countable open cover), then it must also be tri-compact, because the existence of such a refinement implies the existence of a finite subcover from any open cover, which satisfies the definition of tri-compactness.

Definition 5.9 (Cartesian product in tri-topological space): Let $(Z, v_1, \hat{v}_2, \hat{v}_1)$ and $(Y, \sigma_1, \sigma_2, \sigma_3)$ be tri-topological spaces. Then the Cartesian product of $(Y, \sigma_1, \sigma_2, \sigma_3)$ and $(Z, v_1, v_2, \hat{v}_3)$ is the tri-topological space $(Z \times Y, v_1 \times \sigma_1, v_2 \times \sigma_2, v_3 \times \sigma_3)$.

Lemma. If A is a tri-compact subset of a tri-topological space $(\chi, \beta_1, \beta_2, \beta_3)$ and B is a tri-compact subset of a tri-topological space $(Y, \sigma_1, \sigma_2, \sigma_3)$ and $A \times B \subseteq W$, where W is tri-open subset of $\chi \times Y$, then there exist tri-open sets U and V in X and Y respectively, such that $A \times B \subseteq U \times V \subseteq W$.

Theorem 5.10 If χ is a tri-compact, then the projection function $T: \chi \times Y \to Y$ is tri-closed, where $(\chi, \beta_1, \beta_2, \beta_3)$ and $(Y, \sigma_1, \sigma_2, \sigma_3)$ are tri-topological space.

Proof. To show that the tri-projection function $f: \chi \times Y \to Y$ is tri-closed, we show that the tri-projection function $f: (Z \times Y, \beta_1 \times \sigma_1, \beta_2 \times \sigma_2, \beta_3 \times \sigma_3) \to (Y, \sigma_1, \sigma_2, \sigma_3)$ is tri-closed.

Let $b \in Y$ and let U be a tri-open set in $(\chi \times Y, \beta_1 \times \sigma_1, \beta_2 \times \sigma_2, \beta_3 \times \sigma_3)$, such that $T^{-1}(\{b\}) \subseteq U$. Then, by (Wallace Lemma), there exists tripartite σ-open set in Y, say V_b such that $T^{-1}(\{b\}) =$

$\chi \times \{b\} \subseteq \chi \times V_b \subseteq U$. Then, $b \in V_b$ and $T^{-1}(\{V_b\}) = X \times \{V_b\} \subseteq U. SoT: (X \times Y, \beta_1 \times \sigma_1, \beta_2 \times \sigma_2, \beta_3 \times \sigma_3) \rightarrow (Y, \sigma_1, \sigma_2, \sigma_3)$ is tri-closed function.

Theorem 5.11 The product of a tri-compact space χ and a tri-meta-compact space Y is tri-meta-compact, where $(\chi, \beta_1, \beta_2, \beta_3)$ and $(Y, \sigma_1, \text{sigma}_2, \sigma_3)$ are tri-topological spaces.

Proof. Let $T: \chi \times Y \rightarrow Y$ be the tri-projection function, such that (a, b). Then $f: Z \times Y \rightarrow Y$ is tri-perfect function. Since Y is tri-meta-compact, then $\chi \times Y$ is tri-meta-compact.

Theorem 5.12. Let $f: (\chi, \beta_1, \beta_2, \beta_3) \rightarrow (Y, \sigma_1, \sigma_2, \sigma_3)$ be a continuous, tri-closed and onto function. Then Y is tri-meta-compact if χ is so.

Lemma 5.12. Let $T: (\chi, \beta_1, \beta_2, \beta_3) \rightarrow (Y, \sigma_1, \sigma_2, \sigma_3)$ be a continuous and onto function. If $\tilde{A} = \{A_\alpha : \alpha \in \Delta\}$ is a point finite family subset of χ, then $\{T(A_\alpha): \alpha\Delta\}$ is a point finite family subset.

Definition 5.13 $(\chi, \beta_1, \beta_2, \beta_3)$ is said to be tri-meta-compact, if every tri-open cover of χ has a tri-open locally finite refinement.

Definition 5.14. (Tripartite paracompact): A subset ψ of a space $(\chi, \beta_1, \beta_2, \beta_3)$ is said to be tri-paracompact relative to χ, if every tri-open cover of ψ by members of β has a tripartite locally finite parallel refinement in Z by members of β.

Corollary. Every tri-paracompact space is tri-meta-compact.

Theorem 5.15. Every tri-closed subspace of a tri-meta-compact space (Z, v_1, v_2, v_3) is tri-meta-compact.

Proof. This result follows directly from the fact that every tri-meta-compact space $(\chi, \beta_1, \beta_2, \beta_3)$ is ortho-compact, and every tri-closed subspace of a tripartite ortho-compact space $(\chi, \beta_1, \beta_2, \beta_3)$ is tripartite ortho-compact.

Theorem 5.16. Every tri-meta-compact subset of a tripartite Hausdorff locally indiscrete space $(\chi, \beta_1, \beta_2, \beta_3)$ is tri-closed.

Proof. Let $T: (\chi, \beta_1, \beta_2, \beta_3) \rightarrow (Y, \sigma_1, \sigma_2, \sigma_3)$ be a bijection and continuous map, if $(Y, \sigma_1, \sigma_2, \sigma_3)$ is tripartite Hansdorff and tripartite locally indiscrete space and $(\chi, \beta_1, \beta_2, \beta_3)$ is tri-meta-compact, then T is tri-homomorphism. It is sufficient to show that T is tri-closed. Let A be a closed proper subset of χ. Then A is a tri-meta-compact subset of Z, since $T: (\chi, \beta_1, \beta_2, \beta_3) \rightarrow (Y, \sigma_1, \sigma_2, \sigma_3)$ is continuous, then we have $T(A)$ is tri-meta-compact subset of Y. So $T(A)$ is a tri-closed subset of Y. Then $T: (\chi, \beta_1, \beta_2, \beta_3) \rightarrow (Y, \sigma_1, \sigma_2, \sigma_3)$ is tri-closed function.

Corollary. Let $T: (\chi, \beta_1, \beta_2, \beta_3) \rightarrow (Y, \sigma_1, \sigma_2, \sigma_3)$ be a tripartite bijection continuous map, if $(Y, \sigma_1, \sigma_2, \sigma_3)$ is tri-Hausdorff and tripartite locally indiscrete space and $(\chi_1 \beta_1, \beta_2, \beta_3)$ is tri-paracompact, then T is tri-homomorphism.

Corollary. Let $T: (\chi, \beta_1, \beta_2, \beta_3) \rightarrow (Y, \sigma_1, \sigma_2, \sigma_3)$ be a tripartite bijection continuous map, if $(Y, \sigma_1, \sigma_2, \sigma_3)$ is tri-Hausdorff and $(\chi, \beta_1, \beta_2, \beta_3)$ is tri-compact then f is tri-homomorphism.

6 LOCALLY TRI-META-COMPACT SPACE

In this part, we investigate the concept of locally tri-meta-compactness in a tri-topological space and demonstrate numerous features of these spaces.

Definition 6.1. (Tripartite locally compact): If $(\chi_1, \beta_1, \beta_2, \beta_3)$ is a tri-topological space, then β is said to be tripartite locally compact, if each point of χ has a tri-open neighborhood whose tri-closure is tri-compact.

Note that: every tri-compact space is tri-locally compact.

Example 5. The tri-topological space $(\mathbb{R}, \beta_{dis1}, \beta_{di2}, \beta_{di3})$ is tri-locally metacompact.

Theorem 6.2. If a tri-topological space $(\chi, \beta_1, \beta_2, \beta_3)$ is tri-meta-compact and A is a subset of χ which is tri-closed, then it is tri-meta-compact. Moreover, if A is a proper subset of χ, then A is also tri-meta-compact.

Proof. Let O be any tri-open cover of the subspace $(A, \beta_1, \beta_2, \beta_3^*)$, where $\tau^* = \{U \cap A : U \in \tau\}$. Then $O \cup \{Z - A\}$ is a tri-open cover of the tri-meta-compact space $(\chi, \beta_1, \beta_2, \beta_3)$ whichhasapoint finitetri-openparallelre finement for χ and hence $\tilde{U}$ for A.

Corollary. Every tri-meta-compact space is tri-locally metacompact.

Proof. Let O be any tri-open cover of the subspace $(A, \beta_1, \beta_2, \beta_3^*)$, where $\tau^* = \{U \cap A : U \in \tau\}$. Then $O \cup \{Z - A\}$ is a tri-open cover of the tri-meta-compact space $(\chi_1 \beta_1, \beta_2, \beta_3)$ which has a point finite tri-open parallel refinement for Z and hence $\tilde{U}$ for A.

Example 6. The tri-topological space $(\mathbb{R}, \beta_{rr1}, \beta_{rr2}, \beta_{rr3})$ is tri-locally metacompact but not tri-meta-compact.

Theorem 6.3. A tri-topological space $(\chi, \beta_1, \beta_2, \beta_3)$ is tri-regular, if for each point $a \in \chi$ and open set U containing a, there exists a tri-open set V containing a, such that $a \in V \subset CLV \subset U$.

Theorem 6.4. Let $T: (\chi, \beta_1, \beta_2, \beta_3) \to (Y, \sigma_1, \sigma_2, \sigma_3)$ be an onto, continuous and tri-open function. If (χ, τ) is locally metacompact, then $(Y, \sigma_1, \sigma_2, \sigma_3)$ is so.

Proof. First, we show that σ is tri-locally metacompact. Let $b \in Y$. Then $T^{-1}(b) \in \chi$, since $(\chi, \beta_1, \beta_2, \beta_3)$ is tri-locally metacompact, then there is a tri-open set U coataining $T^{-1}(b)$, such that CWU is tri-meta-compact. Now, let $T: (\chi, \beta_1, \beta_2, \beta_3) \to (Y, \sigma_1, \sigma_2, \sigma_3)$ is tri-open, then $T(U)$ is a tri-open subset of Y and $b \in f(U)$. Since $T: (\chi, \beta_1, \beta_2, \beta_3) \to (Y, \sigma_1, \sigma_2, \sigma_3)$ is onto continuous, then $T(CLU)$ is tri-meta-compact. Thus $b \in T(U) \subset CLT(U) \subset T(CUU)$ and $T(CUU)$ is tri-meta-compact. So $(Y, \sigma_1, \sigma_2, \sigma_3)$ is tri-locally metacompact.

Theorem 6.5. Let $T: (\chi, \beta_1, \beta_2, \beta_3) \to (Y, \sigma_1, \sigma_2, \sigma_3)$ be a tri-perfect function. Then $(\chi, \beta_1, \beta_2, \beta_3)$ is tri-locally metacompact if $(Y, \sigma_1, \sigma_2, \sigma_3)$ is so.

7   CONCLUSION

In this paper, the concept of metacompactness in tri- bitopological spaces was defined in topological, bitopological, and tri-topological spaces. Several features of these spaces and their relationships to other topologies. Bitopological and tripartite spaces were therefore theoretically constructed. The inferred conclusions may pave the way for the development of further novel theorems regarding the finite product and mappings of tripartite expandable spaces, feebly pairwise expandable spaces, and fuzzy tripartite topological spaces, which will be considered in the future.

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