Numerical Simulation of Burgers' Equation

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ABSTRACT

An exponential finite difference technique is first presented by Bhattacharya for one-dimensional unsteady state. In this study, the exponential finite difference technique was used to solve the Burgers' equation in one-dimensional with different value of h (step size). Burgers' equation is considered in this study because the equation governing simple nonlinear diffusion process. Since the Burgers' equation is nonlinear, the Hopf-Cole transformation is applied to the linear heat equation which was converted from Burgers' equation. Then, the exponential finite difference methods are used to obtain numerical solution. Three techniques have been implemented namely explicit exponential finite difference method, implicit exponential finite difference method and modified Burgers' equation using explicit exponential finite difference method. In the solution process, the explicit exponential finite difference method used a direct to solve the Burgers' equation while the implicit exponential finite difference method leads to a system of nonlinear equation. At each time-level, Newton's method is used to solve the nonlinear system. The solution of the one-dimensional modified Burgers' equation is using the explicit exponential finite difference method. The solution process has discretized the time derivative and spatial derivative using exponential finite difference technique. Numerical solutions for each method are compared with exact solution and the results obtained using the three methods are precise and reliable. The percent errors are computed and found to be sufficiently small.

Keywords: Burgers' equation, explicit exponential finite difference method, implicit exponential finite difference method, modified Burgers' equation

1. INTRODUCTION

Burgers' equation is an important and simple model in understanding the physical flows. Burgers' equation described various kinds of phenomena such as mathematical model of turbulence and the approximate theory of flow through a shock wave travelling in a viscous fluid [1]. Burgers' equations provide the simplest nonlinear models of turbulence in the phenomena process. The reaction diffusion and convection diffusion systems have important features namely the existence of a time delay and relaxation time [2]. The solution of Burgers' equation series converges very slowly for small values of viscosity [3]. Numerical techniques based on finite difference method [4-7], finite element method [8-11], and boundary element method [12] have been developed in the efforts to solve Burgers' equation numerically. The modified Burgers' equation (MBE) is called the nonlinear advection-diffusion equation. The MBE equation has been solved by several researchers by analytically and/or numerically [13]. A study of Burgers' equation is important since it appears in the approximate theory of flow through a shock wave propagating in a viscous fluid and in the modelling of turbulence. Burgers' equation and the Navier-Stokes equation are similar in the form of their nonlinear terms and in the occurrence of higher order derivatives with small coefficients. The exact solutions of the one-dimensional Burgers' equation have been surveyed by Benton and Platzman [14].

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In this study, two types of finite difference method are used to solve Burgers' equation which is the explicit exponential finite difference method (E-EFDM) and implicit exponential finite difference method (I-EFDM). Besides that, modified Burgers' equation also will be solve by using E-EFDM and I-EFDM. In an addition modified E-EFDM will solve the modified equation.

2. BURGERS' EQUATION

In this study, the one-dimensional non-linear burgers' equation is given as follows

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - v\frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

with initial condition

$$u(x,0) = g(x)$$

and boundary conditions

$$u(d,t) = h_1(t)$$
$$u(e,t) = h_2(t)$$

For $t > 0$, where v is the positive coefficient of kinematic viscosity and g, h_1 and h_2 are variables.

2.1 Linearization and exact solution

The exact solution of the burgers' equation is generated by reducing the equation to linear heat equation using hopf-cole transformation. The hopf-cole transformation is defined by the following theorem [15].

Theorem 1

$\theta(x,t)$ is any solution of the one-dimensional linear heat equation,

$$\frac{\partial \theta}{\partial t} = v\frac{\partial^2 \theta}{\partial x^2} \tag{2}$$

Then, the following hopf-cole transformation is assumed as the solution to equation (1).

$$u(x,t) = -2v\frac{\theta_x}{\theta} \tag{3}$$

In this study, two problems were solved by E-EFDM and I-EFDM.

Problem 1

Firstly, we solved the equation (1) with the initial condition

$$u(x,0) = \sin(\pi x), \ \ 0 < x < 1$$

and boundary conditions

$$u(0,t) = u(1,t) = 0, \ \ t > 0$$

Problem 2

The second problem with initial condition is

$$u(x,0) = 4x(1-x), \quad 0 < x < 1$$

and boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t > 0$$

3. E-EFDM

The discretization equation (2) for E-EFDM are as follow

$$\frac{\partial u}{\partial t} = \frac{u_i^{j+1} - u_i^j}{\Delta t} \tag{4}$$

$$\frac{\partial^2 u}{\partial x^2} = v \frac{u_{i+1}^j - 2u_i^j - u_{i-1}^j}{(\Delta x)^2} \tag{5}$$

Then, substitute the equation (4) and (5) into equation (2) which is the E-EFDM.

$$u_i^{j+1} = \frac{v}{\Delta x}\left(\frac{u_{i+1}^j - u_{i-1}^j}{u_i^j}\right)\exp\left\{\frac{rv}{\Delta x}\left(\frac{u_{i+1}^j - u_{i-1}^j}{u_i^j}\right)\left[-\frac{2v}{\Delta x}\left(\frac{u_{i+1}^j - u_{i-1}^j}{u_i^j}\right) - u_{i-1}^j - u_{i+1}^j\right]\right\} \tag{6}$$

3.1 Result and Discussion

Problem 1

The graph in Figure 1 shows the numerical solution at $t = 0.01$ for $v = 1$ and $k = 10^{-6}$ for $h = 0.05$ and $h = 0.025$. All the numerical points fall on the same line because they are very close to each other. Both values of h show good agreement with the exact solution.
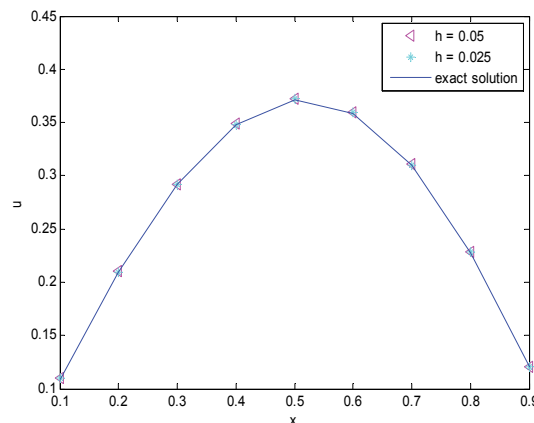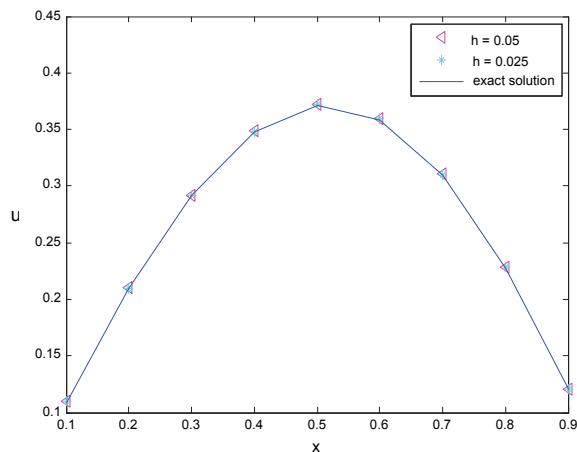

FIGURE 1. Numerical solution generated by E-EFDM with $h = 0.05$, $h = 0.025$ and exact solution for Problem 1.

Problem 2

The graph in Figure 2 shows the numerical solution at $t = 0.01$ for $v = 1$ and $k = 10^{-6}$ for $h = 0.05$ and $h = 0.025$. All the numerical points fall on the same line because they are very close to each other. Both values of h show good agreement with the exact solution.

Figure 2. Numerical solution generated by E-EFDM with $h = 0.05$, $h = 0.025$ and exact solution for Problem 2.

The percent error of $h = 0.025$ from both Problem 1 and Problem 2 are smaller than percent error for $h = 0.05$. However, the percent error from Problem 2 with $h = 0.025$ is smaller than the percent error from Problem 1. Therefore, we concluded that the smaller the values of h will produce smaller percent of error, which indicate more accurate numerical result.

4. I-EFDM

The discretization equation (2) for I-EFDM are as follow

$$\frac{\partial u}{\partial t} = \frac{u_i^{j+1} - u_i^j}{\Delta t} \tag{7}$$

$$\frac{\partial u}{\partial x} = -A \frac{u_i^{j+1} - u_i^j}{2\Delta x} \tag{8}$$

$$\frac{\partial^2 u}{\partial x^2} = v \frac{u_{i+1}^j - 2u_i^j - u_{i-1}^j}{(\Delta x)^2} \tag{9}$$

Then, substitute the equation (7), (8) and (9) into equation (2) which is the I-EFDM

$$u_i^{j+1} = u_i^j \exp \left\{ r \left[-\frac{\Delta x u_i^j}{2v} \frac{(u_{i+1}^{j+1} - u_{i-1}^{j+1})}{u_i^j} + \frac{u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1}}{u_i^j} \right] \right\} \tag{10}$$

4.1 Result and Discussion

Problem 1

The graph in Figure 3 shows the numerical solution at *t=0.01* for *v=1* and *k=10⁻⁶* for $h = 0.05$ and $h = 0.025$. All the numerical results seem to locate on the same line because they are very close to each other. Both values of *h* show good agreement with the exact solution.
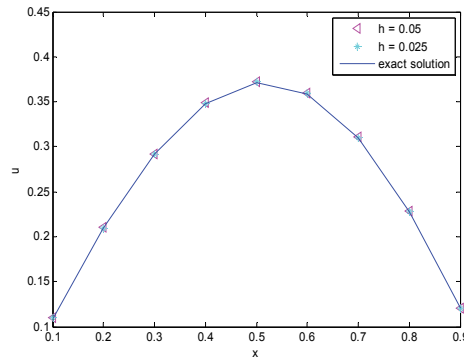

Figure 3. Numerical solution generated by I-EFDM with $h = 0.05$, $h = 0.025$ and exact solution for Problem 1.

Problem 2

The graph in Figure 4 shows the numerical solution at $t = 0.01$ for $v = 1$ and $k = 10^{-6}$ for $h = 0.05$ and $h = 0.025$. All the numerical results seem to locate on the same line because they are very close to each other. Both values of *h* show good agreement with the exact solution.
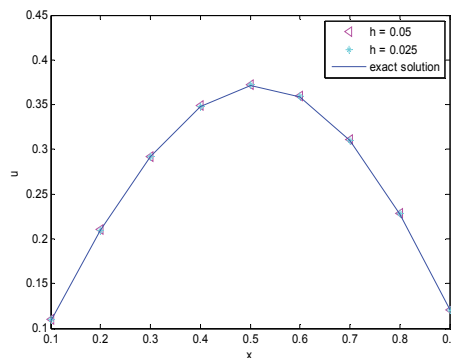

Figure 4. Numerical solution generated by I-EFDM with $h = 0.05$, $h = 0.025$ and exact solution for Problem 2.

The percent error for $h = 0.05$ is slightly higher than the percent error for $h = 0.025$. Therefore, we concluded that smaller h will produce smaller percent of errors, which mean more accurate numerical result.

4.2 Comparison between E-EFDM and I-EFDM

Problem 1

The graph in Figure 5 (a) and (b) display the comparison of the percent errors between E-EFDM and I-EFDM for Problem 1. The percent error of E-EFDM is higher than I-EFDM.
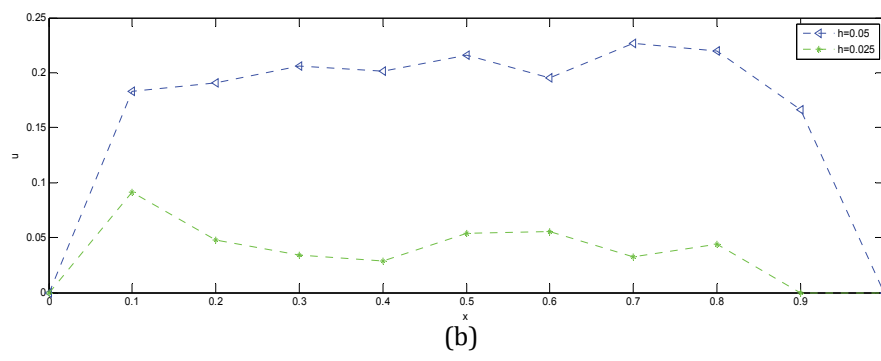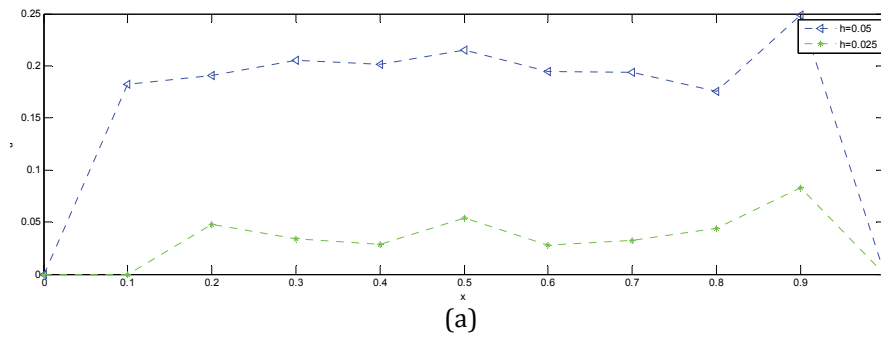
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(a)

(b)

Figure 5. Comparison of the percent errors between E-EFDM and I-EFDM using $h = 0.05$ and $h = 0.025$ for Problem 1.

Problem 2

The graph in Figure 6 (a) and (b) display the comparison of the percent error between E-EFDM and I-EFDM for Problem 2. The percent error of E-EFDM is higher compare than I-EFDM.
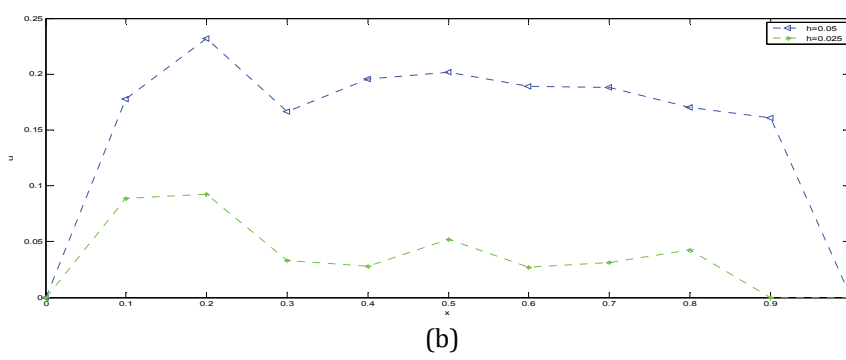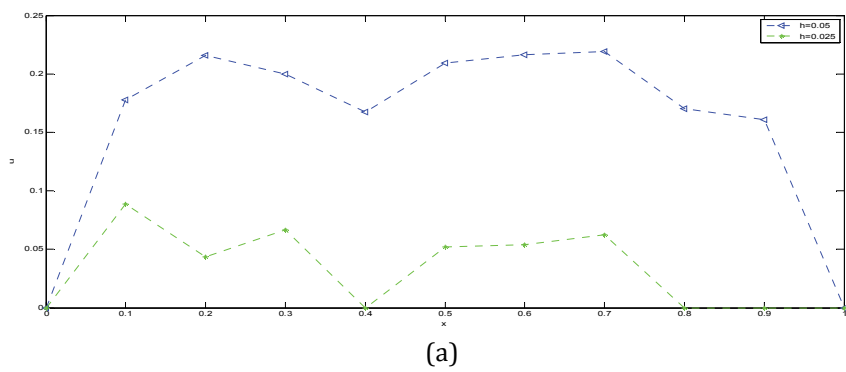

(a)

(b)

Figure 6. Comparison of the percent errors between E-EFDM and I-EFDM using $h = 0.05$ and $h = 0.025$ for Problem 2.

All numerical results generated by E-EFDM and I-EFDM are converging to the exact solution for $h = 0.05$ and $h = 0.05$. Both method give accurate numerical solutions that are in good agreement with the exact solution. The error decreases when the value of h decreases.

5. MODIFIED BURGERS' EQUATION

The modified Burgers' Equation using E-EFDM

$$\frac{\partial u}{\partial t} = -u^p \frac{\partial u}{\partial x} + v \frac{\partial^2 u}{\partial x^2} \tag{11}$$

The discretization equation (11)

$$\frac{\partial u}{\partial t} = \frac{u_i^{j+1} - u_i^j}{\Delta t} \tag{12}$$

$$\frac{\partial u}{\partial x} = -u^p \frac{u_i^{j+1} - u_i^j}{2\Delta x} \tag{13}$$

$$\frac{\partial^2 u}{\partial x^2} = v \frac{u_{i+1}^j - 2u_i^j - u_{i-1}^j}{\left(\Delta x\right)^2} \tag{14}$$

Then, substitute the equation (12), (13) and (14) into equation (11) which is the modified Burgers' equation using E-EFDM

$$u_i^{j+1} = u_i^j \exp\left\{ \frac{v\Delta t}{(\Delta x)^2} \left[-\frac{\Delta x}{2v} \left[u_i^j\right]^{p-1} \left(u_{i+1}^j - u_{i-1}^j\right) + \frac{u_{i-1}^j - 2u_j^j + u_{i+1}^j}{u_i^j} \right] \right\} \tag{15}$$

5.1 Result and Discussion

We solved the modified Burgers' equation (11) with the initial condition and the boundary conditions

$$u(x,t) = \frac{x}{1 + \left(1/c_0\right)\exp\left(x^2/4v\right)}, \; 0 < x < 1$$

and the boundary conditions

$$u(0,t) = u(1,t) = 0, \; t > 0$$

The graph in Figure 7 is display the numerical solution at $v = 0.001$, $\Delta t = 0.01$, and $\Delta x = 0.05$ with exact solution. The graph for exact solution is higher than the graph of numerical result. All the numerical results seem to locate on the same line from $x = 0.35$ until $x = 0.50$ because they are very close to each other.
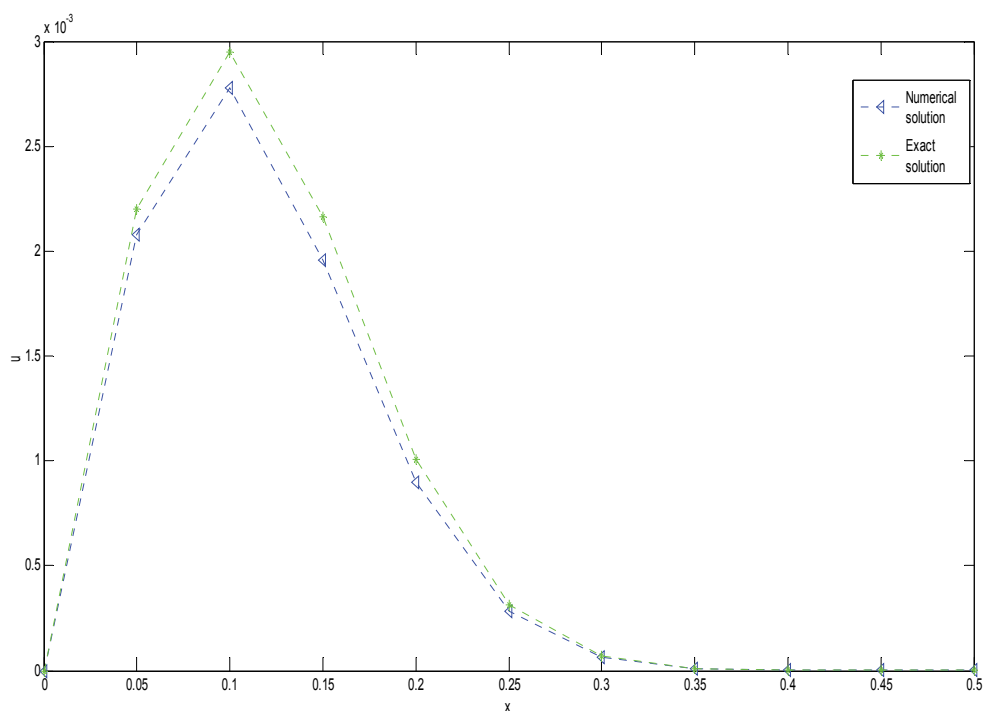
Figure 7. Numerical solution generate by modified Burgers' equation using E-EFDM with exact solution.

6. CONCLUSIONS

This study applied two exponential finite difference methods to the one-dimension Burgers' equation. The numerical solution using two different values of *h* are given. The results of this study show that the E-EFDM and I-EFDM offer a better accuracy in the numerical solution of the one-dimensional Burgers' equation. However, no significant changes found from comparison of percent error between E-EFDM and I-EFDM with exact solution. For modified Burgers' equation using E-EFDM, the numerical solution is compared to exact solution. It can be concluded that when the value of *x* increases, the heat flow will be decreases.

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APPENDIX

If any, the appendix should appear directly after the references without numbering, and on a new page.