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Closed Form Boundary Feedback for the Time Fractional Order Partial Integro Differential Equations

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ABSTRACT

This paper focuses on the application of backstepping control scheme for the time fractional order partial integro differential equation (FPIDE). The fractional derivative is presented by using Caputo fractional derivative. We show how the FPIDE is converted into a Mittag-Leffler stability by designing invertible coordinate transformation. Numerical simulation is used to demonstrate the effectiveness of the proposed control scheme.

Keywords: time fractional order partial integro differential equation

1. INTRODUCTION

Fractional calculus is a generalization of integer order integral and differential calculus to any arbitrary real or even complex order. This generalization extends a traditional definition of integration and differentiation to non-integer order and combines them into one definition where the operator depends on the order sign. The popularity of this subject has increased during the last decades in several fields of science and engineering. Development of algorithms and methods for solving fractional equations and special functions allows the fractional calculus to become a very useful tool for precise description of real-world phenomena [1-4]. Recently, the solution of fractional partial differential equations (FPDEs) and fractional order partial integro differential equation (FPIDEs) of physical interest have gain a huge amount of attention because of the fact that a realistic modelling of a physical phenomenon having dependence not only on the time instant but also on the previous time history that can be achieved by fractional calculus [5]. However, there is not many results on the case when the boundary feedback stabilisation of an unstable FPIDE. The boundary stabilisation for one dimensional fractional diffusion wave equation, based on numerical solution techniques, has been studied in [6,7]. In those studies, the focus was to use the fractional order boundary controller and derive the boundary controller of a Caputo fractional order wave equation. In addition, in 1D system of the heat conduction process, Fourier law and the connection between anomalous diffusion are not satisfied [8]. In [9], an inverse optimal control problem is studied with state function governed be FPDE.

One of the main advantages of the fractional order models over the integer order ones is that many real life applications can be described by applying the notation of fractional order [10,11]. Motivated by the arguments above, in this article, we consider the FPIDE

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$$ {}_0^c D_t^q s(x,t) = \xi s_{xx}(x,t) + d s_x(x,t) + \gamma(x) s(x,t) + \int_0^x f(x,\zeta) s(x,\zeta)\, d\zeta \quad \text{in } (0,1)\times(0,\infty) \tag{1} $$

with the Dirichlet boundary condition

$$ s(0,t) = 0, \quad s(1,t) = U(t), \qquad t \ge 0. \tag{2} $$

and under the assumption

$$ q \in (0,1],\ \xi, d > 0, \gamma(x) \in \mathrm{C}^1[0,1], f(x,\zeta) \in \mathrm{C}^1([0,1]\times[0,1]) \tag{3} $$

where ${}_0^c D_t^q$ is the Caputo time fractional order derivative and $U(t)$ is the control input. The control objective is to stabilise the equilibrium $s(x,t) \equiv 0.$ When q=1, equation (1) is reduced to the integer order unstable PIDE which was studied in [13].

Recently, we proposed the backstepping method for stabilising time fractional order PDE. The semi-discretised fractional-order backstepping approach was introduced to find the boundary controller function which stabilises the time fractional order PDE by transforming it into an equivalent stable closed loop [12]. In this article, we propose an infinite dimensional backstepping method for stabilising FPIDE. To the best of our knowledge, this is the first time in the literature that the backstepping is being used for stabilising FPIDE. The approach developed in this article allows us to prove that the kernel is not only bounded but twice continuously differentiable. This cannot be even seen, much less proved, from discretisation.

The rest of this article is organized as follows: the Dirichlet boundary feedback stabilisation of unstable FPIDE is presented in Section 2. Section 3 provides a numerical simulation to illustrate the effectiveness of proposed control scheme. The conclusions are devoted in the last section.

2. FPIDE FOR THE KERNEL

Consider the following coordinate transformation

$$ r(x,t) = s(x,t) - \int_0^x k(x,\zeta) s(\zeta,t)\, d\zeta, \qquad x \in (0,1),\ t \in (0,\infty) \tag{4} $$

that transforms the system

$$ \begin{aligned} &{}_0^c D_t^q s(x,t) = \xi s_{xx}(x,t) + d s_x(x,t) + \gamma(x) s(x,t) + \int_0^x f(x,\zeta) s(x,\zeta)\, d\zeta && \text{in } (0,1)\times(0,\infty) \\ &s(0,t) = 0,\ s(1,t) = U(t) && \text{in } (0,\infty) \\ &s(x,0) = s_0(x) && \text{in } (0,1) \end{aligned} \tag{5} $$

into the target system

$$ \begin{aligned} &{}_0^c D_t^q r(x,t) = \xi r_{xx}(x,t) - a\, r(x,t) && \text{in } (0,1)\times(0,\infty), \\ &r(0,t) = r(1,t) = 0 && \text{in } (0,\infty), \\ &r(x,0) = r_0(x) && \text{in } (0,1), \end{aligned} \tag{6} $$

where $r_0(x) = s_0(x) - \int_0^x k(x,\zeta)s_0(\zeta)\,d\zeta$ and a is a positive constant. The boundary condition gives the controller,

$$s(1,t) = U(t) = \int_0^1 k(1,\zeta)s(\zeta,t)\,d\zeta. \tag{7}$$

The following compatible conditions is introduced for initial condition

$$s_0(0) = 0, \quad s_0(1) = \int_0^1 k(1,\zeta)s_0(\zeta)\,d\zeta \tag{8}$$

The following notations is introduced before we find out what conditions $k(x,\zeta)$ has to satisfy

$$k_x(x,\zeta) = \frac{\partial}{\partial x}k(x,\zeta)\Big|_{\zeta=x},\ k_\zeta(x,\zeta) = \frac{\partial}{\partial \zeta}k(x,\zeta)\Big|_{\zeta=x},\ \frac{d}{dx}k(x,x) = k_x(x,x) + k_\zeta(x,x).$$

The q- fractional order derivative of (4) with respect to time is

$$\begin{aligned}
{}_0^cD_t^q r(x,t) &= {}_0^cD_t^q s(x,t) - \int_0^x k(x,\zeta)\,{}_0^cD_t^q s(\zeta,t)\,d\zeta \\
&= \xi s_{xx}(x,t) + ds_x(x,t) + \gamma(x)s(x,t) + \int_0^x f(x,\zeta)s(\zeta,t)d\zeta - \xi k(x,x)s_x(x,t) + \xi k(x,0)s_x(0,t) \\
&\quad + \xi k_\zeta(x,x)s(x,t) - \xi k_\zeta(x,0)s(0,t) - \xi\int_0^x k_{\zeta\zeta}(x,\zeta)s(\zeta,t)\,d\zeta - dk(x,x)s(x,t) + dk(x,0)s(0,t) \\
&\quad + d\int_0^x k_\zeta(x,\zeta)s(\zeta,t)d\zeta - \int_0^x \gamma(\zeta)k(x,\zeta)s(\zeta,t)d\zeta - \int_0^x s(\zeta,t)\Big(\int_\zeta^x k(x,p)f(p,\zeta)\,dp\Big)d\zeta
\end{aligned}$$

and differentiating (4) with respect to x obtained

$$r_x(x,t) = s_x(x,t) - k(x,x)s(x,t) + k(x,0)s(0,t) - \int_0^x k_x(x,\zeta)s(\zeta,t)\,d\zeta$$

$$r_{xx}(x,t) = s_{xx}(x,t) - \frac{d}{dx}k(x,x)s(x,t) - k(x,x)s_x(x,t) - k_x(x,x)s(x,t) - \int_0^x k_{xx}(x,\zeta)s(\zeta,t)\,d\zeta$$

Then we get that ${}_0^cD_t^q r(x,t) - \xi r_{xx}(x,t) + a\,r(x,t) = 0$

$$\begin{aligned}
0 &= (\gamma(x) + a - d\,k(x,x) + 2\xi\frac{d}{dx}k(x,x))s(x,t) + ds_x(x,t) + (-\xi k_\zeta(x,0) + d\,k(x,0))s(0,t) + \xi k(x,0)s_x(0,t) \\
&+ \int_0^x \left(\xi k_{xx}(x,\zeta) - \xi k_{\zeta\zeta}(x,\zeta) + d\,k_\zeta(x,\zeta) - (a+\gamma(\zeta))k(x,\zeta) + f(x,\zeta) - \int_\zeta^x k(x,p)f(p,\zeta)dp \right) s(\zeta,t)d\zeta
\end{aligned}$$

For the right-hand side to be zero for all s, the following conditions have to be satisfied

$$
\begin{aligned}
&\xi k_{xx}(x,\zeta)-\xi k_{\zeta\zeta}(x,\zeta)=(a+\gamma(\zeta))k(x,\zeta)-f(x,\zeta)+\int_{\zeta}^{x}k(x,p)f(p,\zeta)dp, && 0\le\zeta\le x\le 1\\
&k(x,0)=0, && 0\le x\le 1\\
&\frac{d}{dx}k(x,x)=\frac{-1}{2\xi}(\gamma(x)+a), && 0\le x\le 1
\end{aligned}
\tag{9}
$$

To prove the well posedness of (9) we need to prove the following Lemma

Lemma 1

Suppose that $\gamma\in C^1[0,1]$ and $f\in C^1([0,1]\times[0,1])$ then problem (9) has a unique solution which is twice continuously differentiable in $T=\{0\le\zeta\le x\le 1\}$.

Proof

Introducing new variables $\varsigma=x+y,\ \eta=x-y,$ and denoting $G(\varsigma,\eta)=k(x,\zeta)=k\left(\frac{\varsigma+\eta}{2},\frac{\varsigma-\eta}{2}\right)$ we have

$$
\begin{aligned}
k_x&=G_\varsigma(\varsigma,\eta)+G_\eta(\varsigma,\eta),\ k_{xx}=G_{\varsigma\varsigma}(\varsigma,\eta)+2G_{\varsigma\eta}(\varsigma,\eta)+G_{\eta\eta}(\varsigma,\eta),\\
k_\zeta&=G_\varsigma(\varsigma,\eta)-G_\eta(\varsigma,\eta),\ k_{\zeta\zeta}=G_{\varsigma\varsigma}(\varsigma,\eta)-2G_{\varsigma\eta}(\varsigma,\eta)+G_{\eta\eta}(\varsigma,\eta).
\end{aligned}
$$

Problem (9) is transformed to

$$
\begin{aligned}
&G_{\varsigma\eta}(\varsigma,\eta)=\frac{1}{4\xi}\left(a+\gamma\left(\frac{\varsigma-\eta}{2}\right)\right)G(\varsigma,\eta)-\frac{1}{4\xi}f\left(\frac{\varsigma+\eta}{2},\frac{\varsigma-\eta}{2}\right)+\frac{1}{4\xi}\int_{\frac{\varsigma-\eta}{2}}^{\frac{\varsigma+\eta}{2}}G\left(\frac{\varsigma+\eta}{2}+\tau,\frac{\varsigma+\eta}{2}-\tau\right)f\left(\tau,\frac{\varsigma-\eta}{2}\right)d\tau, && 0\le\eta\le\varsigma\le 2,\\
&G(\varsigma,\varsigma)=0, && 0\le\varsigma\le 2,\\
&G(\varsigma,0)=\frac{-1}{4\xi}\int_0^{\varsigma}\left(\gamma\left(\frac{\tau}{2}\right)+a\right)d\tau, && 0\le\varsigma\le 2,
\end{aligned}
\tag{10}
$$

Integrating (10) with respect to η from 0 to η gives

$$
\begin{aligned}
G_\varsigma(\varsigma,\eta)=&-\frac{1}{4\xi}\left(\gamma\left(\frac{\varsigma}{2}\right)+a\right)+\frac{1}{4\xi}\int_0^{\eta}\left(\gamma\left(\frac{\varsigma-s}{2}\right)+a\right)G(\varsigma,s)\,ds-\frac{1}{4\xi}\int_0^{\eta}f\left(\frac{\varsigma+\tau}{2},\frac{\varsigma-\tau}{2}\right)d\tau\\
&+\frac{1}{4\xi}\int_0^{\eta}\int_{\varsigma}^{\varsigma+\eta-s}G(\tau,s)f\left(\frac{\tau-s}{2},\varsigma-\frac{\tau+s}{2}\right)d\tau\,ds.
\end{aligned}
\tag{11}
$$

Integrating (11) with respect to ς from η to ς gives

$$
\begin{aligned}
G(\varsigma,\eta)=&-\frac{1}{4\xi}\int_{\eta}^{\varsigma}\left(\gamma\left(\frac{\tau}{2}\right)+a\right)d\tau+\frac{1}{4\xi}\int_{\eta}^{\varsigma}\int_0^{\eta}\left(\gamma\left(\frac{\tau-s}{2}\right)+a\right)G(\tau,s)\,ds\,d\tau-\\
&\frac{1}{4\xi}\int_{\eta}^{\varsigma}\int_0^{\eta}f\left(\frac{s+\tau}{2},\frac{s-\tau}{2}\right)d\tau\,ds+\frac{1}{4\xi}\int_{\eta}^{\varsigma}\int_0^{\eta}\int_{\mu}^{\mu+\eta-s}G(\tau,s)f\left(\frac{\tau-s}{2},\mu-\frac{\tau+s}{2}\right)d\tau\,ds\,d\mu.
\end{aligned}
\tag{12}
$$

To show that Eq. (12) has a unique continuous solution, the method of successive approximation were used. Let

$$G_0(\varsigma,\eta) = -\frac{1}{4\xi}\int_{\eta}^{\varsigma}\left(\gamma\left(\frac{\tau}{2}\right)+a\right)d\tau - \frac{1}{4\xi}\int_{\eta}^{\varsigma}\int_{0}^{\eta} f\left(\frac{s+\tau}{2},\frac{s-\tau}{2}\right)d\tau\,ds$$

$$G_{n+1}(\varsigma,\eta) = \frac{1}{4\xi}\int_{0}^{\eta}\left(\gamma\left(\frac{\tau-s}{2}\right)+a\right)G_n(\tau,s)\,ds\,d\tau + \frac{1}{4\xi}\int_{\eta}^{\varsigma}\int_{0}^{\eta}\int_{\mu}^{\mu+\eta-s} G_n(\tau,s) f\left(\frac{\tau-s}{2},\mu-\frac{\tau+s}{2}\right)d\tau\,ds\,d\mu.$$

and denote $\bar{\gamma} = \sup_{x\in[0,1]}|\gamma(x)|$, $\bar{f} = \sup_{(x,\zeta)\in[0,1]\times[0,1]}|f(x,\zeta)|$, we estimate now $G_n(\varsigma,\eta)$

$$\begin{aligned} |G_0(\varsigma,\eta)| &= \left|-\frac{1}{4\xi}\int_{\eta}^{\varsigma}\left(\gamma\left(\frac{\tau}{2}\right)+a\right)d\tau - \frac{1}{4\xi}\int_{\eta}^{\varsigma}\int_{0}^{\eta} f\left(\frac{s+\tau}{2},\frac{s-\tau}{2}\right)d\tau\,ds\right| \\ &\le \frac{1}{4\xi}\int_{\eta}^{\varsigma}\left|\left(\gamma\left(\frac{\tau}{2}\right)+a\right)d\tau\right| + \frac{1}{4\xi}\int_{\eta}^{\varsigma}\int_{0}^{\eta}\left|f\left(\frac{s+\tau}{2},\frac{s-\tau}{2}\right)d\tau\,ds\right| \\ &\le \frac{1}{4\xi}(\bar{\gamma}+a)(\varsigma-\eta) + \frac{1}{4\xi}\bar{f}\eta(\varsigma-\eta) \\ &\le \frac{1}{\xi}(\bar{\gamma}+a+\bar{f}) \equiv M \end{aligned} \tag{13}$$

Suppose that

$$|G_n(\varsigma,\eta)| \le M^{n+1}\frac{(\varsigma+\eta)^n}{n!} \tag{14}$$

Then, we have the following estimate

$$\begin{aligned} |G_{n+1}(\varsigma,\eta)| &\le \frac{M^{n+1}}{4n!\xi}\left\{(\bar{\gamma}+a)\int_{\eta}^{\varsigma}\int_{0}^{\eta}(\tau+s)^n\,ds\,d\tau + \bar{f}\int_{\eta}^{\varsigma}\int_{0}^{\eta}\int_{\eta}^{\mu+\eta-s}(\tau+s)^n\,d\tau ds d\mu\right\} \\ &\le \frac{M^{n+1}}{4n!\xi}\left\{2(\bar{\gamma}+a)+2\bar{f}\right\}\frac{(\varsigma+\eta)^{n+1}}{n+1} \\ &\le M^{n+2}\frac{(\varsigma+\eta)^{n+1}}{(n+1)!} \end{aligned} \tag{15}$$

So, by induction, (14) is proved.

Lemma 2

[14] If $k(x,\zeta)$ is the solution of problem (9) and the linear bound operator is defined $K: H^i(0,1)\to H^i(0,1)$ $(i=0,1,2)$ by

$$r(x,t) = (Ks)(x) = s(x,t) + \int_0^x k(x,\zeta)s(\zeta,t)\,d\zeta \tag{16}$$

Then we get that

1. K has a linear bounded inverse $K^{-1}: H^i(0,1)\to H^i(0,1)\,(i=0,1,2),$ and
2. K converts the Eq. (1) with boundary feedback control (7) into the target system (6).

Where ($H^i(0,1)$ is usual Sobolev space)

Lemma 3

[15] Suppose that $x:[0,\infty)\to\Re$ is a continuous and differentiable function, then for any given $t\geq 0,$ we have

$$\frac{1}{2}\,{}_0^cD_t^q x^2(t)\leq x(t)\,{}_0^cD_t^q x(t),\quad \forall q\in(0,1]. \tag{17}$$

Theorem 1

Suppose that $\gamma\in C^1[0,1], f\in C^1([0,1]\times[0,1]), a>0,\ r(.,t)$ is a continuous and differentiable function on $[0,\infty)$ and the Laplace transform of $r^2(.,t)$ exists.

1. For any initial data $s_0\in H^1(0,1)$ that satisfy the compatible condition (8), then system (1) with boundary control (7) has a unique solution and is Mittage-Leffler stable at the origin $s(x,t)\equiv 0$

$$\left\|s(.,t)\right\|^2_{H^1(0,1)}\leq n_1\left\|s_0\right\|^2_{H^1(0,1)}E_q(-2at^q),\ t\in[0,\infty) \tag{18}$$

2. For any initial data $s_0\in L^2(0,1)$ that satisfy the compatible condition (8), then system (1) with boundary control (7) has a unique solution and is Mittage-Leffler stable at the origin $s(x,t)\equiv 0$

$$\left\|s(.,t)\right\|^2\leq n_2\left\|s_0\right\|^2 E_q(-2at^q),\ t\in[0,\infty) \tag{19}$$

Proof: (1)

Define $v(t)=\int_0^1 r_x(x,t)^2\,dx$, multiply the first equation of (6) by r_{xx} and integrating from 0 to 1, we have

$$\int_0^1 r_{xx}(x,t)\,{}_0^cD_t^q r(x,t)\,dx=\zeta\int_0^1 r_{xx}(x,t)^2\,dx-a\int_0^1 r_{xx}(x,t)r(x,t)\,dx$$

$$r_x(1,t)\,{}_0^cD_t^q r(1,t)-r_x(0,t)\,{}_0^cD_t^q r(0,t)-\int_0^1 r_x(x,t)\,{}_0^cD_t^q r_x(x,t)\,dx=\zeta\int_0^1 r_{xx}(x,t)^2\,dx-$$
$$a\left(w(1,t)w_x(1,t)-w(0,t)w_x(0,t)\right)+a\int_0^1 r_x(x,t)^2\ dx$$

Since $r(0,t)=r(1,t)=0,\forall t\geq 0,$ we have ${}_0^cD_t^q r(0,t)={}_0^cD_t^q r(1,t)=0,\forall t\geq 0,$ with $r\in H^1(0,1)$ implies that

$$\int_0^1 r_x(x,t)\,{}_0^cD_t^q r_x(x,t)\,dx=-av(t)\text{, and }{}_0^cD_t^q v(t)=\int_0^1 {}_0^cD_t^q r_x(x,t)^2\,dx\leq 2\int_0^1 r_x(x,t)\,{}_0^cD_t^q r_x(x,t)dx\leq -2av(t)$$

by Lemma 2, there exist a positive constant $e_1\,\&\,e_2$ such that

$$\left\|s(.,t)\right\|_{H^1(0,1)}\leq e_1\left\|r(.,t)\right\|_{H^1(0,1)},\ \left\|r_0\right\|_{H^1(0,1)}\leq e_1\left\|s_0\right\|_{H^1(0,1)} \tag{20}$$

$$\|s(.,t)\| \le e_2 \|r(.,t)\|, \ \|r_0\| \le e_2 \|s_0\| \tag{21}$$

Let

$$W(t) = -2a\,v(t) - {}_0^cD_t^q v(t) \tag{22}$$

The Laplace transform of (22) is $\hat{v}(s) = \dfrac{s^{q-1}v(0) - \hat{W}(s)}{s^q + 2a}$, where $v(0) = (1/2)\int_0^1 r_0^2(x)\,dx \ge 0$

From [4], and inverting the Laplace transform of (22) gives the unique solution

$$v(t) = E_q(-2at^q)v(0) - v(t) * [t^{q-1}E_q(-2at^q)], \ t \ge 0. \tag{23}$$

Since t^{q-1} and $E_q(-2at^q)$ are nonnegative functions, we have $v(t) \le E_q(-2at^q)v(0), \ t \ge 0.$

Implies that $\|s(.,t)\|^2 \le e_1^{\,2} \|s_0\|^2 \, E_q(-2at^q), \ 0 \le t < \infty$

(2) Let $R(t) = (1/2)\int_0^1 r^2(.,t)\,dx$ by Lemma 3, we have

$$\begin{aligned}
{}_0^cD_t^q R(t) &= \frac{1}{2}\int_0^1 {}_0^cD_t^q r^2(x,t)\,dx \le \int_0^1 r(x,t)\,{}_0^cD_t^q r(x,t)\,dx \\
&= \int_0^1 r(x,t)\,r_{xx}(x,t)\,dx - a\int_0^1 r^2(x,t)\,dx \\
&= -\int_0^1 r_x(x,t)^2\,dx - a\int_0^1 r^2(x,t)\,dx \\
&\le -2aR(t)
\end{aligned}$$

Similarly, we see that $R(t) \le E_q(-2at^q)R(0), \ t \ge 0$

3. NUMERICAL SIMULATION

In this section, we present the result of numerical simulation for the control plant we take $\xi = 1, \ a = 5, \ \gamma = 10 \ f(x,\zeta) \equiv 0.$ Let the initial condition be $s_0(x) = 4(1-x)\sin(\pi x)$

According to [13], it follows that

$$k(x,\zeta) = -\gamma_0 \zeta \frac{I_1\left(\sqrt{\gamma_0(x^2 - \zeta^2)}\right)}{\sqrt{\gamma_0(x^2 - \zeta^2)}} \tag{24}$$

where $\gamma_0 = \dfrac{\gamma + a}{\xi}, I_1$ is a modified Bessel function of order one. In Figure 1 the kernels $k(x,\zeta)$ is plotted for several values of γ_0. Obviously, as γ_0 gets larger, the system becomes more unstable, the maximum value of $|k(x,\zeta)|$ moves to the left indicating that it requires more control effort.

The system was discretised using BTCS finite difference method with 100 steps. The comparison of L_2 norms is given in Figure 2, while the results of simulation are presented in Figures 3 & 4.

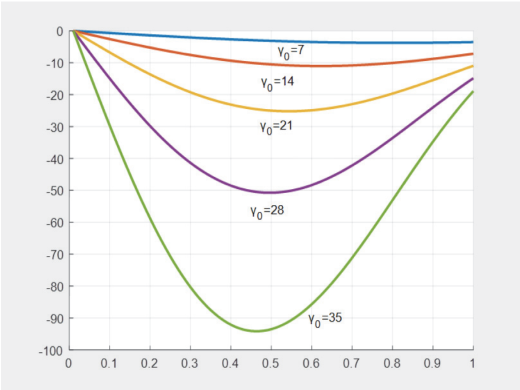

Figure 1: Control gain for different values of γ_0

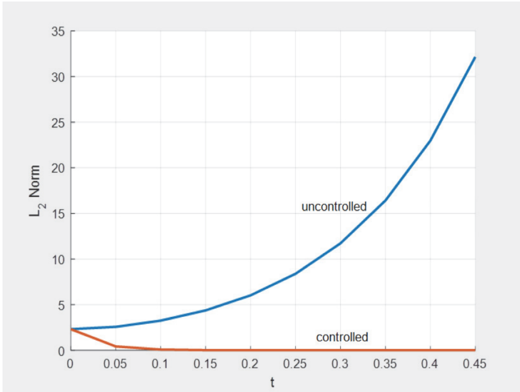

Figure 2: Comparison of L_2 norms when $q = 0.8$

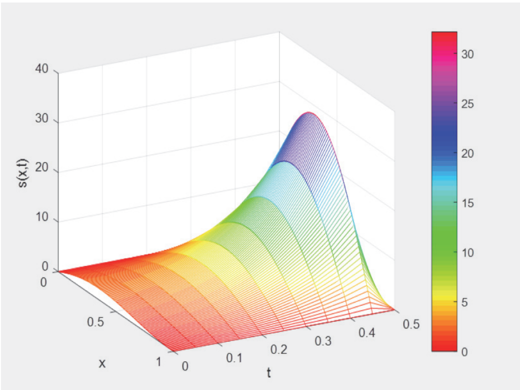

Figure 3: Solution of the system without control when q=0.8

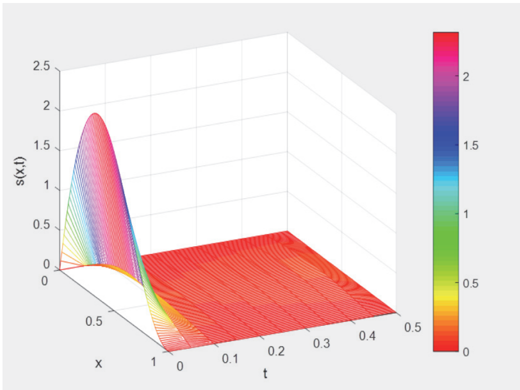

Figure 4: Approximation of controlled system when q=0.8

4. CONCLUSION

In this article, the backstepping technique was proposed for designing boundary feedback controller for FPIDE with Dirichlet boundary conditions. Numerical simulation shows that results in this research are in satisfactory agreement in dealing with unstable FPIDE. We hope that the result provides some insights into the qualitative analysis of the design of fractional order controller.

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APPENDIX

If any, the appendix should appear directly after the references without numbering, and on a new page.