Applied Mathematics and Computational Intelligence
Volume 8, No.1, Dec 2019 [61-66]

An Economical First Order Runge-Kutta Method for Solving Ordinary Differential Equations

Lim Tian Hwee[1*] and Nor'aini Aris[2]

[1]*Foundation, Studies and Language Institute, University of Reading Malaysia, Persiaran Graduan, Educity, 79200 Iskandar Puteri, Johor, Malaysia.*
[2]*Department of Mathematical Sciences, Faculty of Science, University Technologi Malaysia, 81310 Johor Bahru, Johor, Malaysia.*

ABSTRACT

In this paper, we presented a type of Runge-Kutta method to solve initial value problems in Ordinary Differential Equations. Similar to Euler's method, the new method is of order one, easy to implement and only require one function evaluation per step except the initial step. The only different is this method requires the information from the previous step. We studied the stability of the new method and numerical results are presented.

Keywords: Economical Runge-Kutta Method, First Order, Ordinary Differential Equations.

1. INTRODUCTION

There are a lot of methods proposed to solve the following ordinary differential equations (ODEs)

$$\begin{aligned} y' &= f(x, y), \\ y(x_0) &= y_0. \end{aligned} \tag{1}$$

One of the methods is the Runge-Kutta method. Euler's method can be considered the easiest and most economical method in the family of Runge-Kutta method. However, this method also has some disadvantages. For example, Euler's method is well known to be less accurate compare to higher order Runge-Kutta method, it is unstable for stiff problems and its approximated solution converge slower to analytical solution [5].

Higher order Runge-Kutta method is often used rather than Euler's method to improve accuracy. However, higher order method often more computational costly. For example, the famous *Improved Euler method* proposed by Heun is a second order method and it requires two function evaluations [6]. As a result, introduction of additional stage is necessary to improve accuracy of first order method. For example, Ashour and Hanna [1] proposed a two stages first order method, which is more efficient than Euler's method for solving mildly stiff and non-stiff problems.

When talking about economical method for solving ODEs, one would probably think of pseudo Runge-Kutta method. The idea of pseudo Runge-Kutta method is first proposed by Bryrne and Lambert [2]. This method has been studied extensively and modified by Costabile [3] and Nakashima [7]. Unlike Runge-Kutta methods, pseudo Runge-Kutta methods require information from the previous step and they are more economical compare to the conventional method of the same order. However, pseudo Runge-Kutta methods are not self-starting. In other words,

*Corresponding Author: *t.h.lim@reading.edu.my*

they need two initial values where y_1 is normally computed by using Runge-Kutta method [7, 9]. A few years later Costabile et. al. [4] invented a new method for ODEs, called the Economical Runge-Kutta method. This method is presented as:

$$\begin{aligned} y_{n+1} &= y_n + h\sum_{i=1}^{s} b_i K_i^n, \\ K_s^{-1} &= f(x_0, y_0), \\ K_i^n &= f\left(x_n + c_i h, y_n + h\left(a_{i1}K_s^{n-1} + \sum_{j=2}^{i-1} a_{ij}K_j^n\right)\right). \end{aligned} \tag{2}$$

where $b_1 = 0$ and $c_s = 1$. Unlike pseudo Runge- Kutta method, this method is self-starting and is computational cheaper than traditional Runge-Kutta method. They have presented second, third, forth and fifth order methods in their paper [4].

We follow the ideas from Costabile et. al. [4] to construct an economical first order method to solve ODEs. The only different is our method is with $b_1 \neq 0$. The method proposed has the same feature as Euler's method, which is of order one, simple, easy to implement and not computational expensive. Since the proposed method has small stability region, like Euler's method, the proposed method is not suitable for stiff problems. Therefore, the test problems in section 4 are non-stiff problems.

2. DERIVATION OF THE METHOD

Consider the following two stages Runge-Kutta method:

$$\begin{aligned} y_{n+1} &= y_n + h\left(b_1 K_1^n + b_2 K_2^n\right), \\ K_1^n &= f(x_n, y_n), \\ K_2^n &= f\left(x_n + c_2 h, y_n + hc_2 K_1^n\right). \end{aligned} \tag{3}$$

If we set $c_2 = 1$, the necessary condition for the above method to be first order will be

$$b_1 = 1 - b_2, \quad b_2 \neq \frac{1}{2}. \tag{4}$$

As mentioned, Ashour-Hanna method [1] is an example of two stages first order method. Since $c_2 = 1$, it is not hard to see that

$$K_1^n = K_2^{n-1} + O\left(h^2\right). \tag{5}$$

By using (5), method (3) with can be written as

$$\begin{aligned} y_{n+1} &= y_n + h\left(b_1 K_1^{n-1} + b_2 K_2^n\right), \\ K_2^n &= f\left(x_n + h, y_n + hK_2^{n-1}\right). \end{aligned} \tag{6}$$

with $K_2^{-1} = f(x_0, y_0)$. Therefore, method (6) is in the form of Economical Runge-Kutta method as (2). Since this method has the same error bound as method (3), according to Ralston [8], we defined the error bound for method (6)

$$|E| = |1 - 2b_2| ML. \tag{7}$$

Since it satisfies conditions in (4), it is not hard to see that method (6) will become a second order method by selecting $b_2 = \frac{1}{2}$. Our objective is to select a value so that (7) is not too small until it looks like a second order method and it should also be a “nice number”. Finally, we select $b_2 = \frac{3}{5}$ with $|E| = 0.2ML$.

3. STABILITY ANALYSIS

By applying the test equation $y' = \lambda y$ to method (6), we obtained the following stability polynomial

$$y_{n+1} = (1 + h\lambda b_2) y_n + h(b_1 + h\lambda b_2) K_2^{n-1}. \tag{8}$$

Let $u_n = \begin{pmatrix} y_n \\ hK_2^{n-1} \end{pmatrix}$ and $z = \lambda h$, equation (8) can be written as

$$u_{n+1} = Au_n, \tag{9}$$

where $A = \begin{pmatrix} 1 + zb_2 & b_1 + zb_2 \\ z & z \end{pmatrix}$. According to Costabile et. al. [4], the characteristic polynomial for the proposed method is

$$\lambda^2 - (1 + zb_2 + z) - (b_1 z + z^2 b_2) = 0. \tag{10}$$

Solve (10), we obtain the following complex roots:

$$\lambda_1 = \frac{1}{2} + \frac{4}{5}z + \frac{\sqrt{25 + 120z + 124z^2}}{10}, \quad \lambda_2 = \frac{1}{2} + \frac{4}{5}z - \frac{\sqrt{25 + 120z + 124z^2}}{10}. \tag{11}$$

Consider $z = x + yI$, $|\lambda_1| \le 1$ and $|\lambda_2| \le 1$, we obtained the stability region for method (3) as follows:

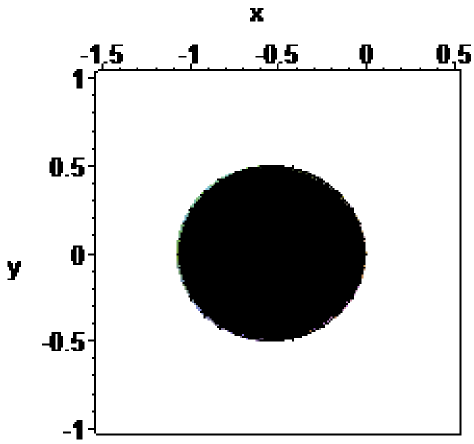

Figure 1. Stability region of the new first order economical Runge-Kutta method.

By definition, a numerical method is said to be A-stable if its region of absolute stability contains the whole of the left-hand half plane. For stiff differential equation, it is desirable to have A-stable numerical method [5, 6]. Since the stability region of the proposed method is small and is not A-stable, we conclude that this method is only suitable for non-stiff problems.

4. NUMERICAL RESULTS

In this section, we compare the method we proposed in section 2 with Euler's method by solving a few ODEs. Since our method is not suitable to solve stiff differential equations, the test problems in this section are all non-stiff problems. Test problem 1 to 3 are obtained from Costabile et. al [4] and test problem 4 is obtained from Nakashima [7].

Problem 1 $y' = -y, y(0) = 1$

Exact solution: $y(x) = e^{-x}$, for $0 \le x \le 10$.

Problem 2 $y' = -\frac{y^3}{2}, y(0) = 1$

Exact solution: $y(x) = \frac{1}{\sqrt{1+x}}$, for $0 \le x \le 10$.

Problem 3 $y' = \frac{y}{4}\left(1 - \frac{y}{20}\right), y(0) = 1$

Exact solution: $y(x) = \frac{20}{1+19e^{-\frac{x}{4}}}$, for $0 \le x \le 10$.

Problem 4 $y_1 = -y_2, \quad y_1(0) = 2$
$y_2 = -3y_1 - 2y_2, \quad y_2(0) = 2$

Exact solution: $y_1(x) = e^{x} + e^{-3x}$, $y_2(x) = 3e^{-3x} - e^{x}$, for $0 \le x \le 2$.

Below is the notation used:

H : Step size
EULER : Euler's method
ECO1 : The proposed method
MAXERROR : Maximum error $| y(x_i) - y_i |$

Table 1 Numerical result for problem 1

H	Method	MAXERROR
0.1	EULER	1.9201×10^{-2}
	ECO1	2.5280×10^{-3}
0.05	EULER	9.3935×10^{-3}
	ECO1	1.5520×10^{-3}
0.01	EULER	1.8471×10^{-3}
	ECO1	3.5641×10^{-4}
0.005	EULER	9.2162×10^{-4}
	ECO1	1.8107×10^{-4}
0.001	EULER	1.8402×10^{-4}
	ECO1	3.6673×10^{-5}

Table 2 Numerical result for problem 2

H	Method	MAXERROR
0.1	EULER	9.6944×10^{-3}
	ECO1	1.3308×10^{-3}
0.05	EULER	4.7190×10^{-3}
	ECO1	7.9124×10^{-4}
0.01	EULER	9.2430×10^{-4}
	ECO1	1.7876×10^{-4}
0.005	EULER	4.6100×10^{-4}
	ECO1	9.0674×10^{-5}
0.001	EULER	9.2016×10^{-5}
	ECO1	1.8342×10^{-5}

Table 3 Numerical result for problem 3

H	Method	MAXERROR
0.1	EULER	9.5325×10^{-2}
	ECO1	2.0381×10^{-2}
0.05	EULER	4.7812×10^{-2}
	ECO1	9.8943×10^{-3}
0.01	EULER	9.5861×10^{-3}
	ECO1	1.9306×10^{-3}
0.005	EULER	4.7945×10^{-3}
	ECO1	9.6224×10^{-4}
0.001	EULER	9.5913×10^{-4}
	ECO1	1.9196×10^{-4}

Table 4 Numerical result for problem 4

H	Method	MAXERROR (y_1)	MAXERROR (y_2)
0.1	EULER	6.6324×10^{-1}	6.5651×10^{-1}
	ECO1	1.8470×10^{-1}	1.8489×10^{-1}
0.05	EULER	3.5004×10^{-1}	3.4614×10^{-1}
	ECO1	8.4086×10^{-2}	8.3661×10^{-2}
0.01	EULER	7.3256×10^{-2}	7.2386×10^{-2}
	ECO1	1.5250×10^{-2}	1.5089×10^{-2}
0.005	EULER	3.6841×10^{-2}	3.6401×10^{-2}
	ECO1	7.5190×10^{-3}	7.4342×10^{-3}
0.001	EULER	7.4027×10^{-3}	7.3137×10^{-3}
	ECO1	1.4866×10^{-3}	1.4689×10^{-3}

5. CONCLUSIONS

By using the idea from Costabile et al [4], we presented an economical first order method to solve ODEs. The only difference is our method does not necessarily require $b_1 = 0$. The proposed method is similar to Euler's method, which is simple, easy to understand and only require one function evaluation per step.

We studied the stability of the new method in section 3. Since this method is not A-stable, we conclude that it is not suitable to solve stiff problems. We compare our method with Euler's method in section 4. Numerical results show that the proposed method is more accurate and more reliable than Euler's method.

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