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Exact Solution for Nonlinear Wave-Like Equation Using Kharrat-Toma Variational Iteration Method

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ABSTRACT

This study introduces a hybrid methodology for solving a class of nonlinear partial differential equations, specifically the nonlinear wave-like equation, which is fundamental in various scientific and engineering applications. These equations model realistic physical phenomena, describe solitons, enhance the understanding of shock waves, and support scientific computing in tackling complex nonlinear problems. The proposed approach integrates the Variational Iteration Method (VIM) with the Kharrat-Toma transform, achieving a balance between accuracy and computational efficiency. The primary objective of this research is to obtain an exact and complete solution for the nonlinear wave-like equation, providing a robust framework for addressing complex nonlinear systems effectively.

Keywords: Nonlinear wave-like equation, Partial differential equation, Variational Iteration Method, Kharrat-Toma transform, Exact solution.

1 INTRODUCTION

In many cases, obtaining an exact solution to nonlinear differential equations poses significant challenges. To address these difficulties, researchers have worked to refine classical solution techniques by modifying them, developing hybrid approaches, or integrating them with various transformation methods. Among the fundamental nonlinear partial differential equations, the wave-like equation stands out as a cornerstone in mathematical physics and applied sciences. This equation has been extensively studied in the works of Aslanov [1], Khalouta [2], and Mohyud [3], emphasizing its importance in diverse scientific applications.

The Variational Iteration Method (VIM) [4] has emerged as a powerful semi-analytical tool for solving a wide range of differential equations, particularly nonlinear systems. For instance, Albert et al. [5] applied VIM to solve coupled nonlinear Klein-Gordon equations, while Bassi et al. [6] employed it to solve linear fractional differential equations. Despite its effectiveness, further refinement is necessary to achieve fully analytical solutions with absolute precision.

In recent years, Kharrat and Toma introduced an innovative hybridization technique that combines new integral transforms with the Homotopy Perturbation Method (HPM) to obtain both exact and approximate solutions for initial and boundary value problems [7-14]. This advancement has significantly enhanced solution methodologies for nonlinear equations. Among these novel approaches, the Kharrat-Toma Transform, introduced in 2020 [15], has gained recognition as a powerful mathematical tool for transforming and solving nonlinear differential equations. For example, Mahmood et al. [16] applied it to solve growth and decay problems, while Patil et al. [17] utilized it for population growth and decay models.

A key advantage of our approach lies in its ability to deliver an exact, analytical solution with zero error. Unlike numerical or approximate methods that require error analysis and validation against other solutions, our method directly yields the true, closed-form solution, eliminating the necessity for comparison with approximate results. This establishes a clear distinction between our approach and other solution techniques, as it guarantees absolute accuracy without reliance on approximations.

Formally, the Kharrat-Toma Transform is defined by the following integral equation:

$$B[g(t)] = G(s) = s^3 \int_0^{\infty} g(t) e^{\frac{-t}{s^2}}\, dt, \quad t \geq 0$$

Integrating the Variational Iteration Method with the Kharrat-Toma Transform offers a promising framework for efficiently and accurately solving nonlinear differential equations. This methodology has the potential to drive advancements in applied physics, engineering, and mathematical sciences, providing a robust and systematic approach for tackling complex nonlinear systems.

This paper explores the effectiveness of integrating VIM with the Kharrat-Toma Transform to develop a robust and efficient hybrid methodology for solving nonlinear wave-like equations.

2 METHODOLOGY AND FORMULATION

Consider the following nonlinear wave like equation

$$v_{tt} = f(t, x, v, v_t, v_x, v_{xx}, \dots) \tag{1}$$

Where f is a general nonlinear partial differential operator.

With the initial conditions

$$v(x,0) = g(x) \;\;, \;\; v_t(x,0) = h(x)$$

By Variational Iteration method technique, we can construct a correction functional as

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda(x,\xi)\big[(v_n(x,\xi))_{\xi\xi} - f\big(\xi, x, \tilde{v}_n(x,\xi), (\tilde{v}_n(x,\xi))_{\xi}, (\tilde{v}_n)_x, (\tilde{v}_n)_{xx}, \dots\big)\big]\, d\xi \quad , \;\; n \geq 0 \tag{2}$$

By taking the Kharrat-Toma transform on (2), finds

$$B[v_{n+1}(x,t)] = B[v_n(x,t)] + B\left\{\int_0^t \lambda(x,\xi)\big[(v_n(x,\xi))_{\xi\xi} - f\big(\xi, x, \tilde{v}_n(x,\xi), (\tilde{v}_n(x,\xi))_\xi, (\tilde{v}_n)_x, (\tilde{v}_n)_{xx}\ , \dots\big)\ \big]\, d\xi\right\}$$

Then

$$B[v_{n+1}(x,t)] = B[v_n(x,t)] + B\{\lambda(x,t) * [(v_n(x,\xi))_{tt} - f(t, x, \tilde{v}_n(x,t), (\tilde{v}_n(x,t))_t, (\tilde{v}_n)_x, (\tilde{v}_n)_{xx}, \dots)]\}$$

Where * is a convolution with respect to t.

$$B[v_{n+1}(x,t)] = B[v_n(x,t)] + \tfrac{1}{s^3}B[\lambda].B[(v_n)_{tt} - f(t, x, \tilde{v}_n, (\tilde{v}_n)_t, (\tilde{v}_n)_x, (\tilde{v}_n)_{xx}, \dots)] \tag{3}$$

Then

$$B[v_{n+1}(x,t)] = B[v_n(x,t)] + \tfrac{1}{s^3}B[\lambda].\left\{\tfrac{1}{s^4}B[v_n] - sv(x,0) - s^3 v_t(x,0) + B[-f(t, x, \tilde{v}_n, (\tilde{v}_n)_t, (\tilde{v}_n)_x, (\tilde{v}_n)_{xx}, \dots)]\right\}$$

We take the variation $\delta$with respect to v_n, yields

$$B[\delta v_{n+1}(x,t)] = B[\delta v_n(x,t)] + \frac{1}{s^3}B[\lambda].\left\{\frac{1}{s^4}B[\delta v_n]\right\}$$

Where $\delta v_{n+1}(x,t) = 0$

Then

$$B[\lambda] = -s^7$$

Then (3) can be written as

$$B[v_{n+1}(x,t)] = B[v_n(x,t)] - s^4.B[(v_n)_{tt} - f(t, x, v_n, (v_n)_t, (v_n)_x, (v_n)_{xx}, \dots)] \tag{4}$$

By the initial conditions, we can be considered

$$v_0(x,t) = g(x) + t.h(x)$$

Then from (4), yields

$$B[v_1(x,t)] = B[v_0(x,t)] - s^4.B[(v_0)_{tt} - f(t, x, v_0, (v_0)_t, (v_0)_x, (v_0)_{xx}, \dots)]$$

By the inverse Kharrat-Toma transform, gets $v_1(x,t)$, $v_2(x,t)$, …

Therefore, the solution is

$$v(x,t) = \lim_{n\to+\infty} v_n(x,t)$$

3 TEST PROBLEMS

Example 1

Consider the following nonlinear wave-like equation [18]

$$v_{tt} = x^2 \frac{\partial}{\partial x}(v_x v_{xx}) - x^2 (v_{xx})^2 - v, v = v(x,t), \;\; 0 \le x \le 1, \;\; t > 0 \tag{5}$$

with the initial conditions

$v(x,0) = 0 \quad , \quad v_t(x,0) = x^2$

By using Variational Iteration method technique, we can construct a correction functional as

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda(x,\xi) \Big[(v_n(x,\xi))_{\xi\xi} - x^2 \frac{\partial}{\partial x}((\tilde{v}_n(x,\xi))_x (\tilde{v}_n(x,\xi))_{xx}) + x^2((\tilde{v}_n(x,\xi))_{xx})^2 + \tilde{v}_n(x,\xi)\Big] d\xi, \;\; n \ge 0 \tag{6}$$

By taking the Kharrat-Toma transform on (6), finds

$$B[v_{n+1}(x,t)] = B[v_n(x,t)] + B\left\{\int_0^t \lambda(x,\xi) \left[(v_n(x,\xi))_{\xi\xi} - x^2 \frac{\partial}{\partial x}((\tilde{v}_n(x,\xi))_x (\tilde{v}_n(x,\xi))_{xx}) + x^2((\tilde{v}_n(x,\xi))_{xx})^2 + \tilde{v}_n(x,\xi)\right] d\xi\right\}$$

Then

$$B[v_{n+1}(x,t)] = B[v_n(x,t)] + B\left\{\lambda(x,t) * \left[(v_n(x,t))_{tt} - x^2 \frac{\partial}{\partial x}((\tilde{v}_n(x,t))_x (\tilde{v}_n(x,t))_{xx}) + x^2((\tilde{v}_n(x,t))_{xx})^2 + \tilde{v}_n(x,t)\right]\right\}$$

where * is a convolution with respect to t.

$$B[v_{n+1}(x,t)] = B[v_n(x,t)] + \frac{1}{s^3} B[\lambda]. B\left[(v_n)_{tt} - x^2 \frac{\partial}{\partial x}((\tilde{v}_n))_x (\tilde{v}_n)_{xx}) + x^2((\tilde{v}_n)_{xx})^2 + \tilde{v}_n\right] \tag{7}$$

$$B[v_{n+1}(x,t)] = B[v_n(x,t)] + \frac{1}{s^3} B[\lambda]. \left\{\frac{1}{s^4} B[v_n] - sv(x,0) - s^3 v_t(x,0) + B\left[-x^2 \frac{\partial}{\partial x}((\tilde{v}_n))_x (\tilde{v}_n)_{xx}) + x^2((\tilde{v}_n)_{xx})^2 + \tilde{v}_n\right]\right\}$$

We take the variation δ with respect to v_n, yields

$$B[\delta v_{n+1}(x,t)] = B[\delta v_n(x,t)] + \frac{1}{s^3} B[\lambda]. \left\{\frac{1}{s^4} B[\delta v_n]\right\}$$

Where $\delta v_{n+1}(x,t) = 0$

Then

$B[\lambda] = -s^7$

Then (7) can be written as

$$B[v_{n+1}(x,t)] = B[v_n(x,t)] - s^4.B\left[(v_n)_{tt} - x^2\frac{\partial}{\partial x}((v_n))_x(v_n)_{xx}) + x^2((v_n)_{xx})^2 + v_n\right] \quad (8)$$

By the initial conditions, we can be considered

$$v_0(x,t) = x^2 t$$

Then from (8), yields

$$B[v_1(x,t)] = B[v_0(x,t)] - s^4.B\left[(v_0)_{tt} - x^2\frac{\partial}{\partial x}((v_0))_x(v_0)_{xx}) + x^2((v_0)_{xx})^2 + v_0\right] = B[v_0] - x^2 s^{11}$$

By the inverse Kharrat-Toma transform, gets

$$v_1(x,t) = x^2\left(t - \frac{t^3}{6}\right)$$

And

$$B[v_2(x,t)] = B[v_1(x,t)] - s^4.B\left[(v_1)_{tt} - x^2\frac{\partial}{\partial x}((v_1))_x(v_1)_{xx}) + x^2((v_1)_{xx})^2 + v_1\right] = B[v_1] + x^2 s^{15}$$

Then

$$v_2(x,t) = x^2\left(t - \frac{t^3}{6} + \frac{t^5}{120}\right)$$

Therefore, the exact solution is

$$v(x,t) = \lim_{n\to+\infty} v_n(x,t) = x^2\left(t - \frac{t^3}{6} + \frac{t^5}{120} + \ldots\right) = x^2\sin t$$

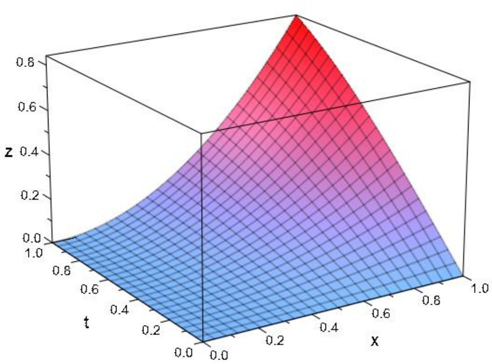

Figure 1: Solution profile for $x \in [0,1]$, $t \in [0,1]$ using the proposed method.

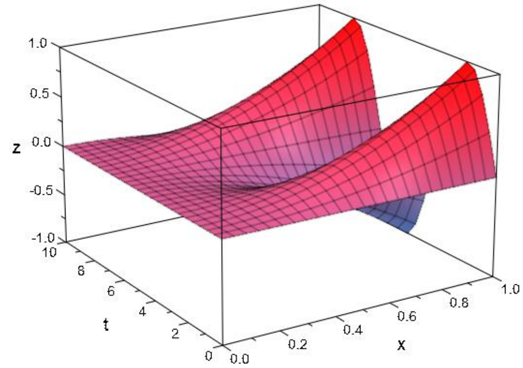

Figure 2: Solution profile for $x \in [0,1]$, $t \in [0,10]$ using the proposed method.

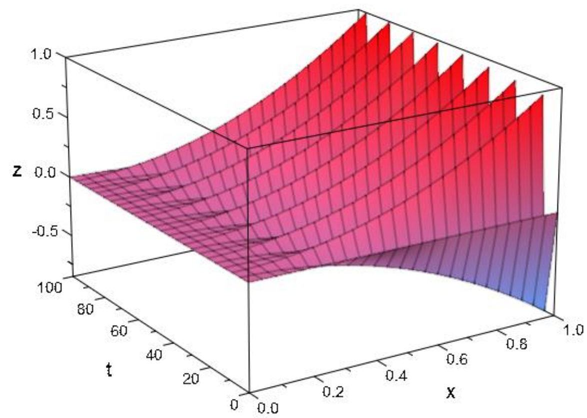

Figure 3: Solution profile for $x \in [0,1]$, $t \in [0,100]$ using the proposed method.

Example 2

Consider the following nonlinear wave-like equation [18]

$$v_{tt} = v^2 \frac{\partial^2}{\partial x^2}(v_x v_{xx} v_{xxx}) + (v_x)^2 \frac{\partial^2}{\partial x^2}((v_{xx})^3) - 18v^5 + v, v = v(x,t),\ 0 \le x \le 1,\ \ t > 0 \tag{9}$$

With the initial conditions

$v(x,0) = e^x$, $v_t(x,0) = e^x$

By using Variational Iteration method technique, we can construct a correction functional as

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda(x,\xi) \begin{bmatrix} (v_n(x,\xi))_{\xi\xi} - \tilde{v}_n{}^2 \frac{\partial^2}{\partial x^2}((\tilde{v}_n)_x(\tilde{v}_n)_{xx}(\tilde{v}_n)_{xxx}) \\ -((\tilde{v}_n)_x)^2 \frac{\partial^2}{\partial x^2}(((\tilde{v}_n)_{xx})^3) + 18\tilde{v}_n{}^5 - \tilde{v}_n \end{bmatrix} d\zeta, n \ge 0 \tag{10}$$

By taking the Kharrat-Toma transform on (10), finds

$$B[v_{n+1}(x,t)] = B[v_n(x,t)] + B\left\{\int_0^t \lambda(x,\xi)\begin{bmatrix}\left(v_n(x,\xi)\right)_{\xi\xi} - \tilde{v}_n{}^2 \dfrac{\partial^2}{\partial x^2}((\tilde{v}_n)_x(\tilde{v}_n)_{xx}(\tilde{v}_n)_{xxx}) \\ -((\tilde{v}_n)_x)^2 \dfrac{\partial^2}{\partial x^2}(((\tilde{v}_n)_{xx})^3) + 18\tilde{v}_n{}^5 - \tilde{v}_n\end{bmatrix} d\zeta\right\}$$

Then

$$B[v_{n+1}(x,t)] = B[v_n(x,t)] + B\left\{\lambda(x,t) * \begin{bmatrix}(v_n(x,\xi))_{tt} - \tilde{v}_n{}^2 \frac{\partial^2}{\partial x^2}((\tilde{v}_n)_x(\tilde{v}_n)_{xx}(\tilde{v}_n)_{xxx}) \\ -((\tilde{v}_n)_x)^2 \frac{\partial^2}{\partial x^2}(((\tilde{v}_n)_{xx})^3) + 18\tilde{v}_n{}^5 - \tilde{v}_n\end{bmatrix}\right\}$$

Then

$$B[v_{n+1}(x,t)] = B[v_n(x,t)] + \tfrac{1}{s^3}B[\lambda].B\begin{bmatrix}(v_n(x,\xi))_{tt} - \tilde{v}_n{}^2 \frac{\partial^2}{\partial x^2}((\tilde{v}_n)_x(\tilde{v}_n)_{xx}(\tilde{v}_n)_{xxx}) \\ -((\tilde{v}_n)_x)^2 \frac{\partial^2}{\partial x^2}(((\tilde{v}_n)_{xx})^3) + 18\tilde{v}_n{}^5 - \tilde{v}_n\end{bmatrix} \tag{11}$$

$$B[v_{n+1}(x,t)] = B[v_n(x,t)] + \tfrac{1}{s^3}B[\lambda].\left\{\tfrac{1}{s^4}B[v_n] - sv(x,0) - s^3v_t(x,0) + \right.$$

$$\left. B\begin{bmatrix}-\tilde{v}_n{}^2 \frac{\partial^2}{\partial x^2}((\tilde{v}_n)_x(\tilde{v}_n)_{xx}(\tilde{v}_n)_{xxx}) \\ -((\tilde{v}_n)_x)^2 \frac{\partial^2}{\partial x^2}(((\tilde{v}_n)_{xx})^3) + 18\tilde{v}_n{}^5 - \tilde{v}_n\end{bmatrix}\right\}$$

We take the variation $\delta$with respect to v_n, yields

$$B[\delta v_{n+1}(x,t)] = B[\delta v_n(x,t)] + \frac{1}{s^3}B[\lambda].\left\{\frac{1}{s^4}B[\delta v_n]\right\}$$

Where $\delta v_{n+1}(x,t) = 0$

$B[\lambda] = -s^7$

Then (11) can be written as

$$B[v_{n+1}(x,t)] = B[v_n(x,t)] - s^4.B\begin{bmatrix}(v_n)_{tt} - v_n{}^2 \frac{\partial^2}{\partial x^2}((v_n)_x(v_n)_{xx}(v_n)_{xxx}) \\ -((v_n)_x)^2 \frac{\partial^2}{\partial x^2}(((v_n)_{xx})^3) + 18v_n{}^5 - v_n\end{bmatrix} \tag{12}$$

By the initial conditions, we can be considered

$v_0(x,t) = e^x(1+t)$

Then from (12), yields

$$B[v_1(x,t)] = B[v_0(x,t)] - s^4.B\left[(v_0)_{tt} - {v_0}^2 \frac{\partial^2}{\partial x^2}((v_0)_x(v_0)_{xx}(v_0)_{xxx}) - ((v_0)_x)^2 \frac{\partial^2}{\partial x^2}(((v_0)_{xx})^3) + 18{v_0}^5 - v_0\right]$$
$$= B[v_0] + e^x(s^9 + s^{11})$$

By the inverse Kharrat-Toma transform, gets

$$v_1(x,t) = e^x\left(1 + t + \frac{t^2}{2} + \frac{t^3}{6}\right)$$

And

$$B[v_2(x,t)] = B[v_1(x,t)] - s^4.B\left[(v_1)_{tt} - {v_1}^2 \frac{\partial^2}{\partial x^2}((v_1)_x(v_1)_{xx}(v_1)_{xxx}) - ((v_1)_x)^2 \frac{\partial^2}{\partial x^2}(((v_1)_{xx})^3) + 18{v_1}^5 - v_1\right]$$
$$= B[v_1] + e^x(s^{13} + s^{15})$$

then

$$v_2(x,t) = e^x\left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120}\right)$$

Therefore, the exact solution is

$$v(x,t) = \lim_{n\to+\infty} v_n(x,t) = e^x\left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + \ldots\right) = e^{x+t}$$

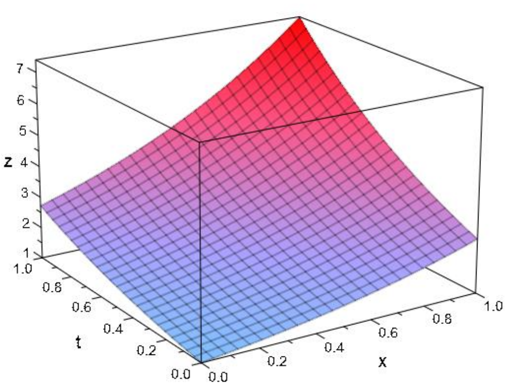

Figure 4: Solution profile for $x \in [0,1]$, $t \in [0,1]$ using the proposed method.

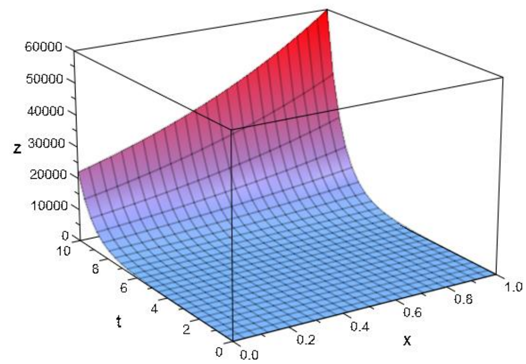

Figure 5: Solution profile for $x \in [0,1]$, $t \in [0,10]$ using the proposed method.

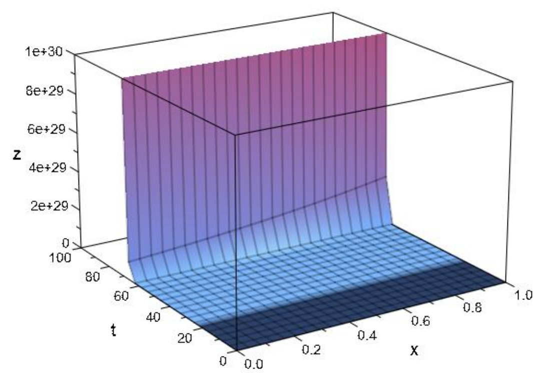

Figure 6: Solution profile for $x \in [0,1]$, $t \in [0,100]$ using the proposed method.

Example 3

Consider the following nonlinear wave-like equation [18]

$$v_{tt} = \frac{\partial^2}{\partial x \partial y}(v_{xx} v_{yy}) - \frac{\partial^2}{\partial x \partial y}\left(xy v_x v_y\right) - v, v = v(x,y,t), 0 \le x, y \le 1, t > 0 \quad (13)$$

With the initial conditions

$v(x,0) = e^{xy}$, $v_t(x,0) = e^{xy}$

By using Variational Iteration method technique, we can construct a correction functional as

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda(x,\xi)\left[(v_n(x,\xi))_{\xi\xi} - \frac{\partial^2}{\partial x \partial y}\left((\tilde{v}_n)_{xx}(\tilde{v}_n)_{yy}\right) + \frac{\partial^2}{\partial x \partial y}\left(xy(\tilde{v}_n)_x(\tilde{v}_n)_y\right) - \tilde{v}_n\right] d\zeta, n \ge 0 \quad (14)$$

By taking the Kharrat-Toma transform on (14), finds

$B[v_{n+1}(x,t)] = B[v_n(x,t)] + B\left\{\int_0^t \lambda(x,\xi)\left[(v_n(x,\xi))_{\xi\xi} - \frac{\partial^2}{\partial x\partial y}\left((\tilde{v}_n)_{xx}(\tilde{v}_n)_{yy}\right) + \frac{\partial^2}{\partial x\partial y}\left(xy(\tilde{v}_n)_x(\tilde{v}_n)_y\right) - \tilde{v}_n\right]d\zeta\right\} B[v_{n+1}(x,t)] = B[v_n(x,t)] + B\left\{\lambda(x,t) * \left[(v_n(x,\xi))_{tt} - \frac{\partial^2}{\partial x\partial y}\left((\tilde{v}_n)_{xx}(\tilde{v}_n)_{yy}\right) + \frac{\partial^2}{\partial x\partial y}\left(xy(\tilde{v}_n)_x(\tilde{v}_n)_y\right) - \tilde{v}_n\right]\right\}$

Then

$$B[v_{n+1}(x,t)] = B[v_n(x,t)] + \frac{1}{s^3}B[\lambda].B\left[(v_n(x,\xi))_{tt} - \frac{\partial^2}{\partial x\partial y}\left((\tilde{v}_n)_{xx}(\tilde{v}_n)_{yy}\right) + \frac{\partial^2}{\partial x\partial y}\left(xy(\tilde{v}_n)_x(\tilde{v}_n)_y\right) - \tilde{v}_n\right] \quad (15)$$

$$B[v_{n+1}(x,t)] = B[v_n(x,t)] + \frac{1}{s^3}B[\lambda].\left\{\frac{1}{s^4}B[v_n] - sv(x,0) - s^3v_t(x,0) + B\begin{bmatrix}(v_n(x,\xi))_{tt} - \frac{\partial^2}{\partial x\partial y}\left((\tilde{v}_n)_{xx}(\tilde{v}_n)_{yy}\right) \\ + \frac{\partial^2}{\partial x\partial y}\left(xy(\tilde{v}_n)_x(\tilde{v}_n)_y\right) - \tilde{v}_n\end{bmatrix}\right\}$$

We take the variation $\delta$with respect to v_n, yields

$$B[\lambda] = -s^7$$

Then (15) can be written as

$$B[v_{n+1}(x,t)] = B[v_n(x,t)] - s^4.B\left[(v_n)_{tt} - \frac{\partial^2}{\partial x\partial y}\left((v_n)_{xx}(v_n)_{yy}\right) + \frac{\partial^2}{\partial x\partial y}\left(xy(v_n)_x(v_n)_y\right) - v_n\right] \quad (16)$$

By the initial conditions, we can be considered

$$v_0(x,t) = e^{xy}(1+t)$$

Then from (16), yields

$$\begin{aligned} B[v_1(x,t)] &= B[v_0(x,t)] \\ &\quad - s^4.B\left[(v_0)_{tt} - {v_0}^2\frac{\partial^2}{\partial x^2}((v_0)_x(v_0)_{xx}(v_0)_{xxx}) - ((v_0)_x)^2\frac{\partial^2}{\partial x^2}(((v_0)_{xx})^3) + 18{v_0}^5 - v_0\right] \\ &= B[v_0] + e^x(s^9 + s^{11}) \end{aligned}$$

By the inverse Kharrat-Toma transform, gets

$$v_1(x,t) = e^{xy}\left(1 + t - \frac{t^2}{2} - \frac{t^3}{6}\right)$$

and

$$v_2(x,t) = e^x\left(1 + t - \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120}\right)$$

Therefore, the exact solution is

$$v(x,t) = \lim_{n\to+\infty} v_n(x,t) = e^{xy}\left(\left(1-\frac{t^2}{2}+\frac{t^4}{24}+\ldots\right)+\left(t-\frac{t^3}{6}+\frac{t^5}{120}+\ldots\right)\right) = e^{xy}(\cos t+\sin t)$$

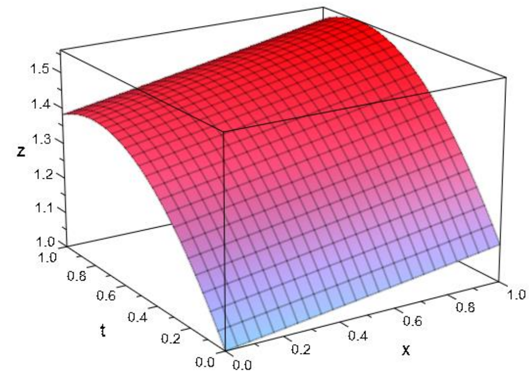

Figure 7: Solution profile for $x \in [0,1]$, $t \in [0,1]$, $y = 0.1$ using the proposed method.

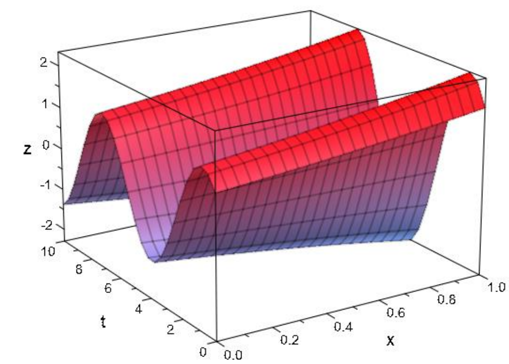

Figure 8: Solution profile for $x \in [0,1]$, $t \in [0,10]$, $y = 0.5$ using the proposed method.

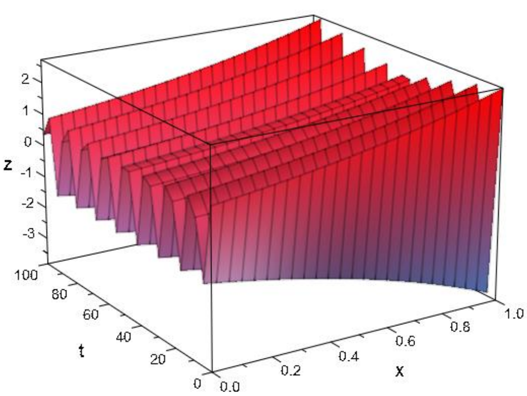

Figure 9: Solution profile for $x \in [0,1]$, $t \in [0,100]$, $y = 1$ using the proposed method.

4 CONCLUSION AND DISCUSSION

In this paper, we introduced a novel hybrid scheme that integrates the Variational Iteration Method (VIM) with the Kharrat-Toma Transform to solve the nonlinear wave-like equation. This approach provides a systematic and efficient framework for obtaining exact solutions to nonlinear partial differential equations. Unlike numerical and approximate methods, the proposed technique guarantees an exact analytical solution with zero error, eliminating the need for validation against approximate solutions. The results confirm that this hybrid method is not only computationally efficient but also provides a powerful and straightforward approach for tackling complex nonlinear wave equations.

The integration of VIM with the Kharrat-Toma Transform enhances both the accuracy and convergence speed of the solution process, positioning it as a valuable alternative to conventional analytical and numerical techniques. Moreover, the simplicity of this hybrid methodology ensures its applicability to a broad range of nonlinear problems in physics, engineering, and applied mathematics. Future research can extend this method to more complex nonlinear systems and investigate its potential in higher-dimensional problems and real-world applications.

The promising outcomes of this study underscore the significance of hybrid analytical techniques in solving nonlinear differential equations. By providing an exact and efficient solution framework, this work contributes to the ongoing development of advanced mathematical tools for modeling and analyzing nonlinear wave phenomena across various scientific disciplines.

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