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# Positivity Preserving for Curve and Surface Interpolation using Rational Cubic Bézier

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**ABSTRACT**

*In [2] the authors have asserted that their proposed partially blended rational bi-cubic spline is shape preserving only for monotone and convex data. In this paper, we prove that the scheme of [2] also positive shape preserving interpolation for positive data. We begin with positive interpolating for curve. Then the sufficient condition for positive surface interpolation is derived on all four boundary curves to form positive interpolating surface. Several numerical results are presented including comparison with some existing scheme.*

**Keywords:** Partially Blended Rational Bi-Cubic, Positivity Preserving, Bézier, Weighted.

## 1. INTRODUCTION

Since rational cubic Bézier has its own limitation i.e. modelling of shape preserving interpolation, Casciola and Romani [2] constructed a NURBS version of rational bi-cubic Bézier. To achieve this, they used partially blended rational bi-cubic Bézier in Hermite form. They proved that the scheme is shape preserving for monotone and convex data provided that all four boundary curves have the shape preserving properties i.e. monotone and convex. Positivity preserving interpolation has received great attention in computer aided geometric design (CAGD) area. For instance, Brodlie *et al*. [1] consider the positivity preserving by using cubic Hermite spline. Unfortunately, their method requires the modification of the first derivative as well as one or two extra knots need to be inserted on the interval in which shape violation is found. Thus this scheme may be fail if the first derivatives are pre-specified at the knots. Hussain and Sarfraz [3] constructed rational bi-cubic spline with 8 parameters. Meanwhile Hussain and Hussain [4] constructed the positive surface by using partially blended rational bi-cubic spline but without any free parameter. The positivity scheme by Hussain *et al*. [5] cannot produce positive surface on entire given domain. Sarfraz *et al*. [6] constructed the rational cubic spline of the form cubic/quadratic with two parameters. But both methods suffer from the fact that, there is no free parameter for shape modification. Furthermore, it can be shown that their scheme may not produce positive interpolating curve and surface everywhere.

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The main objective of this paper is to verify that the partially blended rational bi-cubic Bézier in [2] also can be used for positivity preserving provided that all four boundary curves are positive. To achieve this, we derive the sufficient condition for the positivity of rational cubic Bézier for curve interpolation. Next, we derive the sufficient condition for the positivity of all four boundary curves are derived to produce the positive interpolating surface. This will ensure that the scheme in [2] is also applicable to preserve the positivity of the surface data. Some new results will be presented in this study.

This paper is organized as follows. Section 2 discusses the rational cubic Bézier with one parameter including shape control analysis. Positivity preserving for curve interpolation is discussed in Section 3 including numerical examples. Section 4 is devoted to the construction of the partially blended rational bi-cubic Bézier. In Section 5, the sufficient condition for the positivity of the rational bi-cubic Bézier will be developed. Two positive data sets are used to show that the positivity of data is preserved by using rational bi-cubic Bézier interpolant. Finally, summary and conclusions are given in Section 6.

## 2. RATIONAL CUBIC BÉZIER INTERPOLANT FOR FUNCTIONAL DATA

Given functional data $\{(x_i, f_i), i = 0,1,\ldots,n\}$ such that $x_0 < x_1 < \cdots < x_n$ where $h_i = x_{i+1} - x_i$, $\Delta_i = (f_{i+1} - f_i)/h_i$ and $\theta = (x - x_i)/h_i$ i.e. $0 \le \theta \le 1$. For $x \in [x_i, x_{i+1}], i = 0,1,\ldots,n-1$ the rational cubic Bézier interpolant with parameter or weight $w_i > 0$, $i = 0,1,\ldots,n-1$ is defined by [2]

$$S(x) = S_i(x) = \frac{P_i(\theta)}{Q_i(\theta)}, \tag{1}$$

where,

$$P_i(\theta) = (1-\theta)^3 f_i + 3w_i(1-\theta)^2 \theta V_i + 3w_i(1-\theta)\theta^2 W_i + \theta^3 f_{i+1}$$
$$Q_i(\theta) = (1-\theta)^3 + 3w_i(1-\theta)^2 \theta + 3w_i(1-\theta)\theta^2 + \theta^3$$

The rational cubic Bézier interpolant in (1) satisfies the following $C^1$ condition:

$$\begin{aligned} S(x_i) &= f_i, & S^{(1)}(x_i) &= d_i \\ S(x_{i+1}) &= f_{i+1}, & S^{(1)}(x_{i+1}) &= d_{i+1} \end{aligned} \tag{2}$$

The unknowns $V_i, W_i$ for $i = 0,1,\ldots,n-1$ are given as:

$$V_i = f_i + \frac{h_i d_i}{3w_i}, \quad W_i = f_{i+1} - \frac{h_i d_{i+1}}{3w_i}$$

### 2.1 Shape Control Analysis

Some shape control analysis as well as geometrical interpretation can be made as follows:

(1) When $w_i = 1$ for $i = 0,1,\ldots,n-1$, the rational cubic Bézier interpolant defined by (1) is reduced to standard cubic Bézier

$$S_i(x) = f_i(1-\theta)^3 + 3\left(f_i + \frac{h_i d_i}{3}\right)(1-\theta)^2\theta + 3\left(f_{i+1} - \frac{h_i d_{i+1}}{3}\right)(1-\theta)^2\theta + f_{i+1}\theta^3 \tag{3}$$

(2) Furthermore the piecewise rational cubic Bézier $S_i(x)$ in (1) can be written as

$$S_i(x) = (1-\theta)f_i + \theta f_{i+1} + \frac{h_i(1-\theta)\theta[\Delta_i(2\theta-1)+(1-\theta)d_i - \theta d_{i+1}]}{Q_i(\theta)}. \tag{4}$$

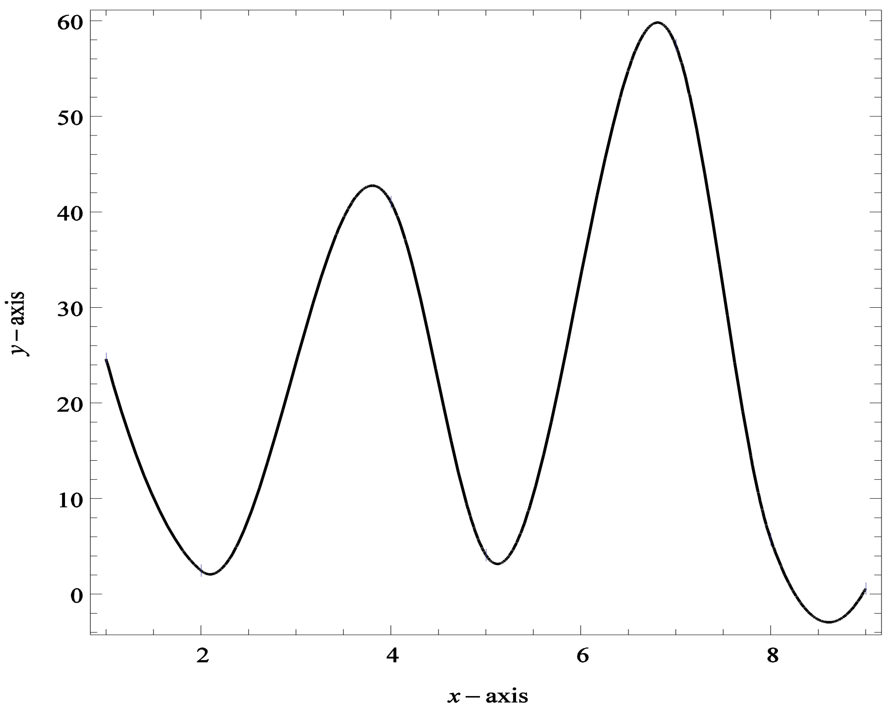
Obviously when $w_i \to \infty, i = 0,1,\ldots,n-1$, the rational cubic Bézier $S_i(x)$ reduce to a straight line on each sub-interval $[x_i, x_{i+1}], i = 0,1,\ldots,n-1$ i.e.

$$\lim_{w_i \to \infty} S_i(x) = (1-\theta)f_i + \theta f_{i+1}. \tag{5}$$

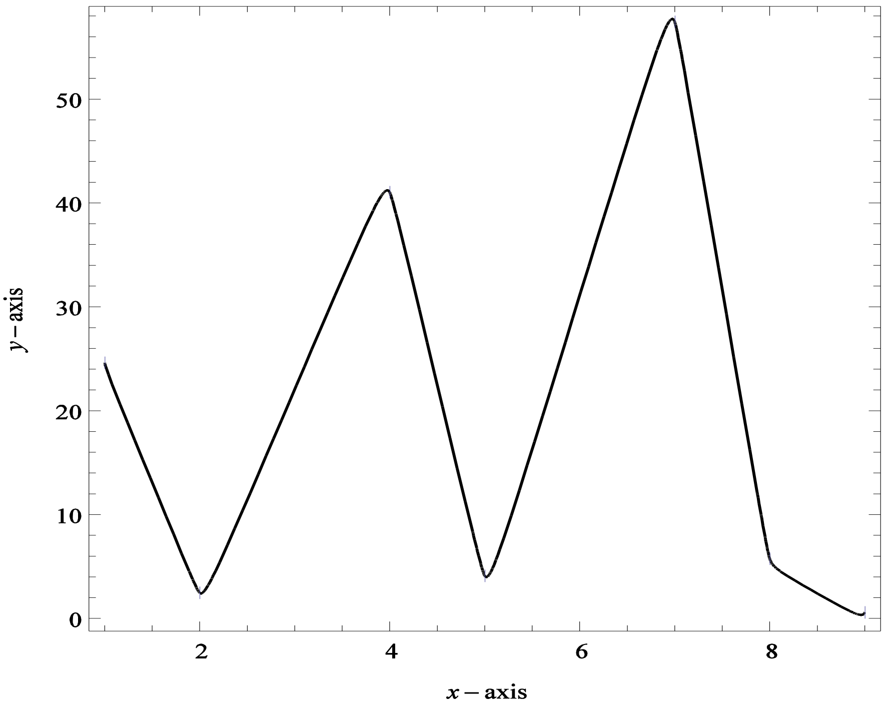
Figures 1 and 2 show the shape control of the rational cubic Bézier defined by (1) for data in Table 1. This shape control analysis is not discussed in [2].

**Table 1** Data set for shape control

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $x_i$ | 1 | 2 | 4 | 5 | 7 | 8 | 9 |
| $f_i$ | 24.6162 | 2.4616 | 41.0270 | 4.1027 | 57.4378 | 5.7438 | 0.5744 |




(A) $w_i = 1$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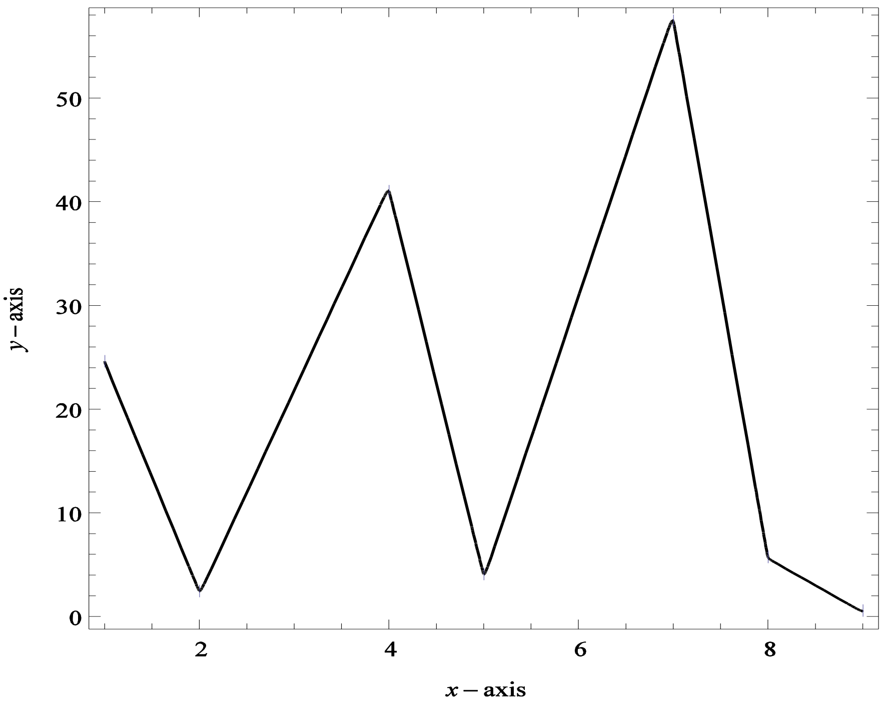
y − axis
x − axis

(b) $w_i = 10$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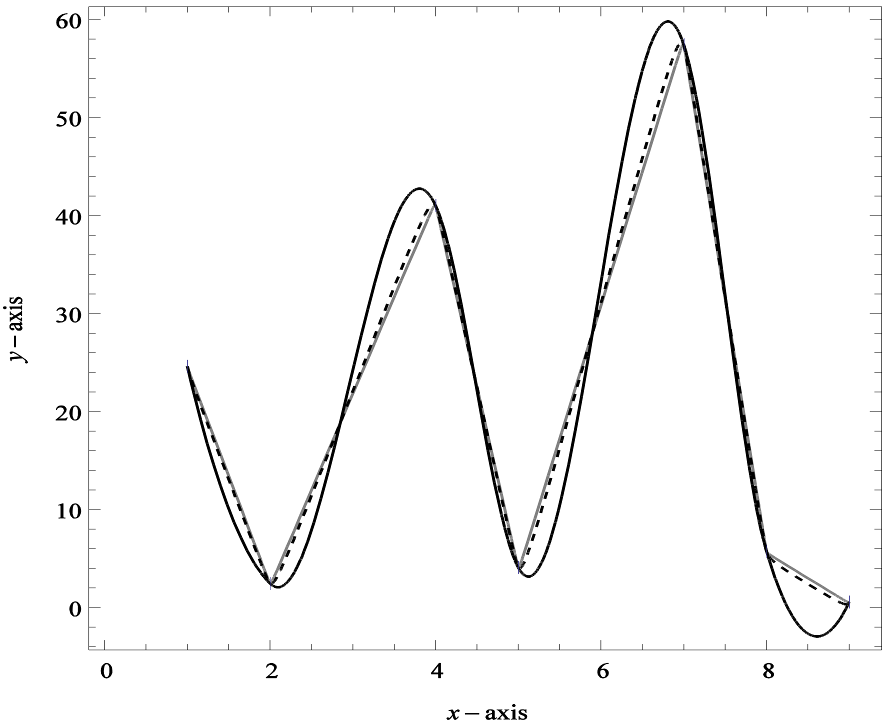
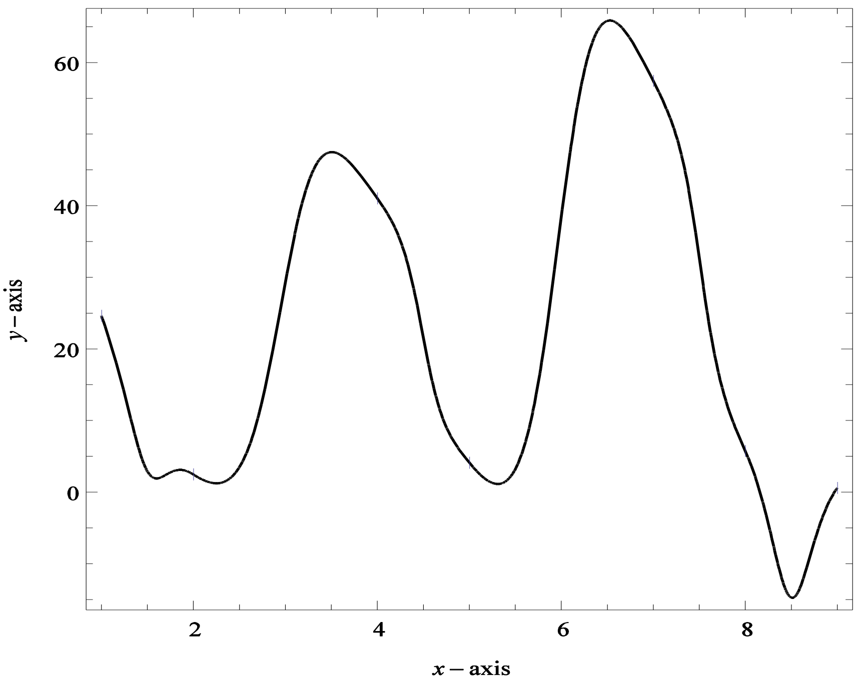
(c) $w_i = 50$

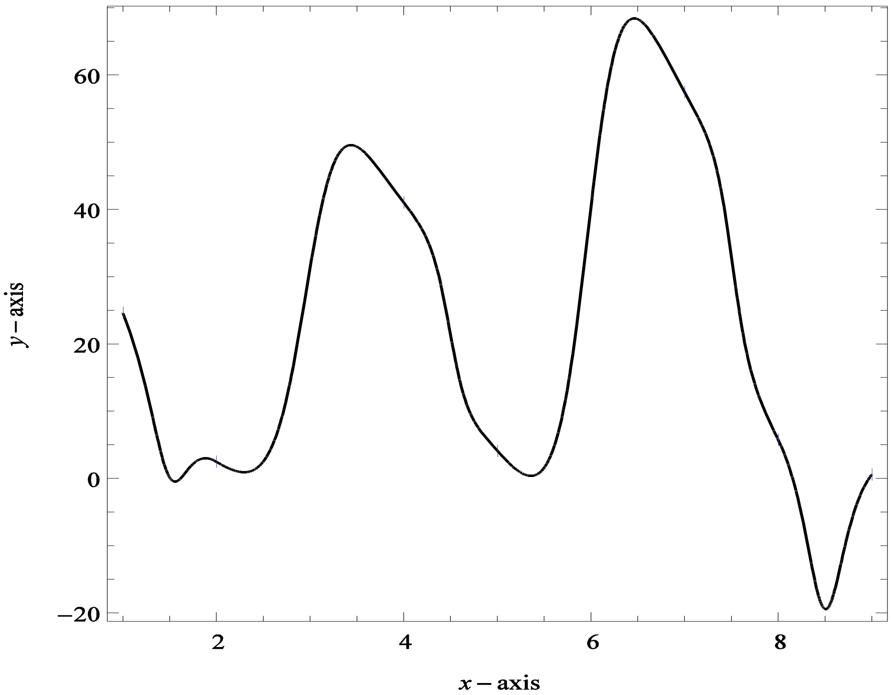



(d)

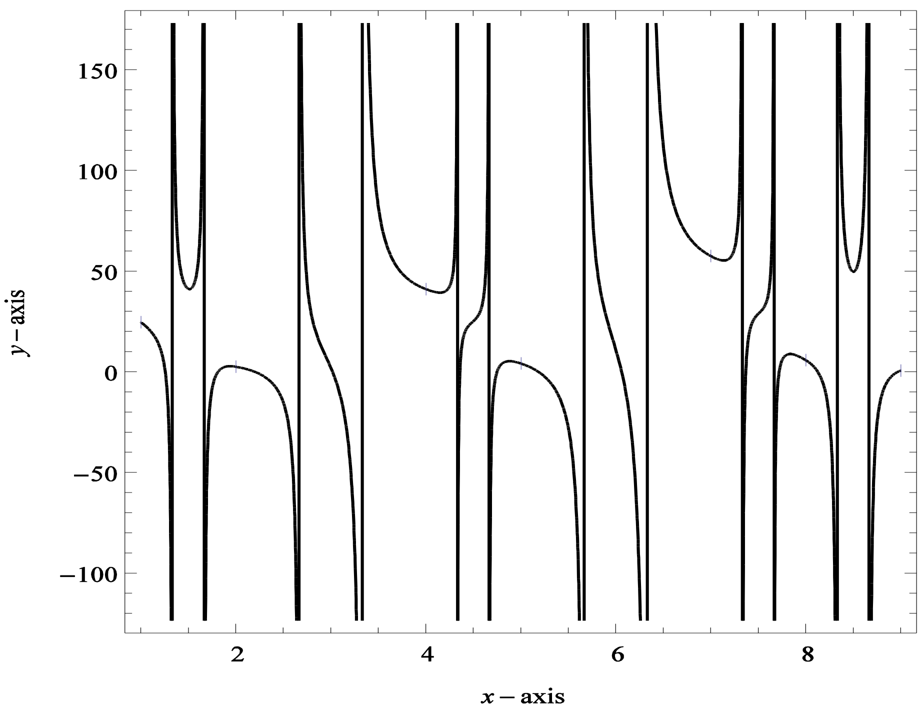
**Figure 1**. Shape control analysis for data in Table 1.




(A) $w_i = 0.1$




(b) $w_i = 0.01$




(c) $w_i = -0.5$

**Figure 2.** Shape control analysis.

Figure 1(a) shows the cubic Bézier interpolation with $w_i = 1$. From Figure 1(b) and 1(c), clearly it can be seen that the interpolating curve is approaching a straight line. This is corresponding to the shape control given in (5). Figure 1(d) shows the combination of Figure 1(a)-black, 1(b)-dashed and 1(c)-gray respectively.

Meanwhile Figure 2 shows the effect when the parameter $w_i$ is approaching zero and what will happen when it has negative value $w_i = -0.5$ i.e. pole at $\theta = 0.5$. This is the main reason why the value of the parameter $w_i$ is chosen such that $w_i > 0$, for all $i = 0,1,\ldots,n-1$.

## 3. POSITIVITY PRESERVING FOR CURVE INTERPOLATION

The authors in [2] are not discussing the shape preserving for positivity interpolation. Motivated by this, in this section, we will construct the positivity curve interpolation by using rational cubic Bézier interpolant defined by (1). We begin with some definition and then later derive the sufficient condition for the positivity.

Given the strictly positive data $\{(x_i, f_i), i = 0,1,\ldots,n\}$ for $x_0 < x_1 < \ldots < x_n$ such that

$$f_i > 0, \quad i = 0,1,\ldots,n-1 \tag{6}$$

The following theorem states the main result for positivity curve interpolation by using rational cubic Bézier defined by (1).

**Theorem 1.** For a strictly positive data in (6), the rational cubic Bézier interpolant $S_i(x)$ is positive if in each subinterval $[x_i, x_{i+1}], i = 0,1,\ldots,n-1$ the parameter $w_i, i = 0,1,\ldots,n-1$ satisfies the following sufficient condition:

$$w_i = \lambda_\mathrm{i} + \mathrm{Max}\left\{0, -\frac{h_i d_i}{3f_i}, \frac{h_i d_{i+1}}{3f_{i+1}}\right\}, \quad \lambda_\mathrm{i} > 0 \tag{7}$$

**Proof.**

The cubic polynomial $P_i(\theta), i = 0,1,\ldots,n-1$ is positive if

$$\left(P_i'(0), P_i'(1)\right) \in R_1 \cup R_2 \tag{8}$$

where,

$$R_1 = \left\{(a,b) : a > \frac{-3P_i(0)}{h_i}, b < \frac{3P_i(1)}{h_i}\right\} \tag{9}$$

$$R_2 = \left\{\begin{matrix}(a,b) : 36 f_i f_{i+1}\left(a^2 + b^2 + ab - 3\Delta_i(a+b) + 3\Delta_i^2\right) + \\ 3(af_{i+1} - bf_i)(2h_i ab - 3af_{i+1} + 3bf_i) + 4h_i\left(a^3 f_{i+1} - b^3 f_i\right) - h_i^2 a^2 b^2\end{matrix}\right\} \tag{10}$$

with $a = P_i'(0)$ $a = P_i'(0)$ and $b = P_i'(1)$.

Usually most author derive the sufficient condition for the positivity by using Condition (9) since it is easy to derive as well as it is sufficiently to produce the positive interpolant i.e. the resulting interpolating curve is positive everywhere. Since the denominator in (1) is always positive, then the sufficient condition for $S_i(x) > 0, i = 0,1,\ldots,n-1$ can be derived from $P_i(\theta) > 0, i = 0,1,\ldots,n-1$. Applying Condition (9), lead us to the following:

$$\frac{-3f_i + 3V_{\mathrm{i}}}{h_i} > \frac{-3f_i}{h_i} \tag{11}$$

and

$$\frac{3f_{i+1} - 3W_{\mathrm{i}}}{h_i} < \frac{3f_{i+1}}{h_i} \tag{12}$$

Which can be written as

$$\frac{-3f_i + 3f_i + \dfrac{h_i d_i}{w_i}}{h_i} > \frac{-3f_i}{h_i}$$

and

$$\frac{3f_{i+1} - 3\,f_{i+1} + \dfrac{h_i d_{i+1}}{w_i}}{h_i} < \frac{3f_{i+1}}{h_i}$$

Both conditions can be further simplified to:

$$w_i > -\frac{h_i d_i}{3f_i} \tag{13}$$

and

$$w_i > \frac{h_i d_{i+1}}{3f_{i+1}} \tag{14}$$

Conditions (13) and (14) give the sufficient condition for the positivity of the rational cubic Bézier as:

$$w_i > \mathrm{Max}\left\{0, -\frac{h_i d_i}{3f_i}, \frac{h_i d_{i+1}}{3f_{i+1}}\right\}, \quad i = 0,1,\ldots,n-1. \tag{15}$$

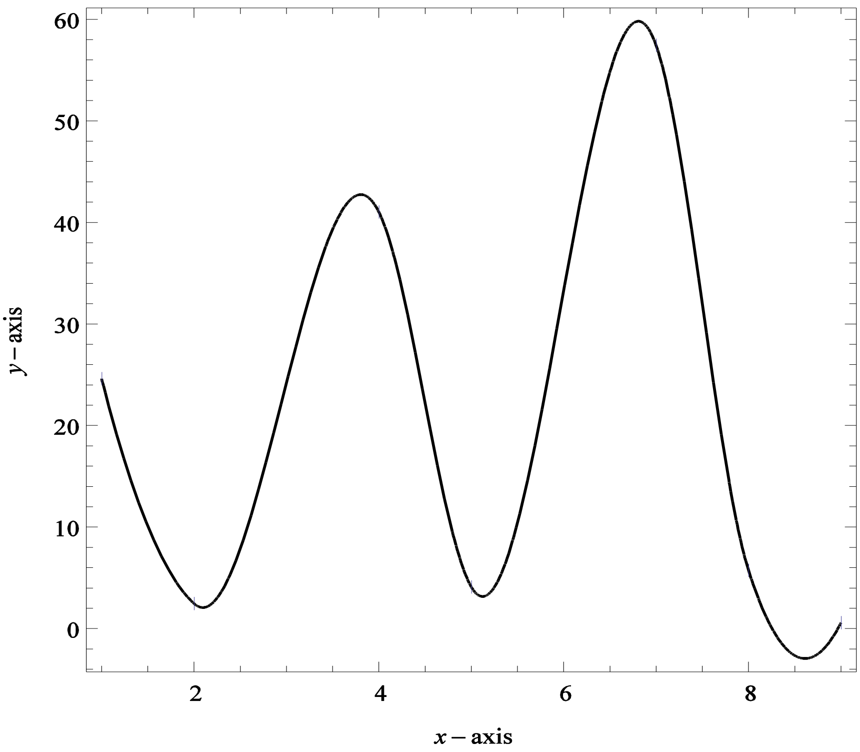
Condition (15) can be reformulated as

$$w_i = \lambda_{\mathrm{i}} + \mathrm{Max}\left\{0, -\frac{h_i d_i}{3f_i}, \frac{h_i d_{i+1}}{3f_{i+1}}\right\}, \quad \lambda_{\mathrm{i}} > 0,\ i = 0,1,\ldots,n-1.$$

### 3.1 Numerical Results

Two data sets are used to test the positivity preserving using $C^1$ rational cubic Bézier developed in in this section.

**Example 1.** Data from Table 1 is used for positivity preserving interpolation.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y − axis
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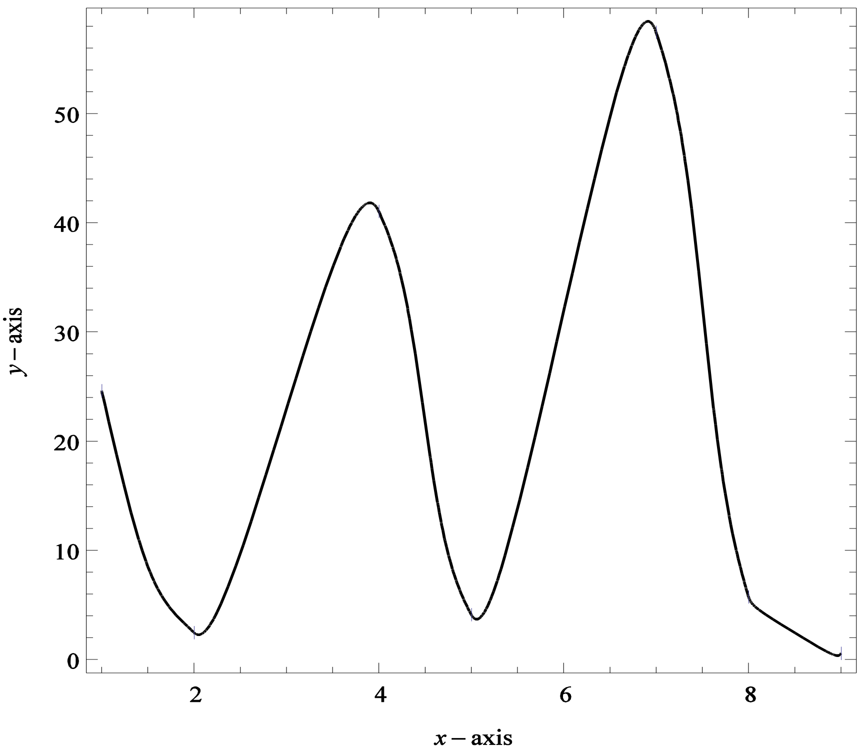
(a) $w_i = 1$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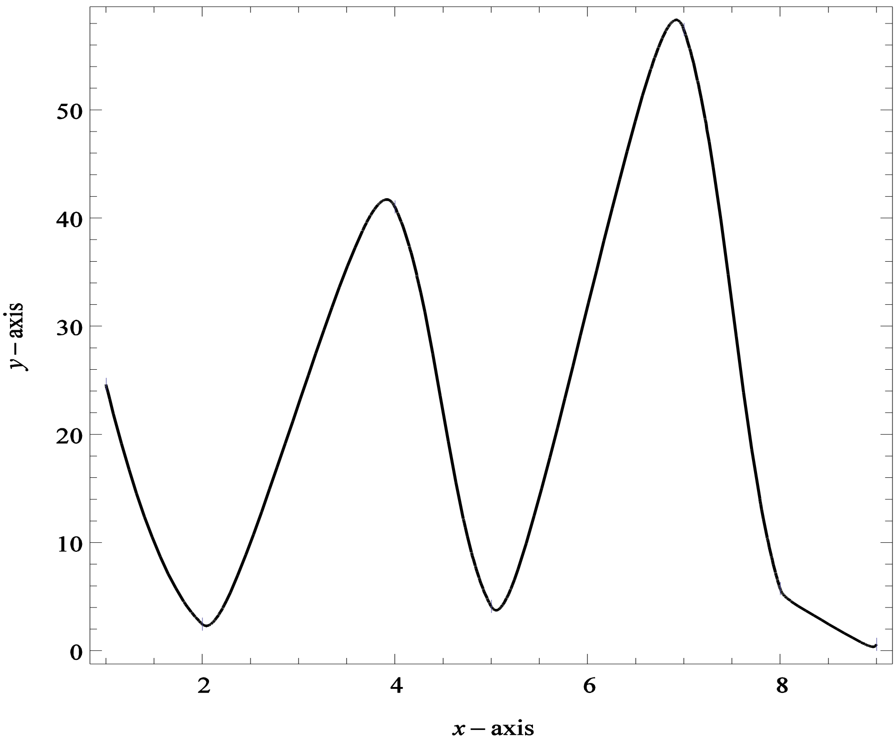
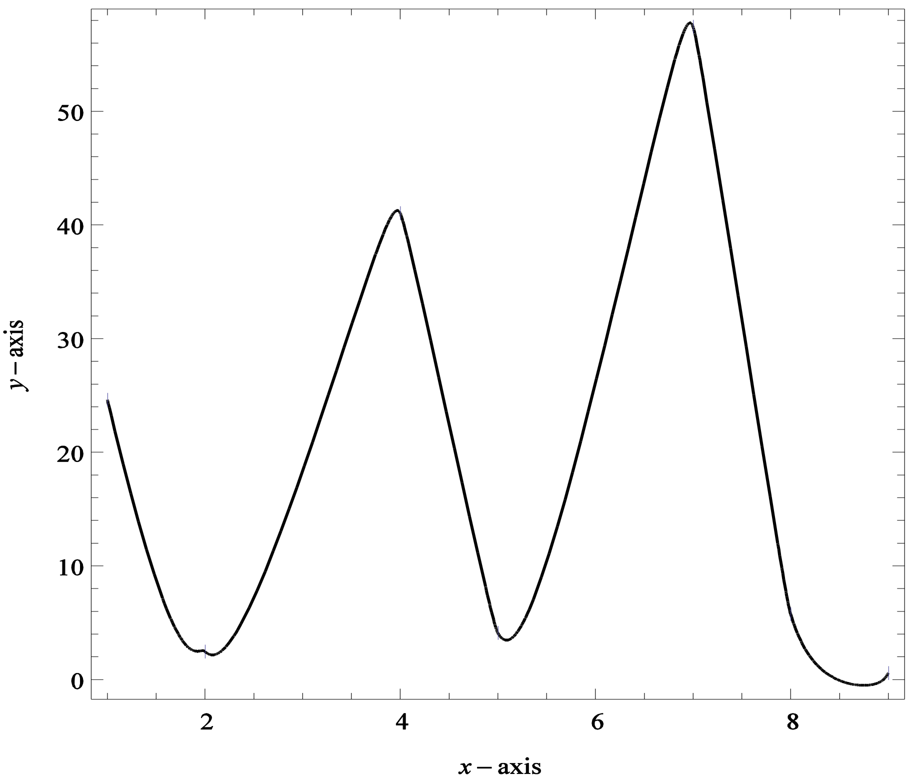
x − axis

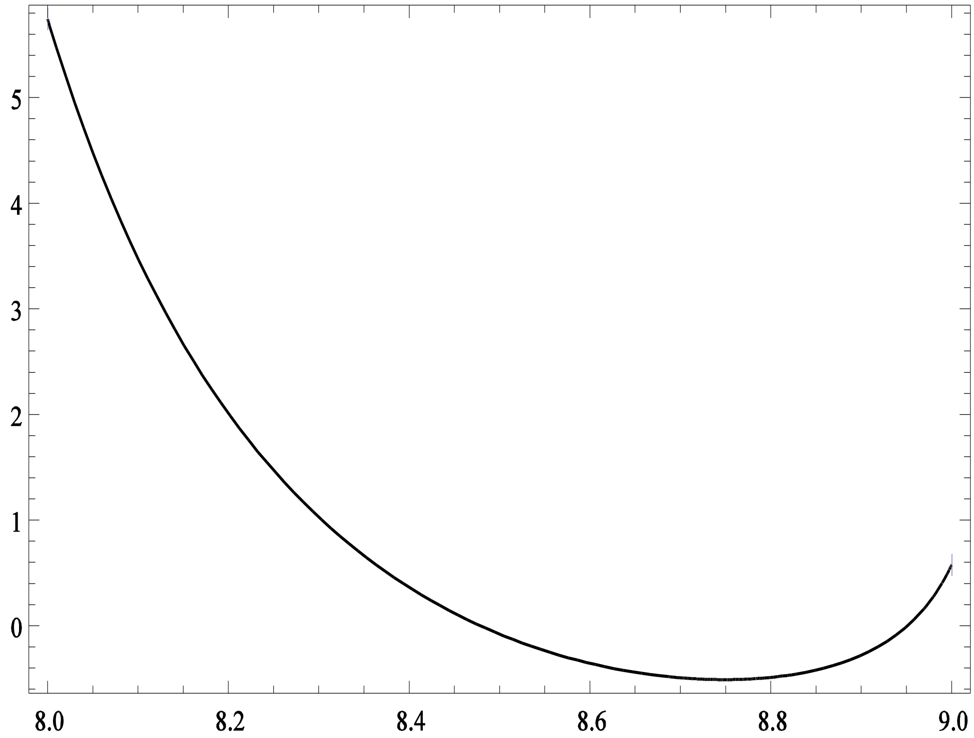
(b)




(c)




(d) Sarfraz *et al*. [6]




(e) Zoom in Figure 1(d)

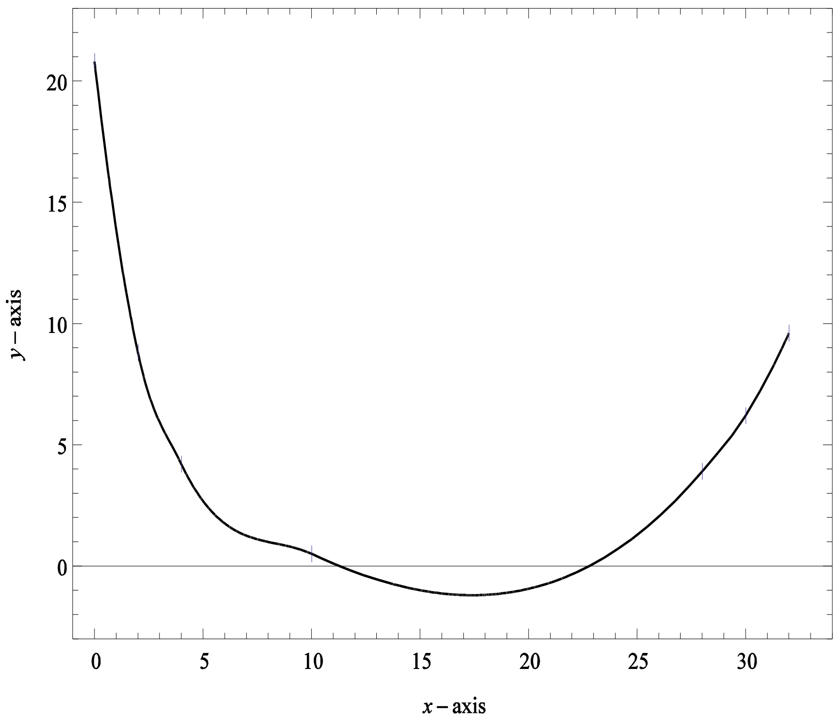
**Figure 3.** Comparison of (a) cubic bézier curve and positivity preserving using the proposed scheme (b) $\alpha_i = \beta_i = 0.1$ and (c) $\alpha_i = \beta_i = 1$ and (d) Sarfraz *et al*. [6] and (e) zoom in for (d).

Figure 3 shows the positivity preserving using the rational cubic Bézier interpolation. Figure 3(a) shows the cubic Bézier interpolation with $w_i = 1$ and without shape preserving property. Figures 3(b) and 3(c) show the positivity preserving interpolation using Condition (15) with $\lambda_i = 0.1$ and $\lambda_i = 0.5$ for all $i = 0,1,\ldots,5.$ We also have implemented the scheme of Sarfraz *et al.* [6]. From Figures 3(d) and 3(e), the scheme presented by [6] is not suitable for positivity preserving interpolation, since on the last interval, the rational interpolant $S_i(x)$ has negative value (below $x-$axis) that will destroy the characteristic of the positive data. It should be noted that, for some positive data sets, Sarfraz *et al.* [6] scheme might be able to produce positive interpolative curve (as shown in their paper). But we would like to stress that, the scheme by Sarfraz *et al.* [6] cannot preserves the positivity of the data sets on the whole interval (or for all types of positive data sets).

**Example 2**. A positive data from [1] is used.

**Table 2** A positive data from [1]

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $x_i$ | 0 | 2 | 4 | 10 | 28 | 30 | 32 |
| $f_i$ | 20.8 | 8.8 | 4.2 | 0.5 | 3.9 | 6.2 | 9.6 |
| $d_i$ | -7.85 | -4.15 | -1.8792 | -0.4153 | 1.0539 | 1.425 | 1.975 |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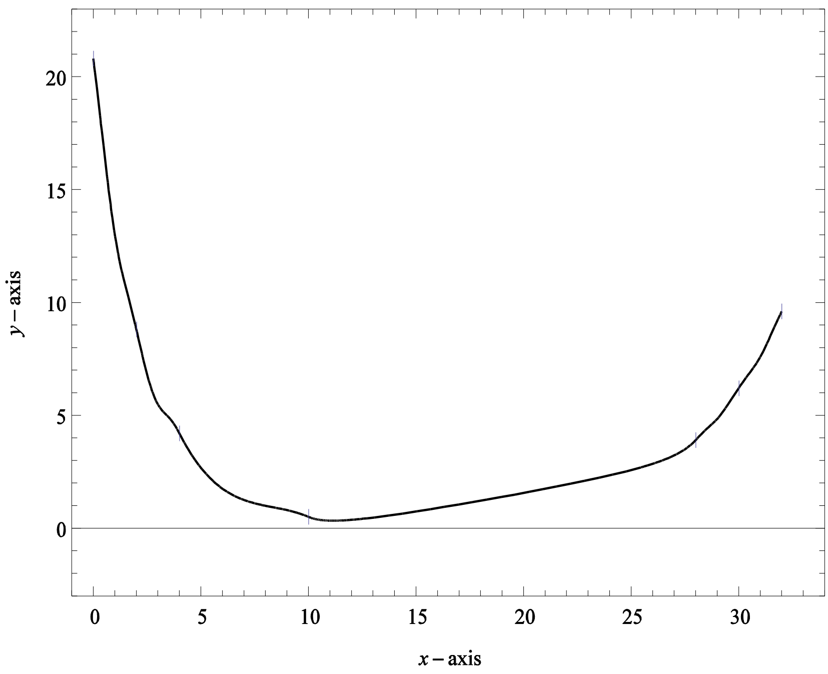
$x-$axis

(a) $w_i = 1$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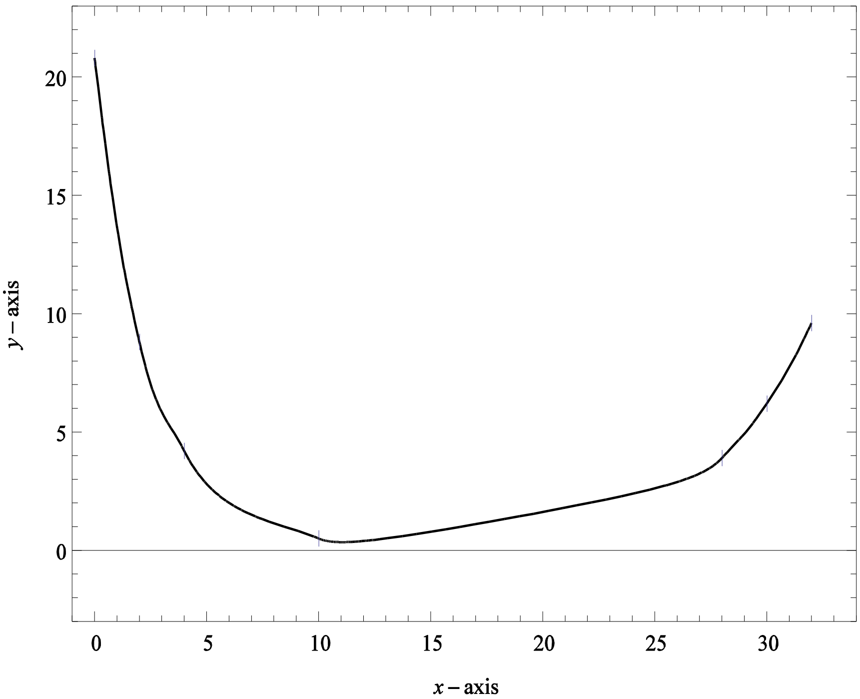
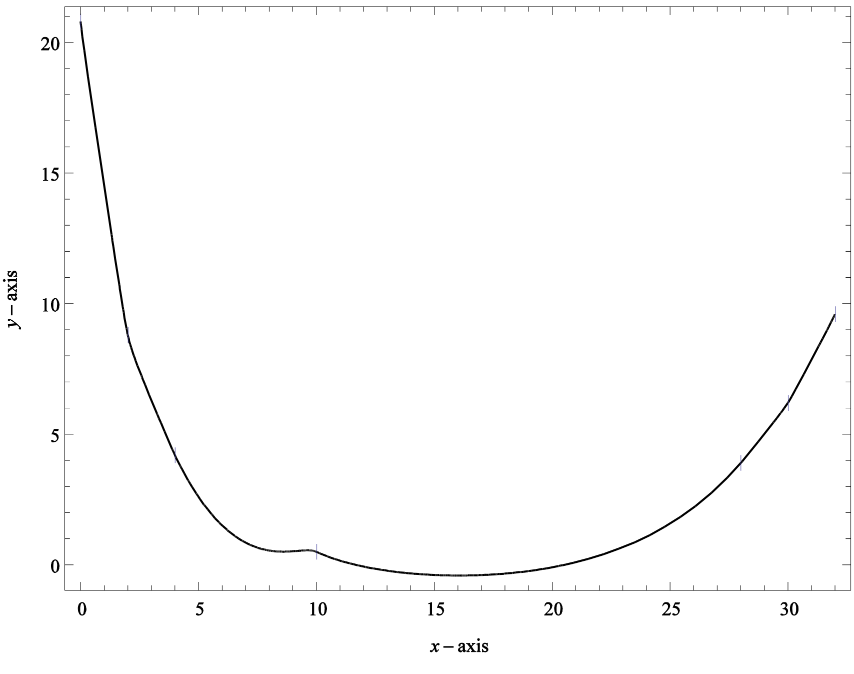
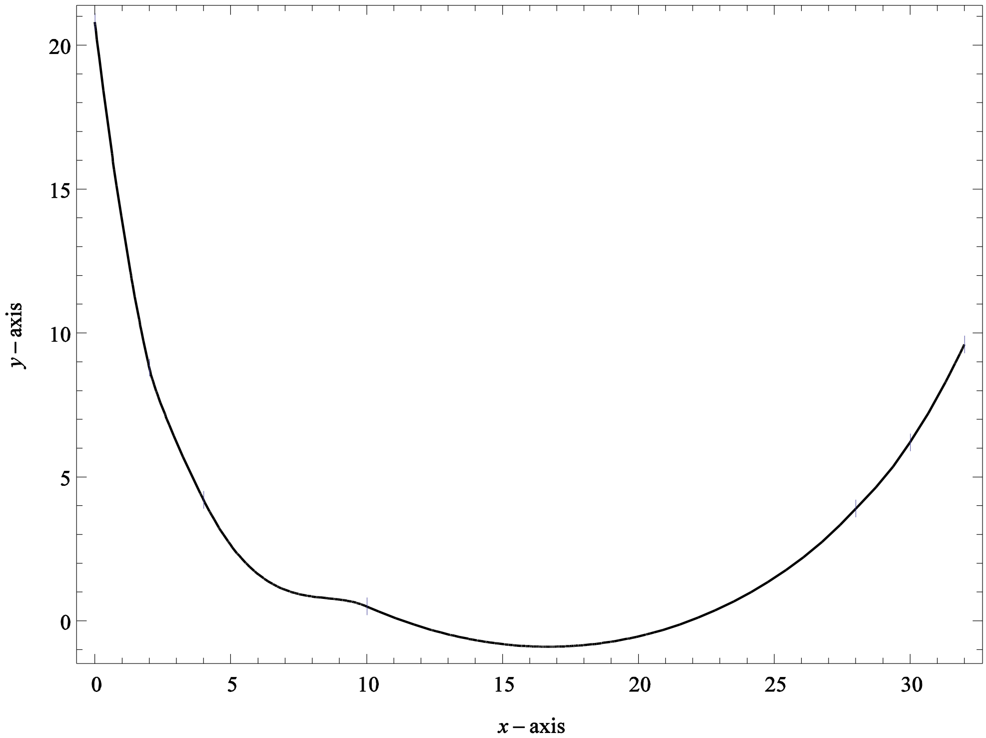
(b)




(c)




(d) Sarfraz *et al*. [6]




(e) Hussain *et al.* [5]

**Figure 4.** Various curve interpolation.

Figure 4 shows the interpolating curve for positive data in Table 2. Figure 3(a) shows that the cubic Bézier interpolation is not shape preserving. Meanwhile Figures 3(b) and 3(c) show the positivity preserving interpolation by applying positivity condition stated in (15) with $\lambda_i = 0.1$ and $\lambda_i = 0.5$ for all $i = 0,1,\ldots,5$, respectively. Finally Figures 3(d) and 3(e) show that, the schemes of [6] and [5] fails to preserve the positivity of the data. Thus the schemes by Hussain *et al.* [5] and Sarfraz *et al.* [6] are not suitable for positivity preserving interpolation since both schemes cannot produce positive interpolating curve everywhere. No proof was given to support this claim, but based on graphical examples, it can clearly be seen that, for data in Table 2, both schemes by Sarfraz *et al.* [6] and Hussain *et al.* [5] fails to preserve the positivity of the data.

## 4. PARTIALLY BLENDED RATIONAL BI-CUBIC BÉZIER

The piecewise rational cubic Bézier defined by (1) can be extended to bi-cubic partially blended rational function $S(x,y)$ over rectangular domain $\Omega = [a,b]\times[c,d]$. The partially blended rational bi-cubic function on each rectangular patch $[x_i, x_{i+1}]\times[y_j, y_{j+1}], i = 0,1,\ldots,n-1; j = 0,1,\ldots,m-1$ is defined as [2]:

$$S(x,y) = -AFB^T, \tag{16}$$

where,

$$F = \begin{pmatrix} 0 & S(x, y_j) & S(x, y_{j+1}) \\ S(x_i, y) & S(x_i, y_j) & S(x_i, y_{j+1}) \\ S(x_{i+1}, y) & S(x_{i+1}, y_j) & S(x_i, y_{j+1}) \end{pmatrix},$$

with

$$A = [-1 \quad a_0(\theta) \quad a_1(\theta)],\ B = [-1 \quad b_0(\phi) \quad b_1(\phi)],$$
$$a_0(\theta) = (1-\theta)^2(1+2\theta),\ \ a_1(\theta) = \theta^2(3-2\theta),$$
$$b_0(\phi) = (1-\phi)^2(1+2\phi),\ \ b_1(\phi) = \phi^2(3-2\phi),$$
$$\theta = \frac{x-x_i}{h_i},\ \ \phi = \frac{y-y_j}{\hat{h}_j},$$

and

$$h_i = x_{i+1} - x_i,\ \ \hat{h}_j = y_{j+1} - y_j.$$

Where the functions $S(x, y_j), S(x, y_{j+1}), S(x_i, y)$ and $S(x_{i+1}, y)$ are rational cubic Bézier interpolant defined by (1) on each boundary of the rectangular patch $[x_i, x_{i+1}] \times [y_j, y_{j+1}], i = 0,1,\ldots,n-1;\ j = 0,1,\ldots,m-1$. Their formula is given as:

$$S(x, y_j) = \frac{(1-\theta)^3 A_0 + 3w_{ij}(1-\theta)^2\theta A_1 + 3w_{ij}(1-\theta)\theta^2 A_2 + \theta^3 A_{i3}}{q_1(\theta)}, \tag{17}$$

with

$$A_0 = F_{i,j},$$
$$A_1 = F_{i,j} + \frac{h_i F^x_{i,j}}{3w_{i,j}},$$
$$A_2 = F_{i+1,j} - \frac{h_i F^x_{i+1,j}}{3w_{i,j}},$$
$$A_3 = F_{i+1,j},$$
$$q_1(\theta) = (1-\theta)^3 + 3w_{i,j}(1-\theta)^2\theta + 3w_{i,j}(1-\theta)\theta^2 + \theta^3$$
$$S(x, y_{j+1}) = \frac{(1-\theta)^3 B_0 + 3w_{i,j+1}(1-\theta)^2\theta B_1 + 3w_{i,j+1}(1-\theta)\theta^2 B_2 + \theta^3 B_3}{q_2(\theta)}, \tag{18}$$

with

$$B_0 = F_{i,j+1},$$
$$B_1 = F_{i,j+1} + \frac{h_i F^x_{i,j+1}}{3w_{i,j+1}},$$
$$B_2 = F_{i+1,j+1} - \frac{h_i F^x_{i+1,j+1}}{3w_{i,j+1}},$$
$$B_3 = F_{i+1,j+1},$$
$$q_2(\theta) = (1-\theta)^3 + 3w_{i,j+1}(1-\theta)^2\theta + 3w_{i,j+1}(1-\theta)\theta^2 + \theta^3$$

$$S(x_i, y) = \frac{(1-\phi)^3 C_0 + 3\hat{w}_{ij}(1-\phi)^2\phi C_1 + 3\hat{w}_{ij}(1-\phi)\phi^2 C_2 + \phi^3 C_3}{q_3(\phi)}, \tag{19}$$

with

$$C_0 = F_{i,j},$$
$$C_1 = F_{i,j} + \frac{\hat{h}_j F_{i,j}^y}{3\hat{w}_{i,j}},$$
$$C_2 = F_{i,j+1} - \frac{\hat{h}_j F_{i,j+1}^y}{3\hat{w}_{i,j}},$$
$$C_3 = F_{i,j+1},$$
$$q_3(\phi) = (1-\phi)^3 + 3\hat{w}_{i,j}(1-\phi)^2\phi + 3\hat{w}_{i,j}(1-\phi)\phi^2 + \phi^3$$
$$S(x_{i+1}, y) = \frac{(1-\phi)^3 D_0 + 3\hat{w}_{i+1,j}(1-\phi)^2\phi D_1 + 3\hat{w}_{i+1,j}(1-\phi)\phi^2 D_2 + \phi^3 D_3}{q_4(\phi)}, \quad (20)$$

with

$$D_0 = F_{i+1,j},$$
$$D_1 = F_{i+1,j} + \frac{\hat{h}_j F_{i+1,j}^y}{3\hat{w}_{i+1,j}},$$
$$D_2 = F_{i+1,j+1} - \frac{\hat{h}_j F_{i+1,j+1}^y}{3\hat{w}_{i+1,j}},$$
$$D_3 = F_{i+1,j+1},$$
$$q_4(\phi) = (1-\phi)^3 + 3\hat{w}_{i+1,j}(1-\phi)^2\phi + 3\hat{w}_{i+1,j}(1-\phi)\phi^2 + \phi^3$$

## 5. POSITIVITY PRESERVING FOR POSITIVE SURFACE DATA

In [2], the authors conclude that their partially blended rational bi-cubic Bézier is shape preserving if all four boundary curves also shape preserving (only for monotonic and convexity). In this section, we will show that the partially blended rational bi-cubic Bézier from [2] also can be used to produce positive interpolating surface provided that all four boundary curves are also positive. In order to derive the sufficient condition for the positivity of the rational bi-cubic Bézier, we begin with some basic definition.

Firstly, assume that the strictly positive data $(x_i, y_{i,} F_{i,j})$ is given over rectangular grid $[x_i, x_{i+1}] \times [y_j, y_{j+1}], i = 0,1,\ldots,n-1; j = 0,1,\ldots,m-1$ such that

$$F_{i,j} > 0, \quad \forall i,j.$$

The bi-cubic partially surface patch in (16) is positive if and only if all boundary curves given in Equations (17) until (20) is positive. Mathematically it can be represented in the form of inequality i.e. $S(x, y_j) > 0$, $S(x, y_{j+1}) > 0$, $S(x_i, y) > 0$ and $S(x_{i+1}, y) > 0$ respectively. By extending the idea from univariate case i.e. Theorem 1, the sufficient condition for the positivity of $S(x, y)$ can be obtained as described below:

We consider each of the boundary curves. Firstly $S(x, y_j) > 0,$ if $A_i > 0,\ i = 0,1,2,3.$ Then $S(x, y_j) > 0,$ if

$$w_{i,j} > -\frac{h_i F^x_{i,j}}{3F_{i,j}} \tag{22}$$

$$w_{i,j} > \frac{h_i F^x_{i+1,j}}{3F_{i+1,j}} \tag{23}$$

This conditions lead to:

$$w_{i,j} > \mathrm{Max}\left\{0, -\frac{h_i F^x_{i,j}}{3F_{i,j}}, \frac{h_i F^x_{i+1,j}}{3F_{i+1,j}}\right\} \tag{24}$$

It can be shown that the sufficient condition for the positivity of the other three boundary curves i.e. $S(x, y_{j+1}) > 0,\ S(x_i, y) > 0$ and $S(x_{i+1}, y) > 0$ are given as:

$$w_{i,j+1} > \mathrm{Max}\left\{0, -\frac{h_i F^x_{i,j+1}}{3F_{i,j+1}}, \frac{h_i F^x_{i+1,j+1}}{3F_{i+1,j+1}}\right\} \tag{25}$$

$$\hat{w}_{i,j} > \mathrm{Max}\left\{0, -\frac{\hat{h}_j F^y_{i,j}}{3F_{i,j}}, \frac{\hat{h}_j F^y_{i,j+1}}{3F_{i,j+1}}\right\} \tag{26}$$

$$\hat{w}_{i+1,j} > \mathrm{Max}\left\{0, -\frac{\hat{h}_j F^y_{i+1,j}}{3F_{i+1,j}}, \frac{\hat{h}_j F^y_{i+1,j+1}}{3F_{i+1,j+1}}\right\} \tag{27}$$

Combining (24), (25), (26) and (27), the sufficient condition for the positivity of the partially blended rational bi-cubic Bézier are.

$$\begin{aligned}
w_{i,j} &= k_{ij} + \mathrm{Max}\left\{0, -\frac{h_i F^x_{i,j}}{3F_{i,j}}, \frac{h_i F^x_{i+1,j}}{3F_{i+1,j}}\right\} \\
w_{i,j+1} &= l_{i,j} + \mathrm{Max}\left\{0, -\frac{h_i F^x_{i,j+1}}{3F_{i,j+1}}, \frac{h_i F^x_{i+1,j+1}}{3F_{i+1,j+1}}\right\} \\
\hat{w}_{i,j} &= m_{i,j} + \mathrm{Max}\left\{0, -\frac{\hat{h}_j F^y_{i,j}}{3F_{i,j}}, \frac{\hat{h}_j F^y_{i,j+1}}{3F_{i,j+1}}\right\} \\
\hat{w}_{i+1,j} &= n_{i,j} + \mathrm{Max}\left\{0, -\frac{\hat{h}_j F^y_{i+1,j}}{3F_{i+1,j}}, \frac{\hat{h}_j F^y_{i+1,j+1}}{3F_{i+1,j+1}}\right\}
\end{aligned} \tag{28}$$

with $k_{i,j} > 0,\quad l_{i,j} > 0,\quad m_{i,j} > 0,\quad n_{i,j} > 0.$

This result is summarized as Theorem 2.

**Theorem 2.** The partially blended rational bi-cubic Bézier function $S(x,y)$ in (16) over the rectangular mesh $[x_i, x_{i+1}]\times[y_j, y_{j+1}], i = 0,1,\ldots,n-1; j = 0,1,\ldots,m-1$ is positive if the parameter satisfies the sufficient condition given in (28).

**Theorem 3.** The positive interpolating surface obtained from Theorem 2 is $C^1$ continuity everywhere.

**Proof.** This result is directly from Casciola and Romani [2].

The following algorithm can be used for computer implementation.

### 5.1 Algorithm for Positivity Preserving using Rational Bi-Cubic Bézier Interpolant

The algorithm for positivity preserving using rational Bi-cubic Bézier is as follows:

**Input:** Positive surface data $(x_i, y_j, F_{i,j}),\ i = 0,1,\ldots,n;\ j = 0,1,\ldots,m.$

**Output:** Interpolating surface with positivity shape preserving.

**Step 1:** Estimate the first partial derivative values $F^x_{i,j}, F^y_{i,j}, F^{xy}_{i,j}$.

**Step 2:** Compute the values of parameters $w_{i,j},\ w_{i,j+1},\ \hat{w}_{i,j}, \hat{w}_{i+1,j}$ by using Condition (28).

**Step 3:** Construct the positive interpolating surface using (16).

Step 2 and 3 can be repeated for difference positive data sets.

### 5.2 Numerical Demonstrations

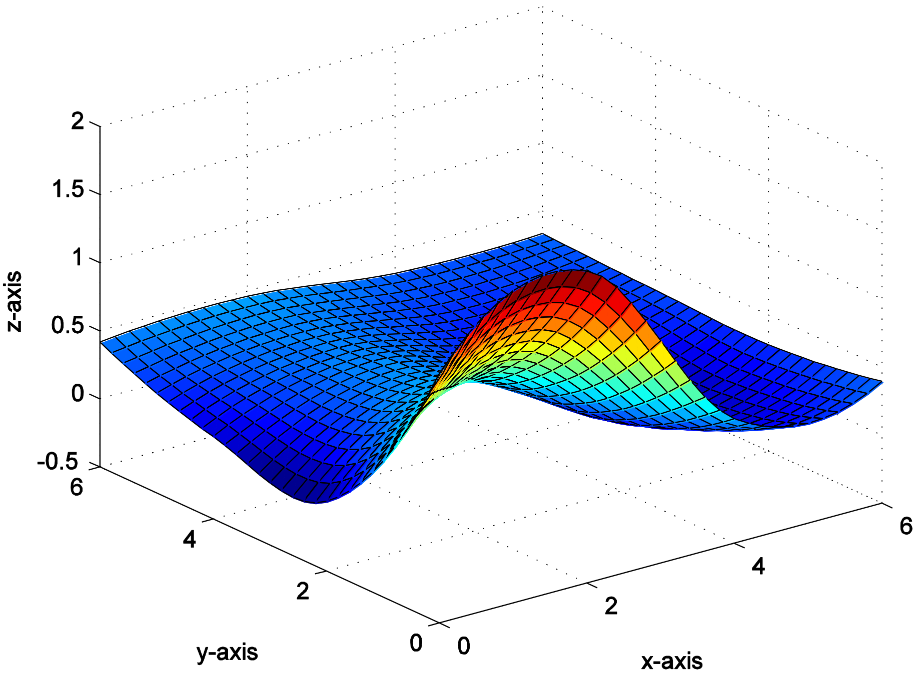
Two positive data sets are used to test the positivity preserving by using the partially blended rational bi-cubic Bézier interpolation.

**Example 3**. A positive data from the following function is truncated to five decimal places.

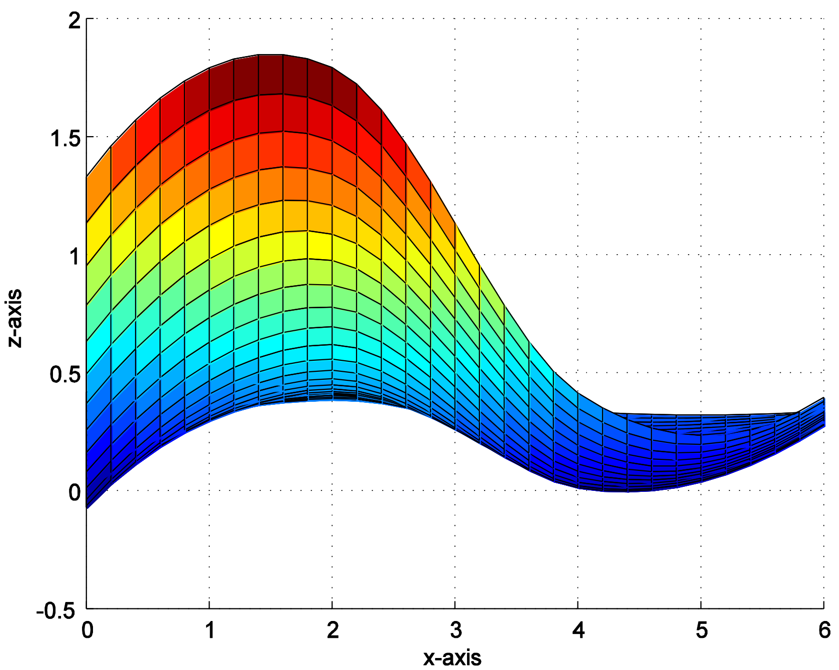
$$F_1(x,y) = \sin\left(ye^{-x}\right) + 1, \quad -3 \le x, y \le 3, \quad x, y \ne 0 \tag{29}$$

**Table 3** Positive surface data from function $F_1(x,y)$

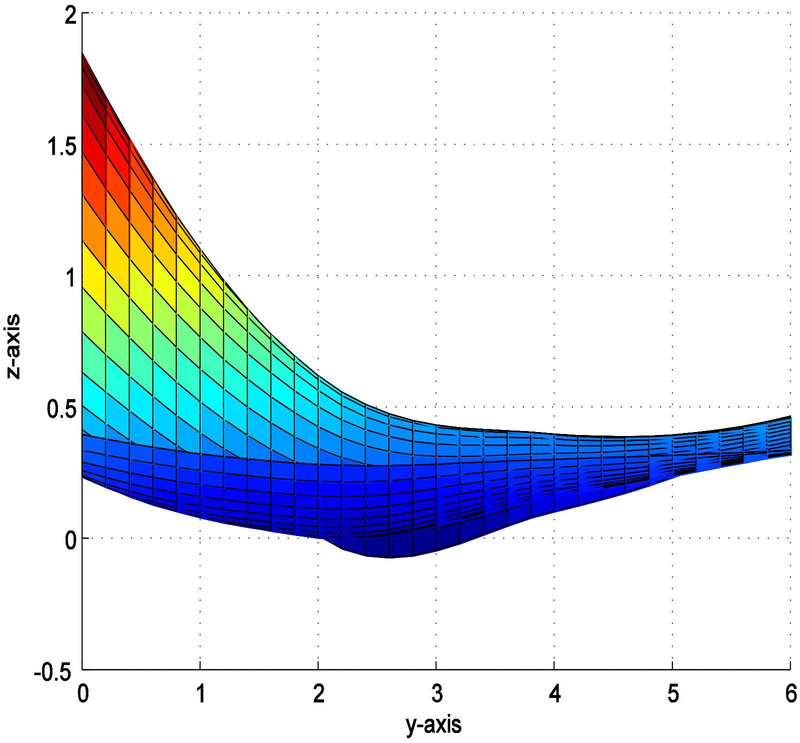
| $y/x$ | -3 | -2 | -1 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|
| -3 | 1.536600 | 0.37930 | 0.055529 | 1.9445 | 1.62070 | 0.46344 |
| -2 | 1.175100 | 0.19847 | 0.106150 | 1.8939 | 1.80150 | 0.82489 |
| -1 | 0.044919 | 1.74900 | 0.589220 | 1.4108 | 0.25095 | 1.95510 |
| 1 | 0.107150 | 0.32885 | 0.640360 | 1.3596 | 1.67110 | 1.89290 |
| 2 | 0.605060 | 0.73262 | 0.865080 | 1.1349 | 1.26740 | 1.39490 |
| 3 | 0.851190 | 0.90059 | 0.950230 | 1.0498 | 1.09940 | 1.14880 |


2
1.5
1
0.5
0
-0.5
z-axis
6
4
2
0
y-axis
0
2
4
6
x-axis

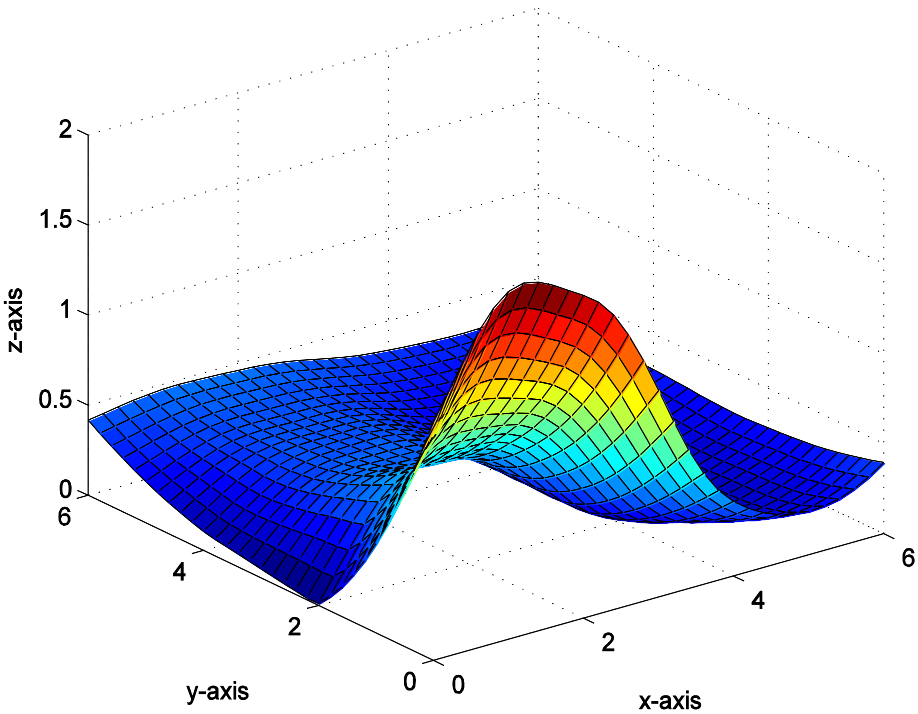
(a)


2
1.5
1
0.5
0
-0.5
z-axis
0
1
2
3
4
5
6
x-axis

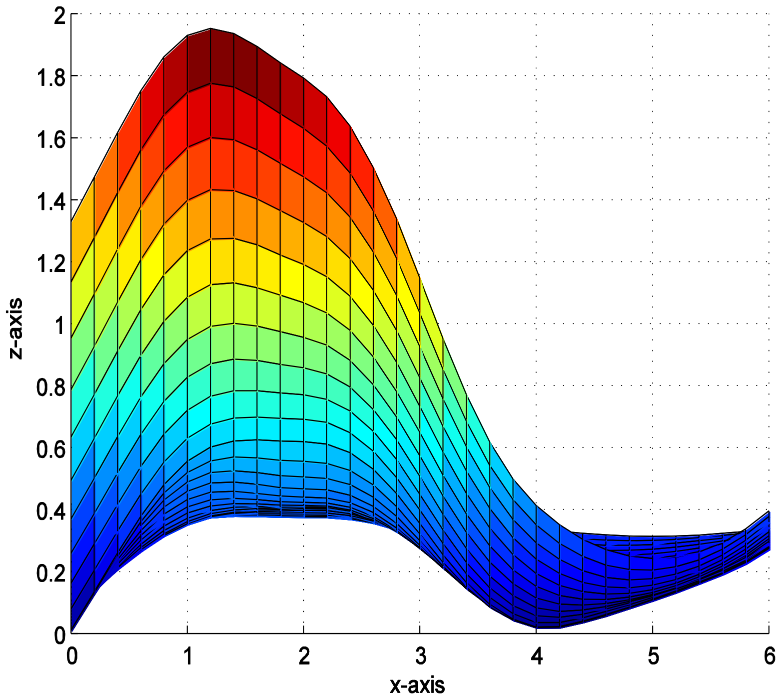
(b) $xz$ – view 4(a)


2
1.5
1
0.5
0
-0.5
z-axis
0
1
2
3
4
5
6
y-axis

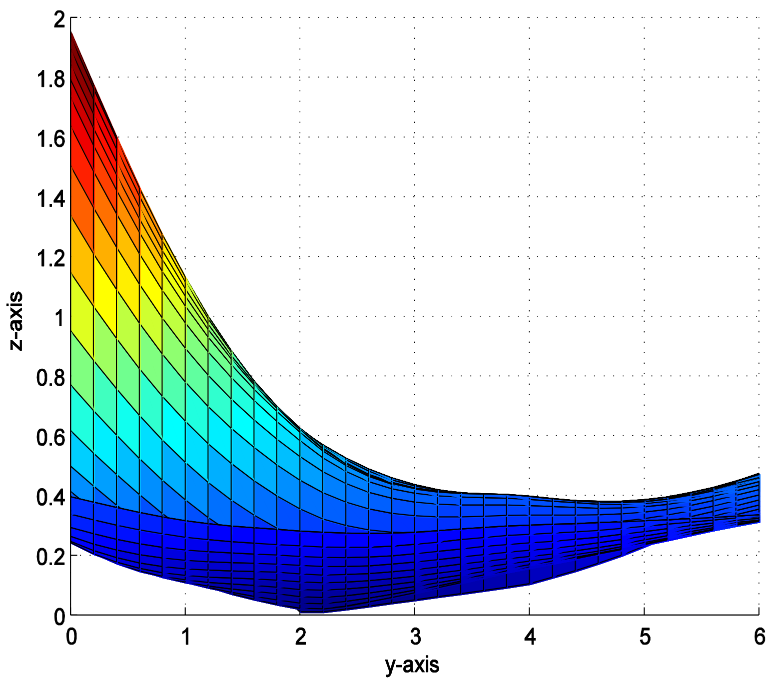
(c) $yz$ – view 4(a)


2
1.5
1
0.5
0
z-axis
6
4
2
0
y-axis
0
2
4
6
x-axis

(d)




(e) $xz$ – view 4(d)

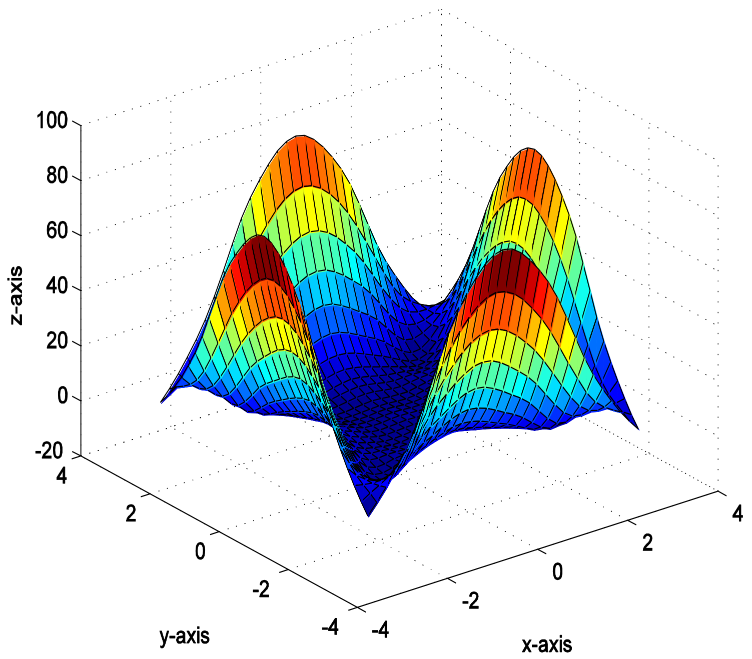



(f) $yz$ – view 4(d)

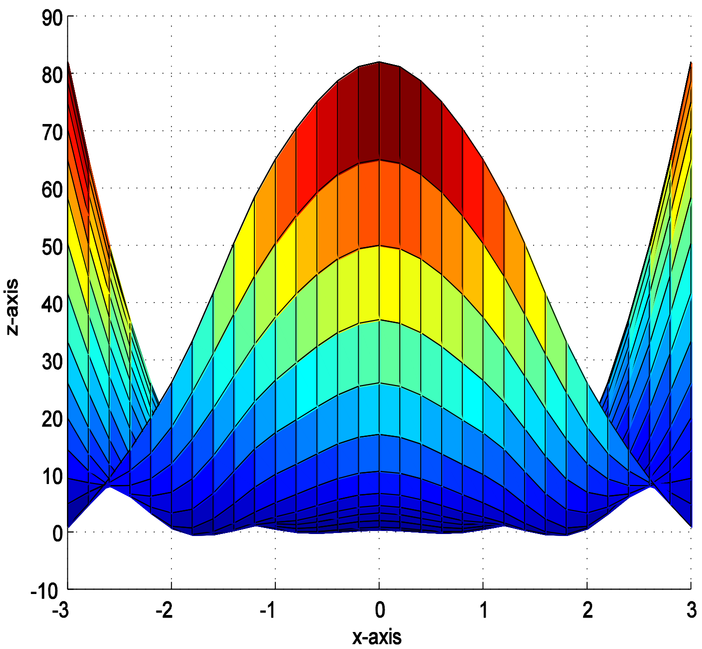
**Figure 5.** Surface interpolation for data in table 2.

Figure 5(a) shows the surface interpolation by using with $w_{i,j} = \hat{w}_{i,j} = 1$ and $w_{i,j+1} = \hat{w}_{i+1,j} = 1$. This is exactly the bi-cubic Bézier interpolation. By visualize on $xz-$ and $yz-$view (shown in Figures 5(b) and 5(c)), the bi-cubic Bézier fail to preserve the positivity of the surface data. Applying the positivity condition in (28), the rational bi-cubic Bézier interpolation is able to preserve the positivity of the data. This statement can be verified through Figures 5(e) and 5(f), respectively.
Example 4. A positive data from the following function [6]

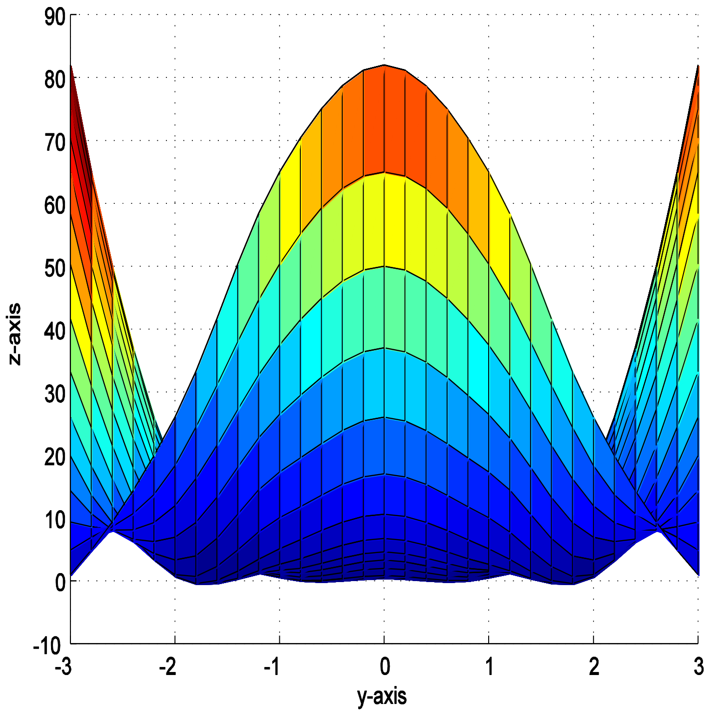
$$F_2(x,y) = \left(\left(x^2 - y^2\right)\right)^2 + 1, \quad -3 \le x, y \le 3 \tag{30}$$



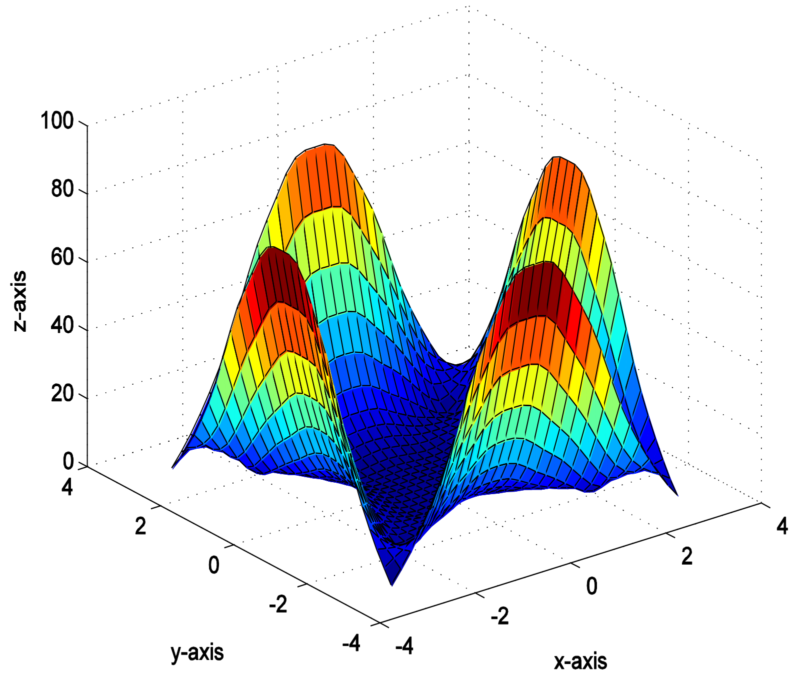

(a)



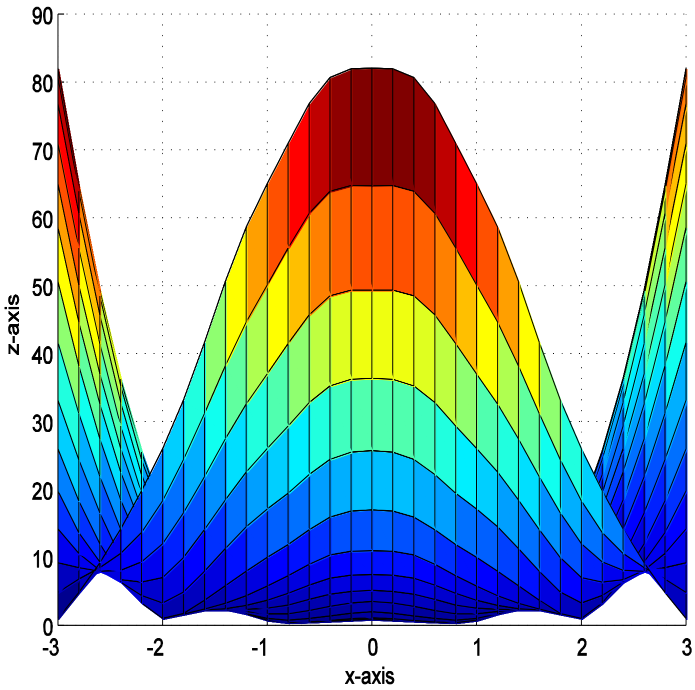

(b) $xz-$view for 5(a)



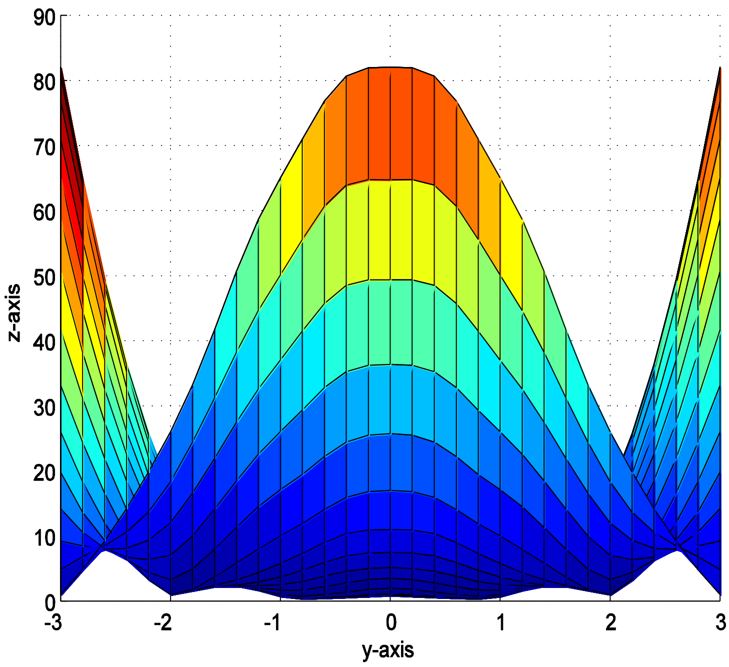

(c) $yz$ – view for 5(a)




(d)




(e) $xz$ – view for 5(d)




(f) $yz$ – view for 5(d)

**Figure 6.** Surface interpolation for function $F_2(x, y)$.

Figure 6(a) shows the surface interpolation using bi-cubic Bézier with $w_{i,j} = \hat{w}_{i,j} = 1$ and $w_{i,j+1} = \hat{w}_{i+1,j} = 1$ . Once again, bi-cubic Bézier fails to preserve the positivity of the data. There are points on the surface that lie below $xy$ plane i.e. Figures 6(b) and 6(c). Now by applying the sufficient condition for positivity, the partially blended rational bi-cubic Bézier preserve the positivity of the data. Figures 6(e) and 6(f) shows that all points on the surface are lies above the $xy$ plane i.e. the surface is positive everywhere including inside the domain.

To measure the effectiveness of the partially blended rational bi-cubic Bézier interpolation [2], we calculate the root mean square error (RMSE) and the total number of point lies below $xy$ plane. Tables 4 summarize the results.

**Table 4** Rmse estimation and total number of points lies below $xy$ plane

| **Scheme** | **RMSE** | | **Number of points below $xy$ plane** | |
|---|---|---|---|---|
| | **Example 2** | **Example 3** | **Example 2** | **Example 3** |
| **Without positivity preserving** | 0.0222 | 4.4359 | 7 | 12 |
| **With positivity preserving** | 0.0198 | 4.7377 | 0 | 0 |

From Table 4, the surface interpolation without using positivity preserving condition have some negative values, for instance, in Examples 3 and 4, there are 7 and 12 points lies below $xy$ plane respectively. In terms of RMSE, Example 2 give smaller value compare with Example 3. Since the RMSE value is totally depend to the data, then if we have a larger value in the data points, RMSE value also will be large and vice versa. We can conclude that the partially blended rational bi-cubic Bézier initiated by [2] can be used for positivity preserving interpolation by applying Condition (28) given in this study.

Remark: We can calculate the value of RMSE since we sample the data from true function. Thus at the interpolating points, there will be no error, but along the whole domain i.e. the entire true surface, there will be error since we only obtain the approximate value except at the sample points i. e. interpolating points. We also calculate the number of point lies below $xy$ plane to show that, the proposed shape scheme is able to produce positive surface everywhere. Since for some schemes, there is possibility that, some points maybe lie below $xy$ plane as happen in positivity preserving for scattered data interpolation.

## 6. SUMMARY AND CONCLUSIONS

This paper has proved that the partially blended rational bi-cubic Bézier interpolant initiated by [2] can be used for positivity preserving both for curve and surface interpolation. The sufficient condition for the positivity of all four boundary curves will ensure that the resulting surface is positive everywhere as stated in Theorem 2. Future research will be the application of the rational cubic Bézier interpolation for parametric data. Besides that, it was shown that the schemes of Hussain *et al*. [5] and Sarfraz *et al*. [6] produce some negative values on some sub-interval for positive data. This show that both schemes are not positivity preserving interpolation. In contrast, the rational cubic Bézier interpolation in this study, is guarantee to produce positive interpolating curve everywhere by applying Theorem 1. Future studies will be the application of the rational bi-cubic Bézier for image processing such as image zooming and image refinement.

### 6.1 Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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