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Comparison of The Trapezoidal and Adam Bashforth Approaches in The Lotka-Volterra Prey-Predator Dynamics

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ABSTRACT

This study primarily focuses on comparing the numerical methods of the Adams-Bashforth and Trapezoidal methods with the exact solution for solving the Lotka-Volterra prey-predator model. These methods are evaluated for their ability to reliably and accurately solve the non-linearity of the model. Based on the results, both methods offer precise solutions, with the Adams-Bashforth method providing a more accurate approximation for short-term predictions and the Trapezoidal method demonstrating better stability for long-term simulations. The study utilizes data from lynx-rabbit and bat-moth interactions to assess the performance of these methods using software tools. For both models, the short-term predictions align closely with observed data, while long-term stability analyses reveal the strengths of the Trapezoidal method. The equilibrium and stability analyses offer critical insights into the long-term behavior and stability of the system. The predator population trails behind the prey population: a rise in prey numbers is followed by a delayed increase in predator numbers as predators consume more prey. The phase portraits show the regularity of these oscillations. The curves move counterclockwise: prey numbers increase when predator numbers are at their lowest, and prey numbers decrease at their highest. These insights are essential for understanding and predicting the dynamics of predator-prey interactions and have significant implications for ecological modeling and conservation strategies.

Keywords: Adams-Bashforth method, ecological modeling, Lotka-Volterra model, numerical methods, stability, Trapezoidal method

1 INTRODUCTION

The study applies and compares two numerical methods, the Adams-Bashforth and Trapezoidal methods, to solve the Lotka-Volterra prey-predator model. This model is essential for understanding the dynamic interactions between predators and their prey, with specific examples drawn from lynx-rabbit and bat-moth interactions. The goal is to comprehend ecosystem stability and the effects of species extinction. The primary challenge is identifying which numerical method, Adams-Bashforth or Trapezoidal, provides the best solution for the Lotka-Volterra model. This includes understanding the behavior and stability of predator-prey interactions in specific species pairs, such as Iberian lynxes, snowshoe hares, and bats and moths. The research objectives are threefold: first, to compare the exact solutions with the Adams-Bashforth and Trapezoidal methods for the Lotka-Volterra prey-predator model; second, to discuss the impact of carrying capacity on the dynamic behaviors of prey-predator interactions between species; and third, to analyze the equilibrium and stability of these interactions. This research holds significance for ecological studies and conservation efforts, offering insights into the dynamics between predator and prey species. Accurate models of these interactions are crucial for making effective conservation decisions and understanding the stability of ecosystems.

Numerical methods involve algorithms that provide approximate solutions to mathematical problems instead of exact solutions derived through symbolic manipulation [1]. These methods are helpful when analytical solutions are difficult or impossible to obtain. Numerical methods are commonly applied to solve linear systems and differential equations and to calculate function derivatives [2]. They help researchers tackle complex mathematical problems across various fields where finding precise solutions is challenging.

Studying the extinction dynamics between predator and prey species, such as the lynx-rabbit and bat-moth interactions, is an intriguing area within ecology and evolutionary biology. Understanding these linkages requires understanding the dynamics of ecosystems, the preservation of biodiversity, and the possible ramifications of species extinctions.

The loss of predator or prey species may majorly affect ecology. In the case of the lynx-rabbit system, the system's collapse would likely result from the extinction of either species. This is because lynxes and rabbits are essential components of the ecosystem, and their extinction would have a detrimental effect on numerous other species. If moths or bats went extinct, it would significantly affect the ecosystem. Losing bats would be harmful to plant reproduction because they are essential pollinators. Nonetheless, many animals rely on moths for sustenance, so their extinction can throw off the balance in the food chain. Understanding the intricacies of these predator-prey interactions could have broader implications for ecological management and preservation. Changes in the population of one species can have a cascading effect on the environment. For example, a decline in bat populations may increase moth populations, affecting other parts of the ecosystem, such as the vegetation. Studying the extinction dynamics in interactions between lynx-rabbit and bat-moth can help us understand the intricate structure of predator-prey relationships and their impact on ecosystem stability.

As demonstrated in recent research on the extinction of wild species, there are few more remarkable instances of two species engaged in a fierce battle for existence than bats and moths. These two creatures represent the epitome of an evolutionary arms race, with one attempting to outperform the other in the fight for survival between predator and prey. This capacity has dramatically increased the strain on the insects that bats prey on, including grasshoppers, lacewings, and crickets. To help them survive, many of these insects have developed an astounding variety of counter-adaptations. And these are nowhere as obvious as they are in months [3]. The lynx population has grown significantly in Andalucía, Spain, and new populations have appeared in Extremadura, Castilla-La Mancha, and Portugal. This has brought economic and ecological benefits to local communities. However, in the 20th century, hunting, agriculture, and industry reduced the lynx population by about 90% and destroyed much of its habitat. By the early 2000s, the lynx was the world's most endangered cat. In 1953, a law by Spanish dictator Francisco Franco led to the killing of thousands of lynxes and their main food, rabbits. Rabbit diseases in the 1950s and 1980s further reduced their food supply. Between 1985 and 2001, the lynx's range shrank by 87%, and the number of breeding females dropped by over 90%. By 2002, only two small populations with 25 breeding females remained. Rabbits make up 90% of the lynx's diet, with males eating one rabbit daily and mothers with kittens needing up to three rabbits daily [4]. Moreover, At the record age of 20, Aura, an Iberian lynx that helped save her species from extinction and whose genes are still present in over 900 spotted and tufty-eared felines, passed away in southern Spain [5].

In the previous study on the Lotka-Volterra Prey-Predator Model using other numerical methods, [6] proposed an ecological model featuring two prey species with different reproduction rates and nutritional values, consumed by generalist and specialist predators. Numerical simulations revealed that specialists and generalists can coexist only if the specialist consumes the more nutritious and reproductive prey. The study suggests that the generalist's ability to switch prey relieves pressure on the limited prey species. Additionally, the generalist can always survive in the environment due to access to external food sources, offering new insights for specialized empirical studies. [7] presented their work on the predator-prey relationship, which is based on biological evolution and eco-epidemiology, characterized by the prey's induced defenses and diseases. Diseases' impact on population stability in the predator-prey relationship in which inducible defense is present is studied. To improve the completeness of analytical research, numerical simulations are carried out. The system's solutions' consistent boundedness and positivity are confirmed by research. Next, the stability and existence of the equilibria are examined. The system has a maximum of nine equilibrium positions. Through studies, researchers investigate inducible defenses and prey sickness, which gives rise to concepts for building mathematical models. Researchers can create more realistic models by altering the defensive trait set, the infectious disease transmission mechanism, and the functional response functions based on natural biological cases. Moreover, [8] addresses the issue of a continuous time delay in a stochastic predator-prey scenario. Methods of Lyapunov analysis are applied in the study. Numerical simulations for a fictitious set of parameter values are provided to support the analytical results. The circumstances leading to the demise of predator populations have been discovered via research. Under certain limitations on noise densities, it is shown that the behavior of the stochastic solutions is like the behavior of the corresponding deterministic delayed predator-prey system. Also, [9] propose a high-order precise method with Neumann boundary conditions to solve the one-dimensional Lotka-Volterra-diffusion problem. An implicit-explicit Runge Kutta scheme should be combined with a fourth-order compact finite difference approach for the spatial component. The basic concept is to employ an implicit-explicit Runge Kutta method of temporal integration and a fourth order near-finite difference scheme to discretize the spatial derivative. This produces a nonlinear system of ordinary differential equations. As a result, the computational cost of the system decreases. The suggested method's usefulness in stability and computation cost is demonstrated. Matlab programming is used in numerical experiments to further validate the proposed method's effectiveness and validity in solving the Lotka-Volterra-diffusion problem.

This previous study on the Lotka-Volterra Competitive Model using numerical methods shows that the study investigates equilibrium and stability in species interactions, focusing on whether species experience competitive exclusion or reach a stable equilibrium. The Runge-Kutta-Fehlberg (RKF) and the Taylor Series methods were used for analysis. The RKF method provided a more accurate approximation than the Taylor Series method, as confirmed by computations using Mathematica 13.2. Due to its accuracy and versatility, the research suggests the RKF method is preferable for solving the Lotka-Volterra competitive model [10]. [11] compare the population model's dynamics with those generated using dependable numerical methods and a precise solution. The study described calculating the numerical solutions to the Lotka-Volterra and the insect population models. The researchers used numerical methods to solve the models in this study, including the RKF method and the Laplace Adomian Decomposition Method (LADM). The researchers employed the intraspecific competition term and the Holling type III functional response to solve modified population models numerically. Compared to the LADM solution, numerical solutions produced using the RKF approach exhibit high accuracy. These numerical findings demonstrate that the RKF approach is an adequate and reliable method for solving linear and nonlinear differential equation models based on population models.

This previous study on the Lotka-Volterra Prey-Predator Model using the Adam Bashforth method [12] developed a predator-prey model using time fractional variable and constant orders, solved numerically with the Adams–Bashforth–Moulton method. They focused on the Liouville–Caputo fractional order and employed the Atangana–Baleanu operator, utilizing a kernel based on the generalized Mittag-Leffler function to model complex predator-prey dynamics. Their findings showed that the variable-order fractional approach better captured the intricate behavior of the system. The model’s results were generated using Matlab on a computer with an Intel Core i7, 2.6 GHz CPU, and 16 GB RAM. [13] conducted a theoretical and numerical study on the fractional Atangana–Baleanu–Caputo prey-predator model. Using the Adams-Bashforth method, they created graphical simulations to analyze the model’s behavior. The study applied nonlinear analysis and fixed-point theory to achieve results on Ulam-Hyers' stability and existence. A fractional Adams-Bashforth method was used to approximate solutions, with specific parameter values illustrating the model's dynamics. The findings contribute to the understanding of fractional dynamics in real-world phenomena.

A previous study on the trapezoidal method of another mathematical model showed that the study discusses various strategies for applying the trapezoidal method to fractional differential equations (FDEs) and multistep methods. Numerical experiments highlight the strengths and limitations of these methods. Several implicit second-order methods for FDEs were explored, focusing on their stability, which varies with the fractional order. The Backward Differentiation Formula (BDF)-based method showed the largest stability regions, similar to the results in ordinary differential equations (ODEs) [14].

Lastly, this previous study on Lotka-Volterra Prey-Predator Model using Analytical Solution Method – Exact solution showed that [15] explored a fractional-order delay differential predator-prey model with Holling-type III infection in the predator population. They analyzed the system using stability criteria, Lyapunov functional, and Laplace transform. Their study found that when time delays exceed critical values, the model undergoes Hopf bifurcation, affecting stability. While fractional order enhances the dynamics, temporal delays significantly impact stability. Critical conditions were identified to ensure local asymptotic stability, with Hopf bifurcation occurring when delays cross crucial thresholds.

2 MATERIAL AND METHODS

2.1 Exact Solution

The logistic equation of the Lotka-Volterra model is used to simulate the development of an isolated population:

$$\frac{dy}{dx} = ry\left(1 - \frac{y}{K}\right) \tag{1}$$

where *y(x)* is the average density (in persons) at time *x* (in generations), *r* is the instantaneous rate of growth (births/deaths), and K is the carrying capacity. Assume continuous linear density dependence (K and r), no time delays, migration, age structure, or restricted resources. Now, a straightforward iterative method for calculating K and r is demonstrated for both lynx-rabbit and bat-moth. Simplify the equation (1) to obtain the solution for the initial condition, which gives:

$$y = \frac{K}{1+\frac{1}{2}e^{-rx}(K-2)} \tag{2}$$

First, the good fit value of r and K is obtained by using curve fitting using equation (2); the fit is reasonable for:

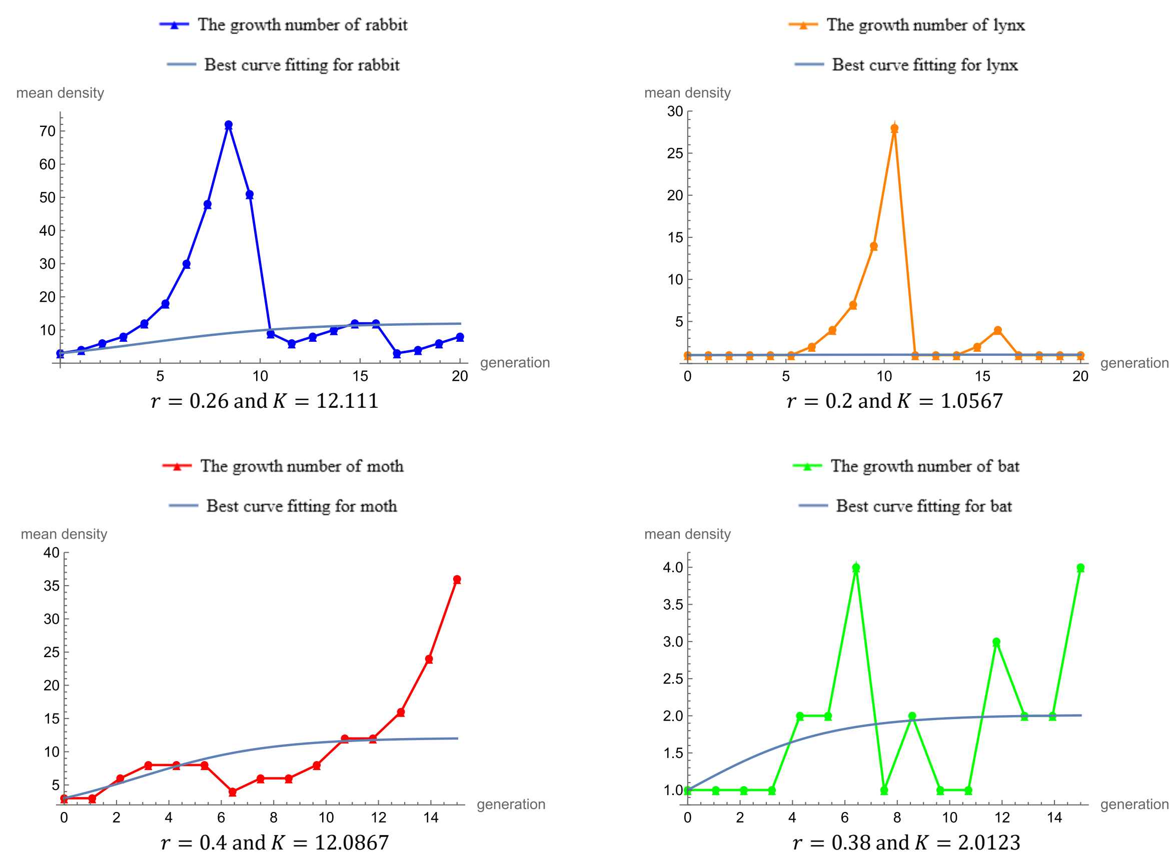

Figure 1: Curve fitting for growth in isolation

Figure 1 shows that the least-squares method fits the data points to the curve. This shows the population trends for two pairs of prey and predators across several generations: rabbits with lynxes and moths with bats. Each graph illustrates how the population size (mean density) changes over time, with fitted curves showing the overall trend. For the rabbits, which have a growth rate (r) of 0.26 and a carrying capacity (K) of 12.111, the population experiences a rapid increase followed by a decline. This pattern is typical for prey species, which grow quickly until predator pressure causes numbers to drop. The lynxes, with $(r = 0.2)$ and $(K = 1.0567)$, show a delayed population increase after the rabbits, reflecting how predator numbers depend on prey availability. In the graph moths, with a growth rate of $(r = 0.4)$ and a carrying capacity of $(K = 12.0867)$, gradually increase in population. Bats, which prey on moths, have a growth rate of $(r = 0.38)$ and a carrying capacity of $(K = 2.0123)$. Their population exhibits noticeable fluctuations, a hallmark of predator-prey cycles, where predator numbers rise and fall in response to prey abundance. Overall, these graphs illustrate the interactions between predators and prey, showing how growth rates and carrying capacities influence population stability and the cyclical nature of their dynamics.

2.2 Trapezoidal Method

The Trapezoidal method is an implicit numerical method for solving ordinary differential equations. It is a second-order method that provides more accurate approximation than first-order methods such as Euler's method. The Trapezoidal method averages the values of the function at the current and next time steps to compute the solution.

For Lotka-Volterra equation:

$$\frac{dx}{dt} = \alpha x - \beta xy \tag{3}$$

$$\frac{dy}{dt} = \delta xy - \gamma y \tag{4}$$

the Trapezoidal method can be written as:

$$x_{n+1} = x_n + \frac{h}{2}[\, f(x_n, y_n) + f(x_{n+1}, y_{n+1})] \tag{5}$$

$$y_{n+1} = y_n + \frac{h}{2}[\, g(x_n, y_n) + g(x_{n+1}, y_{n+1})] \tag{6}$$

where $f(x_n, y_n) = (\alpha x - \beta xy)$, $g(x_n, y_n) = (\delta xy - \gamma y)$, h is the time step size, and x_n and y_n are the populations of prey and predators at the n-th time step, respectively. Solving these equations involves iterating to find x_{n+1} and y_{n+1}, as the method is implicit.

To use the Trapezoidal method, the step size, $h = 1$.

$$k_1 = f[x, y] \tag{7}$$

$$k_2 = f[x, k_1] \tag{8}$$

To find $y_1, y_0 = y$ used, the calculation is:

$$y_{d+1} = y_d + \frac{1}{2}k_1 h + \frac{1}{2}k_2 h \tag{9}$$

2.3 Adam Bashforth Method

The Adams-Bashforth method is an explicit multistep method used for solving ordinary differential equations. It is known for its efficiency and accuracy, particularly when dealing with systems of equations. The Adams-Bashforth method uses information from previous time steps to predict the solution at the next time step.

For the Lotka-Volterra equation in equations (3) and (4) the Adam Bashforth method can be written as:

$$x_{n+1} = x_n + \frac{h}{2}[\,3f(x_n, y_n) - f(x_{n-1}, y_{n-1})] \tag{10}$$

$$y_{n+1} = y_n + \frac{h}{2}[\,3g(x_n, y_n) - g(x_{n-1}, y_{n-1})] \tag{11}$$

where $f(x_n, y_n) = (\alpha x - \beta xy)$, $g(x_n, y_n) = (\delta xy - \gamma y)$, h is the time step size, and x_n and y_n are the populations of prey and predators at the n-th time step, respectively. This method uses the current and previous values of the functions to predict the next value.

For the Adam Bashforth method, the step size, $h = 1$.

$$k_1 = f[x, y] \tag{12}$$

$$k_2 = f[x, k_1] \tag{13}$$

To find $y_2, y_1 = y$ used, the calculation is:

$$y_{d+1} = y_d + \frac{3}{2}k_1 h - \frac{1}{2}k_2 h \tag{14}$$

3 RESULTS AND DISCUSSION

This research compares numerical methods, such as the trapezoidal and Adam Bashforth methods, with the exact solution. Tables 1, 2, 3, and 4 show a comparison of Mean Absolute Error (MAE) and Root Mean Square Error (RMSE) among the Trapezoidal method and Adam Bashforth method compared to the exact solution for the rabbit, lynx, moth, and bat.

Table 1: The Mean Absolute Error (MAE) and Root Mean Square Error (RMSE) for rabbit

Error	Trapezoidal method	Adam Bashforth method
MAE	1.63282959	0.580709946
RMSE	1.744642402	0.656946655

Table 2: The Mean Absolute Error (MAE) and Root Mean Square Error (RMSE) for lynx

Error	Trapezoidal method	Adam Bashforth method
MAE	0.006846623	0.00379602
RMSE	0.007145827	0.004385323

Table 3: The Mean Absolute Error (MAE) and Root Mean Square Error (RMSE) for moth

Error	Trapezoidal method	Adam Bashforth method
MAE	1.188548068	0.476513744
RMSE	1.315281789	0.554266522

Table 4: The Mean Absolute Error (MAE) and Root Mean Square Error (RMSE) for bat

Error	Trapezoidal method	Adam Bashforth method
MAE	0.093013576	0.055706854
RMSE	0.104608473	0.07270618

Tables 1, 2, 3, and 4 present a comparison of the accuracy between two numerical methods, the Trapezoidal and Adams Bashforth methods, using the Mean Absolute Error (MAE) and Root Mean Square Error (RMSE) to evaluate the performance in predicting population dynamics for rabbit, lynx, moth, and bat. These metrics help assess how closely the predicted values match the observed data, with lower values indicating more accurate predictions. The Adams-Bashforth method is much more accurate for rabbits than the Trapezoidal method, as evidenced by its significantly lower MAE (0.5807 compared to 1.6328) and RMSE (0.6569 compared to 1.7446). This indicates that the Adams-Bashforth method predicts rabbit population trends more precisely. Similarly, the Adams-Bashforth method again shows superior accuracy for lynx, with much lower MAE (0.0038 compared to 0.0068) and RMSE (0.0044 compared to 0.0071), suggesting it better captures the lynx population dynamics. The pattern continues with moth, where the Adams-Bashforth method has a lower MAE (0.4765 compared to 1.1885) and RMSE (0.5543 compared to 1.3153), indicating more reliable predictions. Lastly, the Adams-Bashforth method performs better for bats, with lower MAE (0.0557 compared to

0.0930) and RMSE (0.0727 compared to 0.1046). Significant errors, like those exhibited by the Trapezoidal method, could lead to misinterpretations, potentially impacting decision-making or interventions. These tables show that the Adams-Bashforth method consistently provides more accurate predictions for the population dynamics of all four species than the Trapezoidal method.

3.1 Lotka-Volterra Prey-Predator Equation

The Lotka-Volterra predator-prey model is a well-recognized system of differential equations that describes the behavior of prey and predator populations as they change over time.

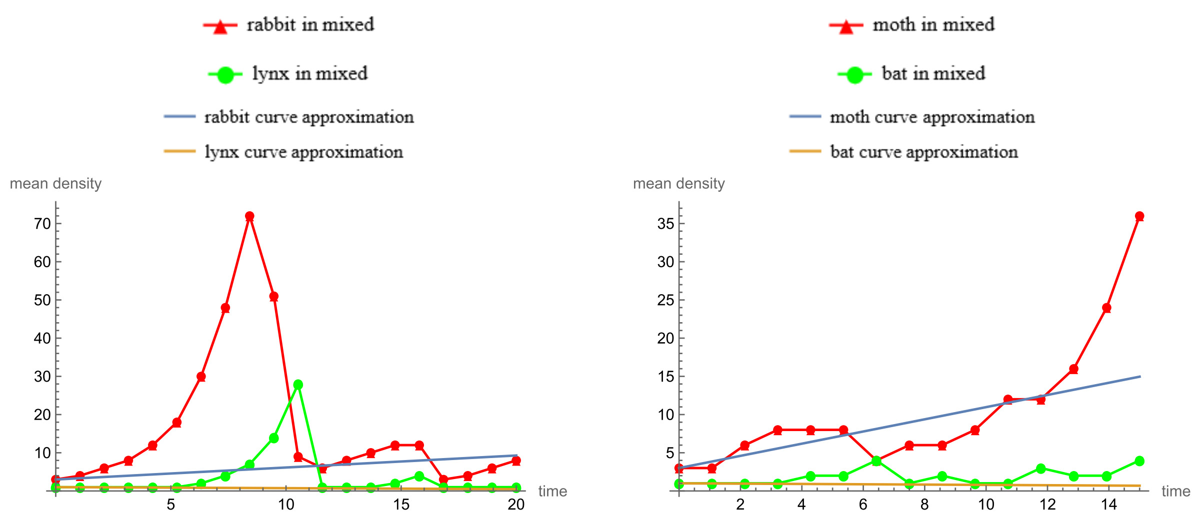

Figure 2: Curve approximation for growth in mixed

Figure 2 shows the population dynamics of two predator-prey pairs, rabbits and lynxes, and moths and bats within a mixed environment. In the rabbit-lynx interaction, the rabbit population initially rises sharply before declining as the lynx population increases. This illustrates a typical predator-prey cycle where predators thrive following prey abundance, leading to a subsequent drop in prey numbers. In the moth-bat interaction, the moth population gradually increases, peaking towards the end of the period. In contrast, the bat population remains relatively stable with minor fluctuations, indicating a less immediate response to prey availability. The curve approximations in both graphs highlight these trends, showing how prey availability drives predator population changes and emphasizing the cyclical nature of predator-prey relationships where predator populations grow in response to prey abundance.

3.2 Equilibrium and Stability

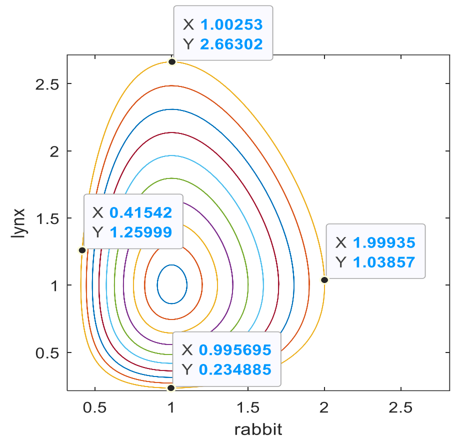

Figure 3: Phase trajectory with $r = 0.67$

Figure 3 illustrates that when $x \approx 1$, There are insufficient lynxes to maintain population balance, increasing the rabbit population. This breeds more lynx; soon, there are so many lynxes that rabbits have difficulty escaping. So, the number of rabbits begins to drop (around $x \approx 2$, When the rabbit population is estimated to reach its peak). This indicates that the rabbit population will eventually decline. As a result, the lynx population gradually increased. This occurs when populations return to their original levels, and the entire cycle begins anew.

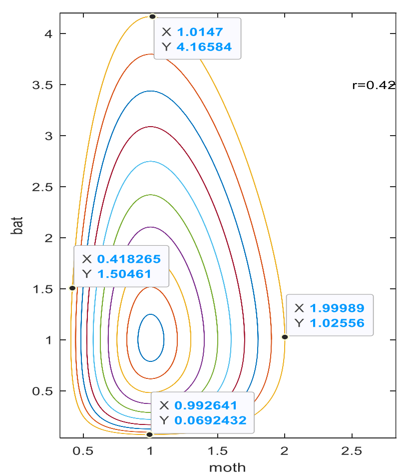

Figure 4: Phase trajectory with $r = 0.42$

Figure 4 illustrates that when $x \approx 1$, There are insufficient bats to maintain population balance, increasing moth populations. That leads to more bats, and finally, there are so many bats that moths have a difficult time evading them. As a result, the number of moths decreases (about at $x = 2$, where the moth population peaks). This indicates that the moth population will eventually decline. As a result, the bat population gradually increases. This occurs when populations return to their original levels, and the entire cycle begins anew.

The Lotka-Volterra model, though simple, makes several assumptions about the environment and the evolution of predator and prey populations, which limit its effectiveness in real-world situations. These assumptions include the prey always having enough food, predators only eating prey, population growth proportional to its current size, a static environment without genetic adaptation, and predators having unlimited appetites. Due to these assumptions, the model doesn't restrict specific parameters, leading to unrealistic predictions. For example, without predators, the prey population grows exponentially, and the rate at which a predator consumes prey increases indefinitely as the prey population grows, implying predators never get full. Thus, the Lotka-Volterra model fails to produce realistic projections for predator and prey dynamics in nature.

4 CONCLUSION

This study uses the Adams-Bashforth and Trapezoidal methods to study the Lotka-Volterra prey-predator model to understand how predator and prey populations interact and maintain ecosystem stability. The Adams-Bashforth method is accurate and efficient for predicting these interactions, while the Trapezoidal method offers a more straightforward but less precise approach. Accurate models of predator-prey dynamics are crucial for ecological balance, as predators help control prey populations and prevent overgrazing. The study focuses on interactions between bats, moths, and lynxes and rabbits, showing how their populations fluctuate in cycles. This research underscores the importance of numerical methods in ecological modeling for effective conservation and ecosystem management. Several steps are recommended to improve the study's results: First, compare different numerical methods to find the most effective one. Using real-world data from various species will make the models more accurate. Implement adaptive step-size control to enhance the stability and precision of the simulations. Broaden research to include more species and interactions to understand ecosystems fully. Develop flexible conservation policies based on continuous monitoring of population changes. Lastly, ecologists and conservationists should be trained in these methods to boost the effectiveness of conservation efforts. While this study focused on theoretical models, future research could benefit from incorporating real-world ecological data to improve the applicability of the findings. For instance, using population data of actual species like lynx and hares or bat-moth dynamics could provide more practical insights into the efficacy of these numerical methods.

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