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A Robust Diagonally Implicit Block Method for Solving First Order Stiff IVP of ODEs

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ABSTRACT

In this work, a block of diagonally implicit backward differentiation method with two off-step points for solving first order stiff initial value problem of ordinary differential equation was derived. In the proposed block method two approximate solution values of y_{n+1} and y_{n+2} with two off-step points $y_{n+\frac{1}{2}}$ and $y_{n+\frac{3}{2}}$ are computed concurrently for each iteration. The properties of the newly proposed method were found to be an A-stable, Zero stable and capable for solving first order Stiff IVPs. To validate the performance of the proposed method, some first order stiff IVPs are solved and the result obtained was compared with other existing numerical schemes. From the tabulated results and the graphs plotted, the proposed method has shown advantages of accuracy in the scale error over the three methods and an advantage of executional time over two of the existing methods considered.

Keywords: Block, Diagonally, Implicit, Ordinary Differential Equation, Stiff IVPs

1 INTRODUCTION

In physical, social and life sciences so many real life problems can be modelled into equation, most often differential equation. Problems in electrical circuits, chemical reactions, mechanics, vibrations, and kinetics and population growth all can be modelled as differential equations and categorized into stiff and non-stiff. A stiff equation is a differential equation for which certain numerical methods for solving the equation are numerically unstable, unless the step size is taken to be extremely small. So, with regard to modern scientific and engineering cases most of the modelled equations derived turns to be stiff problems. Researchers are formulating various methods to obtain analytic and/or numerical solutions of the modelled stiff IVPs. But, stiff problem usually deviates from been solved analytically due its complexities and other phenomena which is found within its solution, the transient and steady state components found in its solution make explicit method difficult to handle with appreciated results. While, numerical solution is much more easier and obtainable in any form stiff IVP of ODEs. Most of the stiff cases have no analytical solutions at all. Hence, preferences are always channels to numerical methods that would solve any sort of stiff IVP of ODEs. The ultimate goal is to get a method with a solution that has absolutely minimum scale error and computational time.

The early work integration of stiff equations [1], the extended backward differential formula and modified extended BDF [2, 3]. Block method was given more recognition through the work of [4, 8, 9, 10, 11, 12]. New fifth order implicit block method for solving first order stiff ordinary differential equations, a new super class of block backward differentiation formula for stiff ordinary differential equations, an accurate block solver for stiff initial value problems [5, 6, 7]. More numerical solutions and Hybrid multistep block methods were developed [19, 20, 21, 22, 23, 28]. Numerical treatment of block method for the solution of ODEs, on the approximate solution of continuous coefficients for solving third order ordinary differential equations, an accurate computation of block hybrid method for solving stiff ordinary differential equations [13, 14, 15]. An A-stable block integrator scheme for the solution of first order system of IVPs of ordinary differential equations, order and convergence of the enhanced 3-point fully implicit super class of block backward differentiation formula for solving first order stiff initial value problems [16, 17]; extended 3-Point super class of block backward differentiation formula for solving first order stiff initial value problems [18]. Other works include the following [24], [25], [26], [27], [29], [30], [31], [32], [33], [34] and [35] all the methods highlighted above demonstrates very good stability properties, at one point or the other, with appreciated results in terms of accuracy and computational time.

This paper considers derivation and implementation of a robust diagonally implicit block method with two off-step points for solving a system of first order initial value problem of ordinary differential equations. The proposed method will generate newly equally spaced solution values concurrently. In addition, the method uses of a lower triangular matrix with identical diagonal entries, as such the coefficients of the upper triangular matrix entries are zero. The method is of the form

$$y' = f\left(x,\ \hat{Y}\right), \quad \hat{Y}(a) = \varphi\eta, \qquad a \leq x \leq b \tag{1}$$

where $\hat{Y} = (y_1, y_2, y_3, \dots \dots \dots y_n)$, $\quad \eta\bar{\varphi} = (\varphi\eta_1, \varphi\eta_2, \varphi\eta_3, \dots, \varphi\eta_n)$.

2 DERIVATION OF A ROBUST 2-POINT BBDF WITH OFF-STEP POINTS

In this section, two approximate solution values y_{n+1} and y_{n+2} with step size h, and two off-step points $y_{n+\frac{1}{2}}$ and $y_{n+\frac{3}{2}}$ which are chosen at the points where the step size is halved are formulated in a block simultaneously. The formulae are computed using two back values y_n and y_{n-1} with step size h. The formulae are derived with the aid of a diagram as shown in Figure 1.

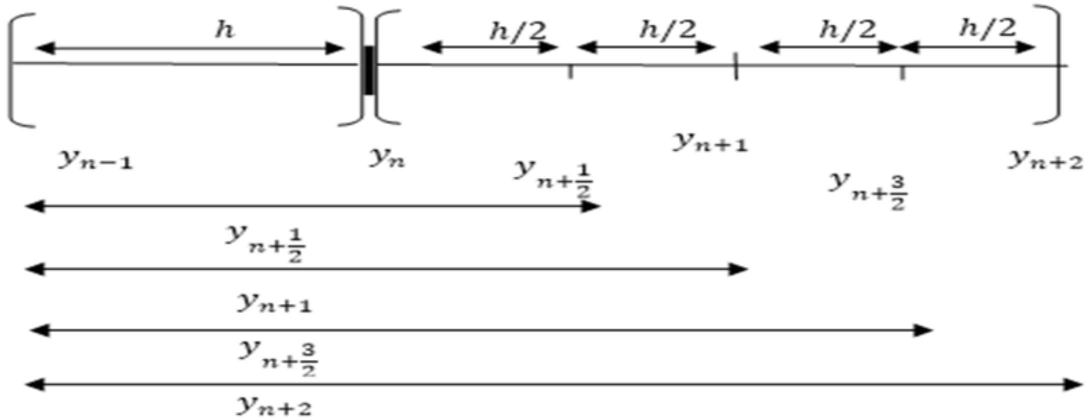

Figure 1: Diagram for RDIBM derivation.

The proposed method (RDIBM) is of the form

$$\sum_{j=o}^{1+k} \alpha_{j,i} y_{n+j-1} = h\beta_{k,i}[f_{n+k} + f_{n+k-1}] k = i = \tfrac{1}{2}, 1, \tfrac{3}{2}, 2 \tag{2}$$

where k and i have the same value. The formula (2) is derived using Taylor's series expansion about x_n

Definition 2.1. According to [26], the linear operator L_i associated with first, second, third and fourth point of the RDIBM with off-step points method is defined as follows:

$$\left.\begin{aligned}
&L_{\nu}[y(x_n), h]: \alpha_{0,\nu} y_{n-1} + \alpha_{1,\nu} y_n + \alpha_{\frac{3}{2},\nu} y_{n+\frac{1}{2}} - h\beta_{\psi,\nu}\left[f_{n+\psi} + f_{n+\psi-1}\right] = 0 \\
&L_{\nu}[y(x_n), h]: \alpha_{0,\nu} y_{n-1} + \alpha_{1,\nu} y_n + \alpha_{\frac{3}{2},\nu} y_{n+\frac{1}{2}} + \alpha_{2,\nu} y_{n+1} - h\beta_{\psi,\nu}\begin{bmatrix} f_{n+\psi} \\ +f_{n+\psi-1}\end{bmatrix} = 0 \\
&L_{\nu}[y(x_n), h]: \alpha_{0,\nu} y_{n-1} + \alpha_{1,\nu} y_n + \alpha_{\frac{3}{2},\nu} y_{n+\frac{1}{2}} + \alpha_{2,\nu} y_{n+1} + \alpha_{\frac{5}{2},\nu} y_{n+\frac{3}{2}} - h\beta_{\psi,\nu}\begin{bmatrix} f_{n+\psi} \\ +f_{n+\psi-1}\end{bmatrix} = 0 \\
&L_{\nu}[y(x_n), h]: \alpha_{0,\nu} y_{n-1} + \alpha_{1,\nu} y_n + \alpha_{\frac{3}{2},\nu} y_{n+\frac{1}{2}} + \alpha_{2,\nu} y_{n+1} + \alpha_{\frac{5}{2},\nu} y_{n+\frac{3}{2}} + \alpha_{3,\nu} y_{n+2} \\
&\qquad -h\beta_{\psi,\nu}\left[f_{n+\psi} + f_{n+\psi-1}\right] = 0
\end{aligned}\right\} \tag{3}$$

Consider the following value of $\psi \,\&\, \nu$'s value in (3) for the cases below:

For cases 1, 2, 3 and 4 as in $\psi = \nu = \frac{1}{2}$, $\psi = \nu = 1$, $\psi = \nu = \frac{3}{2}$ & $\psi = \nu = 2$ for the first, second, third and fourth point respectively, with the associated operator ($L_{\frac{1}{2}}$, L_1 , $L_{\frac{3}{2}}$ & L_2) related to (3) is written as

$$\left.\begin{array}{c}\alpha_{0,\frac{1}{2}}y(x_n-h)+\alpha_{1,\frac{1}{2}}y(x_n)+\alpha_{\frac{3}{2},\frac{1}{2}}y(x_n+\frac{1}{2}h)-h\beta_{\frac{1}{2},\frac{1}{2}}\left[f\left(x_n+\frac{1}{2}h\right)+f\left(x_n-\frac{1}{2}h\right)\right]=0\\ \alpha_{0,1}y(x_n-h)+\alpha_{1,1}y(x_n)+\alpha_{\frac{3}{2},1}y\left(x_n+\frac{1}{2}h\right)+\alpha_{2,1}y(x_n+h)\\ -h\beta_{1,1}[f(x_n+h)+f(x_n)]=0\\ \alpha_{0,\frac{3}{2}}y(x_n-h)+\alpha_{1\frac{3}{2},}y(x_n)+\alpha_{\frac{3}{2},\frac{3}{2}}y\left(x_n+\frac{1}{2}h\right)+\alpha_{2,\frac{3}{2}}y(x_n+h)+\alpha_{\frac{5}{2},\frac{3}{2}}y\left(x_n+\frac{3}{2}h\right)\\ -h\beta_{\frac{3}{2},\frac{3}{2}}\left[f\left(x_n+\frac{3}{2}h\right)+f\left(x_n+\frac{1}{2}h\right)\right]=0\\ \alpha_{0,2}y(x_n-h)+\alpha_{1,2}y(x_n)+\alpha_{\frac{3}{2},2}y\left(x_n+\frac{1}{2}h\right)+\alpha_{2,2}y(x_n+h)+\alpha_{\frac{5}{2},2}y\left(x_n+\frac{3}{2}h\right)\\ +\alpha_{3,2}y(x_n+2h)-h\beta_{2,2}[f(x_n+2h)+f(x_n+h)]=0\end{array}\right\}\quad(4)$$

Expanding (x_n-h) , $y(x_n)$, $y(x_n+\frac{1}{2}h)$,$y(x_n+h)$, $y(x_n+\frac{3}{2}h)$, $y(x_n+2h)$,$f\left(x_n+\frac{1}{2}h\right)$, $f\left(x_n-\frac{1}{2}h\right)$, $f\left(x_n+\frac{3}{2}h\right)$, $f(x_n+h)$, $f(x_n+2h)$ in (4) with a Taylor's series expansion about x_n and collect the like terms gives

$$\left.\begin{array}{c}C_{0,\frac{1}{2}}y(x_n)+C_{1,\frac{1}{2}}hy'(x_n)+C_{\frac{3}{2},\frac{1}{2}}h^2y''(x_n)+\cdots=0\\ C_{0,1}y(x_n)+C_{1,1}hy'(x_n)+C_{\frac{3}{2},1}h^2y''(x_n)+C_{2,1}h^3y'''(x_n)+\cdots=0\\ C_{0,\frac{3}{2}}y(x_n)+C_{1\frac{3}{2},}hy'(x_n)+C_{\frac{3}{2},\frac{3}{2}}h^2y''(x_n)+C_{2,\frac{3}{2}}h^3y'''(x_n)+C_{\frac{5}{2},\frac{3}{2}}h^4y'^{v}(x_n)+\cdots=0\\ C_{0,\frac{3}{2}}y(x_n)+C_{1\frac{3}{2},}hy'(x_n)+C_{\frac{3}{2},\frac{3}{2}}h^2y''(x_n)+C_{2,\frac{3}{2}}h^3y'''(x_n)+C_{\frac{5}{2},\frac{3}{2}}h^4y'^{v}(x_n)+\cdots=0\end{array}\right\}\quad(5)$$

where (5) is evaluated respectively as follows

$$\left.\begin{array}{r}C_{0,\frac{1}{2}}=\ \alpha_{0,\frac{1}{2}}+\alpha_{1,\frac{1}{2}}+\alpha_{\frac{3}{2},\frac{1}{2}}=0\\ C_{1,\frac{1}{2}}=\ -\alpha_{0,\frac{1}{2}}+\frac{1}{2}\alpha_{\frac{3}{2},\frac{1}{2}}-2\beta_{\frac{1}{2},\frac{1}{2}}=0\\ C_{\frac{3}{2},\frac{1}{2}}=\ \ -\frac{1}{6}\alpha_{0,\frac{1}{2}}+\frac{1}{48}\alpha_{\frac{3}{2},\frac{1}{2}}-\frac{1}{16}\beta_{\frac{1}{2},\frac{1}{2}}=0\end{array}\right\}\quad(6)$$

$$\left.\begin{array}{r}C_{0,1}=\ \alpha_{0,1}+\alpha_{1,1}+\alpha_{\frac{3}{2},1}+\alpha_{2,1}=0\\ C_{1,1}=\ -\alpha_{0,1}+\frac{1}{2}\alpha_{\frac{3}{2},1}+\alpha_{2,1}-2\beta_{1,1}=0\\ C_{\frac{3}{2},1}=\frac{1}{2}\alpha_{0,1}+\frac{1}{8}\alpha_{\frac{3}{2},1}+\frac{1}{2}\alpha_{2,1}-\beta_{1,1}=0\\ C_{2,1}=-\frac{1}{6}\alpha_{0,1}+\frac{1}{48}\alpha_{\frac{3}{2},1}+\frac{1}{6}\alpha_{2,1}-\frac{1}{2}\beta_{1,1}=0\end{array}\right\}\quad(7)$$

$$\left.\begin{array}{r}C_{0,\frac{3}{2}} = \alpha_{0,\frac{3}{2}} + \alpha_{1,\frac{3}{2}} + \alpha_{\frac{3}{2},\frac{3}{2}} + \alpha_{2,\frac{3}{2}} + \alpha_{\frac{5}{2},\frac{3}{2}} = 0 \\ C_{1,\frac{3}{2}} = -\alpha_{0,\frac{3}{2}} + \frac{1}{2}\alpha_{\frac{3}{2},\frac{3}{2}} + \alpha_{2,\frac{3}{2}} + \frac{3}{2}\alpha_{\frac{5}{2},\frac{3}{2}} - 2\beta_{\frac{3}{2},\frac{3}{2}} = 0 \\ C_{\frac{3}{2},\frac{3}{2}} = \frac{1}{2}\alpha_{0,\frac{3}{2}} + \frac{1}{8}\alpha_{\frac{3}{2},\frac{3}{2}} + \frac{1}{2}\alpha_{2,\frac{3}{2}} + \frac{9}{8}\alpha_{\frac{5}{2},\frac{3}{2}} - 2\beta_{\frac{3}{2},\frac{3}{2}} = 0 \\ C_{2,\frac{3}{2}} = -\frac{1}{6}\alpha_{0,\frac{3}{2}} + \frac{1}{48}\alpha_{\frac{3}{2},\frac{3}{2}} + \frac{1}{6}\alpha_{2,\frac{3}{2}} + \frac{9}{16}\alpha_{\frac{5}{2},\frac{3}{2}} - \frac{10}{8}\beta_{\frac{3}{2},\frac{3}{2}} = 0 \\ C_{\frac{5}{2},\frac{3}{2}} = \frac{1}{24}\alpha_{0,\frac{3}{2}} + \frac{1}{384}\alpha_{\frac{3}{2},\frac{3}{2}} + \frac{1}{24}\alpha_{2,\frac{3}{2}} + \frac{27}{128}\alpha_{\frac{5}{2},\frac{3}{2}} - \frac{7}{12}\beta_{\frac{3}{2},\frac{3}{2}} = 0\end{array}\right\} \quad (8)$$

and

$$\left.\begin{array}{r}C_{0,2} = \alpha_{0,2} + \alpha_{1,2} + \alpha_{\frac{3}{2},2} + \alpha_{2,2} + \alpha_{\frac{5}{2},2} + \alpha_{3,2} = 0 \\ C_{1,2} = -\alpha_{0,2} + \frac{1}{2}\alpha_{\frac{3}{2},2} + \alpha_{2,2} + \frac{3}{2}\alpha_{\frac{5}{2},2} + 2\alpha_{3,2} - 2\beta_{2,2} = 0 \\ C_{\frac{3}{2},2} = \frac{1}{2}\alpha_{0,2} + \frac{1}{8}\alpha_{\frac{3}{2},2} + \frac{1}{2}\alpha_{2,2} + \frac{9}{8}\alpha_{\frac{5}{2},2} + 2\alpha_{3,2} - 3\beta_{2,2} = 0 \\ C_{2,2} = -\frac{1}{6}\alpha_{0,2} + \frac{1}{48}\alpha_{\frac{3}{2},2} + \frac{1}{6}\alpha_{2,2} + \frac{9}{16}\alpha_{\frac{5}{2},2} + \frac{4}{3}\alpha_{3,2} - \frac{5}{2}\beta_{2,2} = 0 \\ C_{\frac{5}{2},2} = \frac{1}{24}\alpha_{0,2} + \frac{1}{384}\alpha_{\frac{3}{2},2} + \frac{1}{24}\alpha_{2,2} + \frac{27}{128}\alpha_{\frac{5}{2},2} + \frac{2}{3}\alpha_{3,2} - \frac{3}{2}\beta_{2,2} = 0 \\ C_{3,2} = -\frac{1}{120}\alpha_{0,2} + \frac{1}{3840}\alpha_{\frac{3}{2},2} + \frac{1}{120}\alpha_{2,2} + \frac{81}{1280}\alpha_{\frac{5}{2},2} + \frac{4}{15}\alpha_{3,2} - \frac{45}{72}\beta_{2,2} = 0\end{array}\right\} \quad (9)$$

Normalizing the coefficients $\alpha_{\frac{3}{2},\frac{1}{2}}$, $\alpha_{2,1}$, $\alpha_{\frac{5}{2},\frac{3}{2}}$ & $\alpha_{3,2}$ of $y_{n+\frac{1}{2}}$, y_{n+1} , $y_{n+\frac{3}{2}}$ and y_{n+2} respectively to 1, solving equation (6), (7), (8) and (9) with the aids of Maple Software for the coefficients of $\alpha_{j,i}$ and $\beta_{j,i}$ and substituting them in (4) gives the first, second, third and fourth point as

$$\left.\begin{array}{r}y_{n+\frac{1}{2}} = -\frac{1}{4}y_{n-1} + \frac{5}{4}y_n + \frac{1}{8}hf_{n+\frac{1}{2}} + \frac{1}{8}hf_{n-\frac{1}{2}} \\ y_{n+1} = \frac{1}{9}y_{n-1} - 0y_n + \frac{8}{9}y_{n+\frac{1}{2}} + \frac{1}{3}hf_{n+1} + \frac{1}{3}hf_n \\ y_{n+\frac{3}{2}} = -\frac{163}{257}y_{n-1} + \frac{1472}{257}y_n - \frac{3023}{257}y_{n+\frac{1}{2}} + \frac{1971}{257}y_{n+1} + \frac{273}{514}hf_{n+\frac{3}{2}} + \frac{273}{514}hf_{n+\frac{1}{2}} \\ y_{n+2} = -\frac{11}{65}y_{n-1} + \frac{29}{13}y_n - \frac{64}{13}y_{n+\frac{1}{2}} + \frac{63}{13}y_{n+1} - \frac{64}{65}y_{n+\frac{3}{2}} + \frac{6}{13}hf_{n+2} + \frac{6}{13}hf_{n+1}\end{array}\right\} \quad (10)$$

3 ANALYSIS OF THE METHOD

In this section, order and stability properties of the proposed method (10) will be analyzed.

3.1 Order of the Method

In this section, the order of the proposed method (10) will be derived. The method could be transformed to a general matrix form as

$$\sum_{j=0}^{1} C_j^* Y_{m+j-1} = h\sum_{j=0}^{1} D_j^* Y_{m+j-1}, \tag{11}$$

where C and D are constant coefficient matrices of the method (11) is equivalent to the following form

$$\left.\begin{aligned}
-\tfrac{1}{4}y_{n-1} + \tfrac{5}{4}y_n + y_{n+\frac{1}{2}} &= -\tfrac{1}{8}hf_{n+\frac{1}{2}} - \tfrac{1}{8}hf_{n-\frac{1}{2}}\\
\frac{1}{9}y_{n-1} - 0y_n + \frac{8}{9}y_{n+\frac{1}{2}} + y_{n+1} &= -\frac{1}{3}hf_{n+1} - \frac{1}{3}hf_n\\
-\tfrac{163}{257}y_{n-1} + \tfrac{1472}{257}y_n - \tfrac{3023}{257}y_{n+\frac{1}{2}} + \tfrac{1971}{257}y_{n+1} + y_{n+\frac{3}{2}} &= -\tfrac{273}{514}hf_{n+\frac{3}{2}} - \tfrac{273}{514}hf_{n+\frac{1}{2}}\\
-\frac{11}{65}y_{n-1} + \frac{29}{13}y_n - \frac{64}{13}y_{n+\frac{1}{2}} + \frac{63}{13}y_{n+1} - \frac{64}{65}y_{n+\frac{3}{2}} + y_{n+2} &= -\frac{6}{13}hf_{n+2} - \frac{6}{13}hf_{n+1}
\end{aligned}\right\} \tag{12}$$

also (12) can be written as

$$\left.\begin{aligned}
&\begin{bmatrix} 0 & -\frac{1}{4} & 0 & \frac{5}{4} \\ 0 & \frac{1}{9} & 0 & 0 \\ 0 & -\frac{163}{257} & 0 & \frac{1472}{257} \\ 0 & -\frac{11}{65} & 0 & \frac{29}{13} \end{bmatrix}\begin{bmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{8}{9} & 1 & 0 & 0 \\ -\frac{3023}{257} & \frac{1971}{257} & 1 & 0 \\ -\frac{64}{13} & \frac{63}{13} & -\frac{64}{65} & 1 \end{bmatrix}\begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix}\\
&= h\begin{bmatrix} 0 & 0 & -\frac{1}{8} & 0 \\ 0 & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{bmatrix} + h\begin{bmatrix} -\frac{1}{8} & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ -\frac{273}{514} & 0 & -\frac{273}{514} & 0 \\ 0 & -\frac{6}{13} & 0 & -\frac{6}{13} \end{bmatrix}\begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix}
\end{aligned}\right\} \tag{13}$$

where

$$C_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} C_1 = \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{9} \\ -\frac{163}{257} \\ -\frac{11}{65} \end{bmatrix} C_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} C_3 = \begin{bmatrix} \frac{5}{4} \\ 0 \\ \frac{1472}{257} \\ \frac{29}{13} \end{bmatrix} C_4 = \begin{bmatrix} \frac{1}{8} \\ \frac{9}{} \\ -\frac{3023}{257} \\ \frac{64}{13} \end{bmatrix} C_5 = \begin{bmatrix} 0 \\ 1 \\ \frac{1971}{257} \\ \frac{63}{13} \end{bmatrix} C_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -\frac{64}{13} \end{bmatrix} C_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$D_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} D_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} D_2 = \begin{bmatrix} -\frac{1}{8} \\ 0 \\ 0 \\ 0 \end{bmatrix} D_3 = \begin{bmatrix} 0 \\ -\frac{1}{3} \\ 0 \\ 0 \end{bmatrix} D_4 = \begin{bmatrix} -\frac{1}{8} \\ 0 \\ -\frac{273}{514} \\ 0 \end{bmatrix} D_5 = \begin{bmatrix} 0 \\ -\frac{1}{3} \\ 0 \\ -\frac{6}{13} \end{bmatrix} D_6 = \begin{bmatrix} 0 \\ 0 \\ -\frac{273}{514} \\ 0 \end{bmatrix} D_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{6}{13} \end{bmatrix}$$

Definition 3.1.1 According to [26], the order of the block method (10) and its associated linear operator are given by

$$L[y(x);h] = \sum_{j=0}^{7}[c_j y(x+jh)] - h\sum_{j=0}^{7} D_j y'(x+jh)] \tag{14}$$

where p is unique integer such that $\mathrm{E_q} = 0,\ \mathrm{q} = 0,1,\dots\ \mathrm{p}$ and $\mathrm{E_{p+1}} \neq 0$, where the $\mathrm{E_q}$ are constant matrix with

$$E_0 = \sum\nolimits_{J=0}^{7} C_J = 0$$

$$E_1 = \sum_{J=0}^{7} \left[jC_j - 2D_j \right] = 0$$

$$E_2 = \sum_{J=0}^{7} \left[\frac{1}{2!} j^2 C_j - 2jD_j \right] = 0$$

$$E_3 = \sum_{J=0}^{7} \left[\frac{1}{3!} j^3 C_j - 2\frac{1}{2!} j^2 D_j \right] = 0$$

$$E_4 = \sum_{J=0}^{7} \left[\frac{1}{4!} j^4 C_j - 2\frac{1}{3!} j^3 D_j \right] = 0$$

$$E_5 = \sum_{J=0}^{7}\left[\frac{1}{5!}j^5C_j - 2\frac{1}{4!}j^4D_j\right] = 0$$

$$E_6 = \sum_{J=0}^{7}\left[\frac{1}{6!}j^6C_j - 2\frac{1}{5!}j^5D_j\right] = 0$$

$$E_7 = \sum_{J=0}^{7}\left[\frac{1}{7!}j^7C_j - 2\frac{1}{6!}j^6D_j\right] = \begin{pmatrix} -\frac{121}{9353} \\ -\frac{221}{5065} \\ -\frac{274}{7385} \\ -\frac{192}{5123} \end{pmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the developed method is of order 6, with error constant

$$\mathrm{E}_7 = \begin{pmatrix} -\frac{121}{9353} \\ -\frac{221}{5065} \\ -\frac{274}{7385} \\ -\frac{192}{5123} \end{pmatrix} \tag{15}$$

3.2 Stability Analysis of the Method

In this section, we investigate the Zero and A- stability property of the proposed method (10).

Definition 3.2.1 A linear multistep method is said to be zero stable if no root of the first characteristics polynomial has modulus higher than 1 and that any root with modulus 1 is simple [31].

Definition 3.2.2 A linear multistep method is said to be an A-stable method if its region of stability encloses the entire negative half-plane [31].

The stability of the scheme (10-11) can be obtained by applying the standard test equation of the form

$$y' = \lambda y, \qquad Re(\lambda) < 0 \tag{16}$$

where λ is a complex number.

To get the following solutions

$$\left.\begin{aligned} y_{n+\frac{1}{2}} &= -\tfrac{1}{4}y_{n-1} + \tfrac{5}{4}y_n + \tfrac{1}{8}h\lambda y_{n+\frac{1}{2}} + \tfrac{1}{8}h\lambda y_{n-\frac{1}{2}} \\ y_{n+1} &= \tfrac{1}{9}y_{n-1} - 0y_n + \tfrac{8}{9}y_{n+\frac{1}{2}} + \tfrac{1}{3}h\lambda y_{n+1} + \tfrac{1}{3}h\lambda y_n \\ y_{n+\frac{3}{2}} &= -\tfrac{163}{257}y_{n-1} + \tfrac{1472}{257}y_n - \tfrac{3023}{257}y_{n+\frac{1}{2}} + \tfrac{1971}{257}y_{n+1} + \tfrac{273}{514}h\lambda y_{n+\frac{3}{2}} + \tfrac{273}{514}h\lambda y_{n+\frac{1}{2}} \\ y_{n+2} &= -\tfrac{11}{65}y_{n-1} + \tfrac{29}{13}y_n - \tfrac{64}{13}y_{n+\frac{1}{2}} + \tfrac{63}{13}y_{n+1} - \tfrac{64}{65}y_{n+\frac{3}{2}} + \tfrac{6}{13}h\lambda y_{n+2} + \tfrac{6}{13}h\lambda y_{n+1} \end{aligned}\right\} \tag{17}$$

(17) can also be written as

$$\begin{bmatrix} 1-\frac{1}{8}h\lambda & 0 & 0 & 0 \\ -\frac{8}{9} & 1-\frac{1}{3}h\lambda & 0 & 0 \\ \frac{3023}{257}-\frac{273}{514}h\lambda & -\frac{1971}{257} & 1-\frac{273}{514}h\lambda & 0 \\ \frac{64}{13} & -\frac{63}{13}-\frac{6}{13}h\lambda & \frac{64}{65} & 1-\frac{6}{13}h\lambda \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{4} & \frac{1}{8}h\lambda & \frac{5}{4} \\ 0 & \frac{1}{9} & 0 & \frac{1}{3}h\lambda \\ 0 & -\frac{163}{257} & 0 & \frac{1472}{257} \\ 0 & -\frac{11}{65} & 0 & \frac{29}{13} \end{bmatrix} \begin{bmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix} \tag{18}$$

From (18) it is given that

$$AY_m = BY_{m-1} \tag{19}$$

If m is the number of block and r is the number of points in the block, then $n = mr$

Here, $r = 2$ and $n = 2m$. It follows that

$$Y_m = \begin{bmatrix} y_{2m+\frac{1}{2}} \\ y_{2m+1} \\ y_{2m++\frac{3}{2}} \\ y_{3m+2} \end{bmatrix} = \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+22} \end{bmatrix},\ Y_{m-1} = \begin{bmatrix} y_{2(m-1)-\frac{3}{2}} \\ y_{2(m-1)-1} \\ y_{2(m-1)-\frac{1}{2}} \\ y_{2(m-1)} \end{bmatrix} = \begin{bmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix}$$

and the coefficient matrices are given as

$$A = \begin{bmatrix} 1-\frac{1}{8}h\lambda & 0 & 0 & 0 \\ -\frac{8}{9} & 1-\frac{1}{3}h\lambda & 0 & 0 \\ \frac{3023}{257}-\frac{273}{514}h\lambda & -\frac{1971}{257} & 1-\frac{273}{514}h\lambda & 0 \\ \frac{64}{13} & -\frac{63}{13}-\frac{6}{13}h\lambda & \frac{64}{65} & 1-\frac{6}{13}h\lambda \end{bmatrix}$$

$$B=\begin{bmatrix} 0 & -\frac{1}{4} & \frac{1}{8}h\lambda & \frac{5}{4} \\ 0 & \frac{1}{9} & 0 & \frac{1}{3}h\lambda \\ 0 & -\frac{163}{257} & 0 & \frac{1472}{257} \\ 0 & -\frac{11}{65} & 0 & \frac{29}{13} \end{bmatrix}$$

The stability polynomial of the proposed method was computed with the aid of Maple Software and the result is found to be

$$\left.\begin{aligned} & det(At-B)= \\ & -\frac{2291}{160368}t h^2+\frac{119933}{150345}t^2-\frac{1044449}{400920}t^2 h-\frac{9791}{53456}t^2 h^2-\frac{2449}{53456}h^3 t^2 \\ & \quad -\frac{442469}{801840}t^3 h^2-\frac{110731}{80184}t^4 h+\frac{108625}{160368}t^4 h^2-\frac{7269}{53456}t^4 h^3+\frac{237}{26728}t^4 h^4 \\ & \quad +\frac{8495}{26728}t^3 h^3+\frac{3996001}{1202760}t^3 h-\frac{237}{26728}h^4 t^3-\frac{92249}{1202760}t h-\frac{1534}{1503}t^3+t^4 \end{aligned}\right\} \quad (20)$$

$$R(t,0)=\frac{119933}{150345}t^2-\frac{1534}{1503}t^3+t^4=0 \quad (21)$$

$t=0\,,0\,,1,-0.2629880097$

3.3 A- stability of the Proposed Method

In this section, the region for the absolute stability of the proposed methods is plotted, by considering the stability polynomials (20). The set of point defined by $t=e^{i\theta}$, $0 \le \theta \le 2\pi$ describes the boundary of the stability region. The following stability region was the complex plot of the proposed method with the aid of Maple Software.

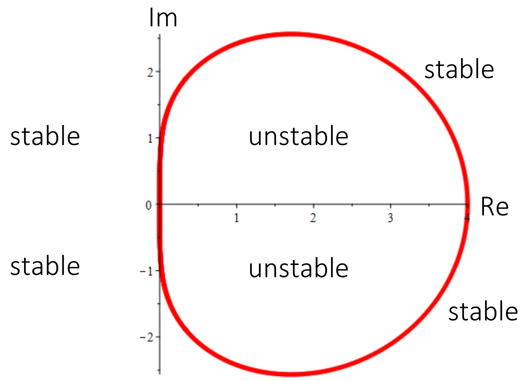

Figure 2: A-stability region of the Proposed Method (RDIBM)

4 IMPLEMENTATION OF THE METHOD

Consider the system of first order initial value problem of ordinary differential equation of the form

$$y' = f(x, \hat{Y}), \quad \hat{Y}(a) = \eta, \qquad a \le x \le b \tag{22}$$

$$\hat{Y} = (y_1, y_2, y_3, \dots \dots \dots y_n), \tag{23}$$

using Newton's iteration to implement the methods (10).

Let y_i and $y(x_i)$ be the approximate and exact solutions of system (22-23) respectively.

Define the absolute error as

$$(error_i)_t = |(y_i)_t - (y(x_i))_t| \tag{24}$$

and the maximum error as

$$\text{MAXE} = \underbrace{max}_{1 \le i \le \tau} (\underbrace{max\ (error_i)_t}_{1 \le i \le \aleph}) \tag{25}$$

τ and $\aleph$ are the total number of step and equations respectively.

From the method (10)

$$\left.\begin{aligned}
F_1 &= y_{n+\frac{1}{2}} - \frac{1}{8}hf_{n+\frac{1}{2}} - \frac{1}{8}hf_{n-\frac{1}{2}} - \varepsilon_1 \\
F_2 &= y_{n+1} - \frac{8}{9}y_{n+\frac{1}{2}} - \frac{1}{3}hf_{n+1} - \frac{1}{3}hf_n - \varepsilon_2 \\
F_3 &= y_{n+\frac{3}{2}} + \frac{3023}{257}y_{n+\frac{1}{2}} - \frac{1971}{257}y_{n+1} - \frac{273}{514}hf_{n+\frac{3}{2}} - \frac{273}{514}hf_{n+\frac{1}{2}} - \varepsilon_3 \\
F_4 &= y_{n+2} + \frac{64}{13}y_{n+\frac{1}{2}} - \frac{63}{13}y_{n+1} + \frac{64}{65}y_{n+\frac{3}{2}} - \frac{6}{13}hf_{n+2} - \frac{6}{13}hf_{n+1} - \varepsilon_4
\end{aligned}\right\} \tag{26}$$

The $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_2$ and $\mathcal{E}_4$ are the back values of (10) as

$$\left.\begin{aligned}
\varepsilon_1 &= -\frac{1}{4}y_{n-1} + \frac{5}{4}y_n \\
\varepsilon_2 &= \frac{1}{9}y_{n-1} - 0y_n \\
\varepsilon_3 &= -\frac{163}{257}y_{n-1} + \frac{1472}{257}y_n \\
\varepsilon_4 &= -\frac{11}{65}y_{n-1} + \frac{29}{13}y_n
\end{aligned}\right\} \tag{27}$$

Let $y_{n+j}^{(i+1)},\ j = \frac{1}{2}, 1, \frac{3}{2}, 2$, denote the $(i+1)^{th}$ iterative values of y_{n+j} and consider

$$e_{n+j}^{(i+1)} = y_{n+j}^{(i+1)} - y_{n+j}^{(i)},\ j = \frac{1}{2}, 1, \frac{3}{2}, 2 \tag{28}$$

Now, the Newton's iteration for the proposed method will have the form

$$y_{n+j}^{(i+1)} = y_{n+j}^{(i)} - \frac{\left(F_j\left(y_{n+j}^{(i)}\right)\right)}{\left(F'_j\left(y_{n+j}^{(i)}\right)\right)}, j = \frac{1}{2}, 1, \frac{3}{2}, 2 \tag{29}$$

$$y_{n+j}^{(i+1)} = y_{n+j}^{(i)} - \left(F'_j\left(y_{n+j}^{(i)}\right)\right)^{-1}\left(F_j\left(y_{n+j}^{(i)}\right)\right), j = \frac{1}{2}, 1, \frac{3}{2}, 2 \tag{30}$$

$$y_{n+j}^{(i+1)} - y_{n+j}^{(i)} = -\left(F'_j\left(y_{n+j}^{(i)}\right)\right)^{-1}\left(F_j\left(y_{n+j}^{(i)}\right)\right), j = \frac{1}{2}, 1, \frac{3}{2}, 2 \tag{31}$$

$$e_{n+j}^{(i+1)} = -\left(F'_j\left(y_{n+j}^{(i)}\right)\right)^{-1}\left(F_j\left(y_{n+j}^{(i)}\right)\right), j = \frac{1}{2}, 1, \frac{3}{2}, 2 \tag{32}$$

It can be written as

$$\left(F'_j\left(y_{n+j}^{(i)}\right)\right)e_{n+j}^{(i+1)} = -\left(F_j\left(y_{n+j}^{(i)}\right)\right), j=\tfrac{1}{2},1,\tfrac{3}{2},2 \tag{33}$$

Equation (33) will also be written in its matrix form as:

$$\underbrace{\begin{pmatrix} 1-\frac{1}{8}\frac{\partial F^{(i)}_{n+\frac{1}{2}}}{\partial y^{(i)}_{n+\frac{1}{2}}} & 0 & 0 & 0 \\ -\frac{8}{9} & 1-\frac{1}{3}\frac{\partial F^{(i)}_{n+1}}{\partial y^{(i)}_{n+1}} & 0 & 0 \\ \frac{3023}{257}-\frac{273}{514}\frac{\partial F^{(i)}_{n+\frac{1}{2}}}{\partial y^{(i)}_{n+\frac{1}{2}}} & -\frac{1971}{257} & 1-\frac{273}{514}\frac{\partial F^{(i)}_{n+\frac{3}{2}}}{\partial y^{(i)}_{n+\frac{3}{2}}} & 0 \\ \frac{64}{13} & -\frac{63}{13}-\frac{6}{13}\frac{\partial F^{(i)}_{n+1}}{\partial y^{(i)}_{n+1}} & \frac{64}{65} & 1-\frac{6}{13}h\frac{\partial F^{(i)}_{n+2}}{\partial y^{(i)}_{n+2}} \end{pmatrix}}_{jacobian\,matrix} \begin{pmatrix} e^{(i+1)}_{n+\frac{1}{2}} \\ e^{(i+1)}_{n+1} \\ e^{(i+1)}_{n+\frac{3}{2}} \\ e^{(i+1)}_{n+2} \end{pmatrix} =$$

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ \frac{8}{9} & -1 & 0 & 0 \\ -\frac{3023}{257} & \frac{1971}{257} & -1 & 0 \\ -\frac{64}{13} & \frac{63}{13} & -\frac{64}{65} & -1 \end{pmatrix}\begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{pmatrix} + h\begin{pmatrix} 0 & 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{pmatrix} +$$

$$h\begin{pmatrix} \frac{1}{8} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ \frac{273}{514} & 0 & \frac{273}{514} & 0 \\ 0 & \frac{6}{13} & 0 & \frac{6}{13} \end{pmatrix}\begin{pmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{pmatrix} + \begin{pmatrix} \varepsilon_{\frac{1}{2}} \\ \varepsilon_1 \\ \varepsilon_{\frac{3}{2}} \\ \varepsilon_2 \end{pmatrix} \tag{34}$$

4.1 Test Problems

To validate the method developed, (RDIBM), a code in 'C' (programming Language) with Equation (34) would be used to solve the following stiff IVPs.

Table 1: Sample of First Order Initial Value Problem of Stiff ODEs

S/n	Problems	Initial Conditions	Interval	Exact Solutions	Eigen Values	Source
1	$y_1' = 198y_1 + 199y_1$ $y_2' = -398y_1 - 399y_2$	$y_1(0) = 1$ $y_2(0) = -1$	$0 \leq x \leq 10$	$y_1(x) = e^{-x}$ $y_2(x) = -e^{-x}$	−1,−200	[4]
2	$y_1' = y_2$ $y_2' = -2y_1$ $y_3' = y_2 + 2y_3$	$y_1(0) = 0$ $y_2(0) = 0$ $y_3(0) = 1$	$0 \leq x \leq 4\pi$	$y_1(x) = 2cosx + 6sinx - 6x - 2$ $y_2(x) = -2sinx + 6cosx - 6$ $y_3(x) = 2sinx - 2cosx + 3$		[25]
3	$y' = 5e^{5x}(y - x)^2 + 1$	$y(0) = 0$	$0 \leq x \leq 1$	$y_1(x) = x - e^{-5x}$		[4]
4	$y_1' = -20y_1 - 19y_2$ $y_2' = -19y_1 - 20y_2$	$y_1(0) = 2$ $y_2(0) = 0$	$0 \leq x \leq 20$	$y_1(x) = e^{-x}$ $y_2(x) = -e^{-x}$	−1,−200	[5]

5 RESULT AND DISCUSSIONS

The sample problems presented in this paper are solved using the proposed methods. The result of the tested problems are tabulated and compared with the existing ones. The graphs highlighting the performance of these methods are plotted. The acronyms below are used in the tables.

h= step-size;
MHTD =Method
MAX-ERR = Maximum Error;
EXEC-TIME= Executional Time in second;
ABISBDF = An A-stable Block Integrator Scheme for the Solution of First Order System of IVP of Ordinary Differential Equations
3ESBBDF = Extended 3-Point Super class of Block Backward Differentiation formula for Solving Stiff Initial Value Problems.
RDIBM = A Robust Diagonally Implicit Block Method with Two Off-Step Points for Solving First Order Stiff IVP of ODEs
3NBBDF = A New Fifth Order implicit block method for Solving First Order Stiff Ordinary Differential Equations
3BDF = Implicit r-point block backward differentiation formula for solving first-order stiff ODEs

Table 2: Comparison of Errors for Problem 1

Numerical Result for Problem 1				
h	**MTHD**	**NS**	**MAX-ERR**	**EXEC-TIME**
10^{-2}	3NBBDF	333	1.94447e-004	1.20394e-002
	ABISBDF	555	5.83217e-003	5.68676e-002
	RDIBM	100	1.52564e-004	3.93719e-003
	3BDF	333	1.07308e-02	31,867μs
10^{-3}	3NBBDF	3,333	2.07993e-006	1.19193e-001
	ABISBDF	5,555	6.05338e-005	5.64515e-001
	RDIBM	1,000	1.76763e-006	1.87573e-002
	3BDF	3,333	1.10060e-03	258,361μs
10^{-4}	3NBBDF	33,333	2.09995e-008	1.19296e+000
	ABISBDF	55,555	6.26692e-007	5.68143e+000
	RDIBM	10,000	1.79766e-008	1.66571e-001
	3BDF	33,333	1.10333e-04	2,582,756μs
10^{-5}	3NBBDF	333,333	2.10257e-010	1.19173e+001
	ABISBDF	555,555	6.32740e-009	5.59821e+001
	RDIBM	100,000	1.82566e-010	1.43458e+000
	3BDF	333,333	1.10361e-05	26,011,417μs
10^{-6}	3NBBDF	3,333,333	1.41029e-011	1.19110e+002
	ABISBDF	5,555555	6.33362e-011	5.53567e+002
	RDIBM	1,000,000	1.85567e-012	1.28786e+001
	3BDF	3,333,333	1.10363e-06	260,435,329μs

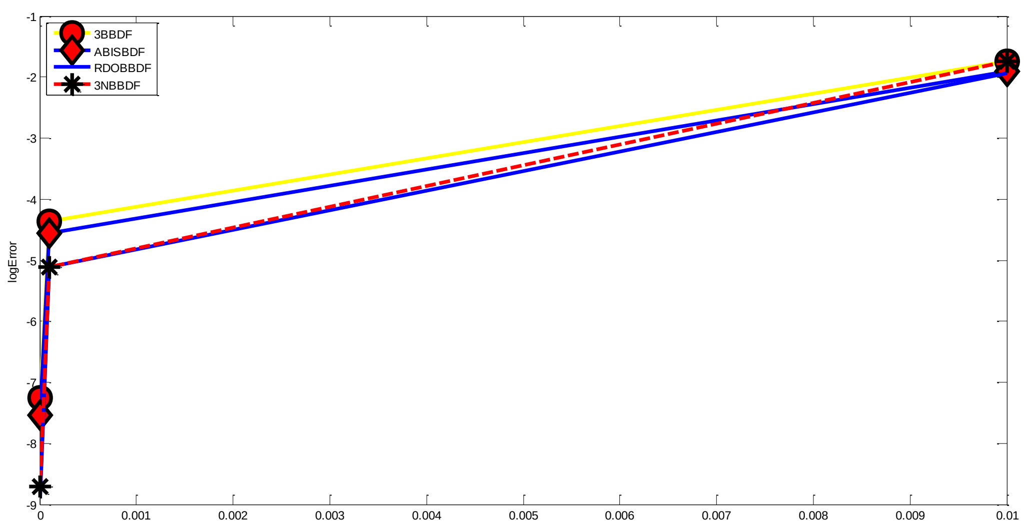

Figure 3: Graph of $\text{Log}_{10}(MAXE)$ against the step size h for Problem 1

Table 3 Comparison of Errors for Problem 2

Numerical Result for Problem 2				
H	**MTHD**	**NS**	**MAX-ERR**	**EXEC-TIME**
10^{-2}	3ESBBDF ABISBDF RDIBM	333 555 100	2.5780e-002 3.8321e-003 2.2979e-003	3.96563e-001 5.58677e-002 2.98278e-002
10^{-3}	3ESBBDF ABISBDF RDIBM	3,333 5,555 1,000	2.2907e-003 4.0533e-005 2.1066e-005	3.66799e+000 5.54512e-001 1.79012e-001
10^{-4}	3ESBBDF ABISBDF RDIBM	33,333 55,555 10,000	2.0972e-005 4.2669e-007 1.9765e-007	3.35906e+000 5.52149e-001 1.49191e+000
10^{-5}	3ESBBDF ABISBDF RDIBM	333,333 555,555 100,000	2.002e-007 4.3274e-009 1.7932e-009	3.01024e+001 5.49867e+000 1.20674e+001
10^{-6}	3ESBBDF ABISBDF RDIBM	3,333,333 5,555555 1,000,000	1.8702e-009 4.3335e-009 1.6004e-011	2.96155e+002 5.35204e+000 1.20678e+001

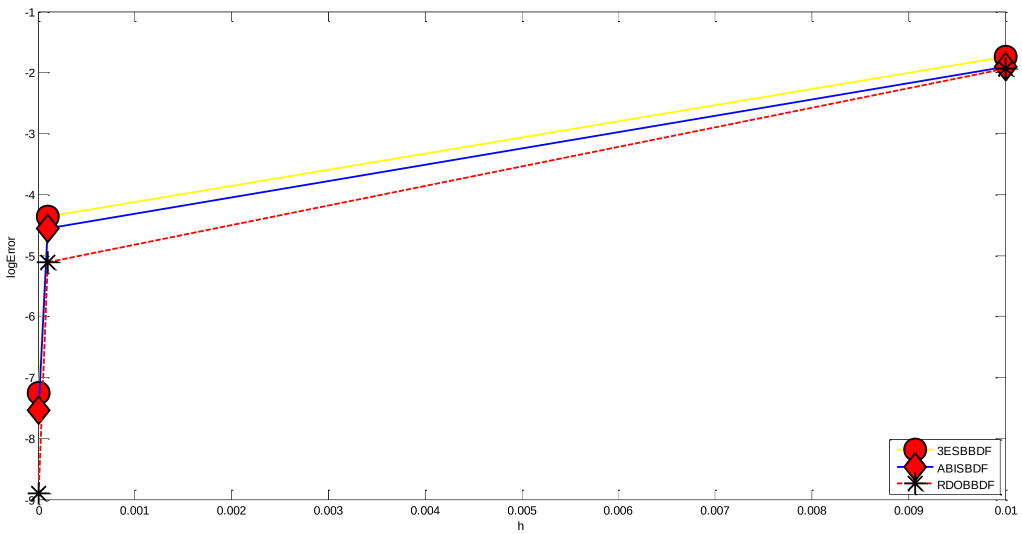

Figure 4: Graph of $\text{Log}_{10}(MAXE)$ against the step size h for Problem 2

Table 4: Comparison of Errors for Problem 3

Numerical Result for Problem 3				
H	**MTHD**	**NS**	**MAX-ERR**	**EXEC-TIME**
10^{-2}	3ESBBDF	666	4.83217e-003	6.23441e-005
	3NBBDF	333	3.51456e-003	5.52416e-004
	RDIBM	555	2.61015e-003	3.11121e-005
10^{-3}	3ESBBDF	6,666	5.95338e-005	6.65467e-004
	3NBBDF	3,333	4.90191e-005	4.50367e-003
	RDIBM	5,555	3.73116e-005	2.96482e-004
10^{-4}	3ESBBDF	66,666	5.95692e-007	6.48433e-003
	3NBBDF	33,333	5.20417e-007	4.36918e-002
	RDIBM	55,555	3.73371e-007	2.94261e-003
10^{-5}	3ESBBDF	666,666	5.95974e-009	6.58687e-002
	3NBBDF	333,333	5.25030e-009	4.34808e-001
	RDIBM	555,555	3.73652e-009	2.92149e-002
10^{-6}	3ESBBDF	6,666,666	6.18636e-011	6.23434e-001
	3NBBDF	3,333,333	5.25648e-011	4.35791e+00
	RDIBM	5,555,555	4.05313e-011	2.90945e-001

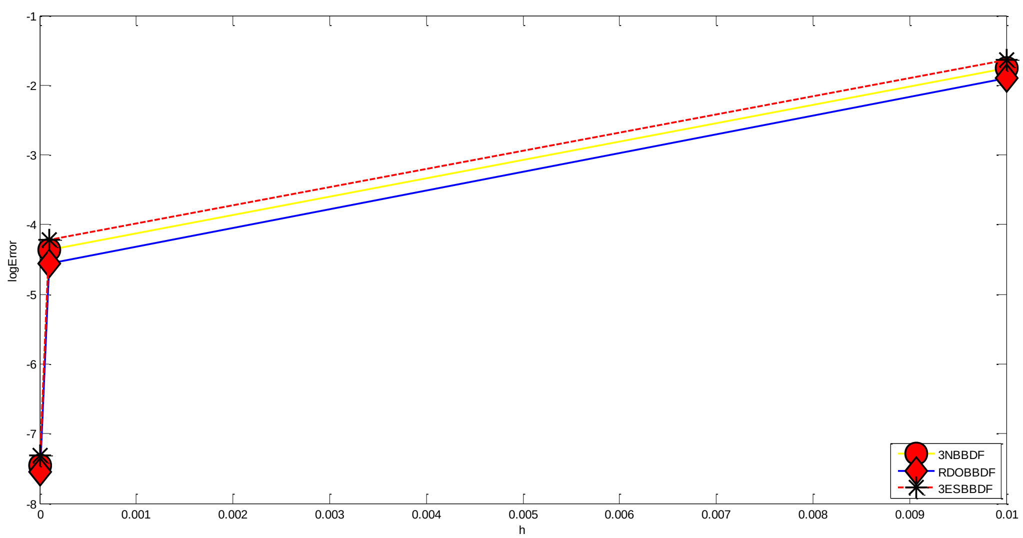

Figure 5: Graph of $\mathrm{Log}_{10}(MAXE)$ against the step size h for Problem 3

Table 5: Comparison of Errors for Problem 4

Numerical Result for Problem 4				
H	MTHD	NS	MAX-ERR	EXEC-TIME
10^{-2}	3ESBBDF	333	2.23033e-002	3.67590e-002
	3NBBDF	666	6.98707e-002	2.63337e-002
	RDIBM	555	4.45713e-003	2.41226e-002
10^{-3}	3ESBBDF	3,333	3.56164e-003	8.56636e-002
	3NBBDF	6,666	5.40956e-003	2.60816e-001
	RDIBM	5,555	3.74938e-005	2.42705e-001
10^{-4}	3ESBBDF	33,333	3.56515e-005	8.54385e-001
	3NBBDF	66,666	3.08942e-005	2.60725e+000
	RDIBM	55,555	3.52727e-007	2.40503e+000
10^{-5}	3ESBBDF	333,333	3.60706e007	8.53788e+000
	3NBBDF	666,666	3.18534e-007	2.60597e+001
	RDIBM	555,555	3.31505e-009	2.40064e+001
10^{-6}	3ESBBDF	3,333,333	3.61122e-009	8.53356e+001
	3NBBDF	6,666,666	3.19872e-009	2.60700e+002
	RDIBM	5,555,555	3.11313e-011	2.40003e+001

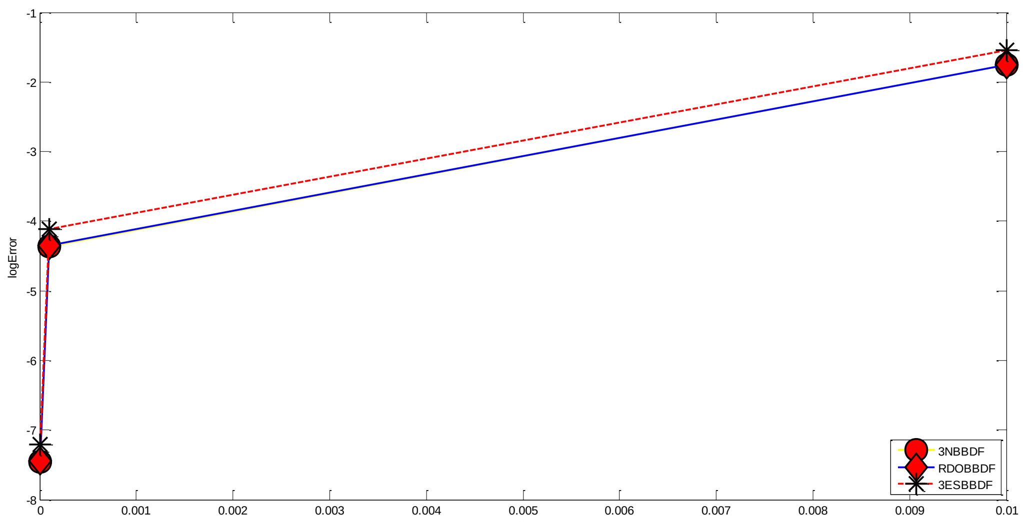

Figure 6: Graph of $\text{Log}_{10}(MAXE)$ against the step size h for Problem 4

Considering the Table 2, 3, 4 and 5 comprising problem 1, 2, 3 and 4, it has been shown that the newly proposed method, RDIBM outperformed the 3BDF, ABISBDF and 3NBBDF in terms of accuracy in problems 1, 2, 3 and 4, and computational time in Problem 1, 3 and 4. While, 3NBBDF has good accuracy and executional time than 3BDF, ABISBDF and 3ESBBDF in problems 1 and 3. However, the 3NBBDF and 3ESBBDF competes closely in terms of accuracy of the scale errors in Problems 4.

Similarly, the accuracy of the scale errors and executional time of the proposed methods, RDIBM found to be better than all the methods compared in Problems 2.

To further depicts visibly the performance of the proposed method, RDIBM with respect to the other methods compared, the graphs of *Log10(MAXE)* against the step size *h* for all the problems tested are plotted (using Matlab) in Figure 3, 4, 5 and 6. From the figures the above statement is validated. The proposed method can be an alternative solver for first order stiff initial value problem of ordinary differential equation.

6 CONCLUSION

A robust diagonally implicit block method with two off-step points for solving stiff initial value problem of ordinary differential equation was derived. The method can generate four solution values at a time per step. The properties of the proposed method has been checked, the method is found to be Zero and A-stable, capable of solving stiff initial value problems of ordinary differential equations. Some selected problem validated the performance of proposed method in terms of accuracy of the scale error and executional time. Hence, the proposed method can be an alternative solver of first order system of stiff IVP of ordinary differential equations.

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