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Some Properties of Group Representation over Modules

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ABSTRACT

Representation theory is the parts of advanced topics in abstract algebra that deal with groups. Reperesentation theory in general facilitate the problems on abstract algebra by transforming into linear algebra form. There are some cases of representation theory which can be expressed as modules over ring. Let G be a group and V be a vector space over field, F. The representation of group G is a homomorphism $\gamma : G \to GL(V)$, where $GL(V)$ is invertible automorphism from V to itself.. In this study, the representation of group was generalized by exchanging the vector spaces with modules. Furthermore, the aim of this study is not only to generalize the representation of group over vector space but also to investigate conditions that formed on representation of group over modules. Results regarding the properties of representation of group over modules have been obtained in this study.

Keywords: group representation; vector spaces; group theory.

1. INTRODUCTION

Representation theory is the parts of advanced topics in abstract algebra that deal with groups. Let G be a group and V be a vector space over F. Set of linear transformation from V to itself is called endomorphism of V denoted by $End(V)$ and if the linear transformation satisfies the bijective condition then it is called automorphism denoted by $Aut(V)$. In the representation theory, the automorphism need to be invertible and denoted by $GL(V)$. The representation of group G over V is a homomorphism $\gamma : G \to GL(V)$.

Based on the definition of representation, it is clear that a representation depends on the groups and the representation space. Some terminology of representation theory depends on the subspaces such as decomposable representation, irreducible representation, and etc. In addition, there are some basic concepts in the representation of groups such as invariant, and equivalence of two representations.

Module is an algebraic structure built from an Abelian group and a ring. Basically, module is generalization of vector space which is the scalar multiplication element of ring rather than field. There are many kinds of modules such as torsion modules, free-torsion modules, free modules, Noetherian modules, and etc. The type of the modules depends on the submodules.

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In this study, the representation space was generalized from vector spaces to modules and then the validity of the basic concept and properties in representation theory over vector spaces for representation theory over modules are proven. The result of this study is expected to facilitate the understanding of the representation theory.

2. PREMILINARY

In this section, some definitions used in this paper are listed. In this paper, the representation of group over vector spaces will be generalized to the representation of group over modules. Thus, every definition of the representation of group over vector spaces will be generalized to modules and some properties related to representation of group over modules are stated.

Let R be a ring and M is $R-$ module, a subset N of M is said to be a submodule of M if N is a subgroup of group M and also an $R-$ module by the scalar multiplication on M [1].

Definition 1: Noetherian Module [2].

A module M is said to be Noetherian module if for every submodules of M satisfy the ascending chain condition.

Definition 2: Free Module [1].

A module M is said to be free module if for a subset N of M satisfy:

i. N generates M as an $R-$ module, that is for every $x \in M$ can be written as

$$x = a_1x_1 + a_2x_2 + \ldots + a_nx_n$$

for some $x_1, x_2, \ldots, x_n \in A$ and $a_1, a_2, \ldots, a \in R$.

ii. N linear independent, that is for $x_1, x_2, \ldots, x_n \in A$ and $a_1, a_2, \ldots, a \in R$ then

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$$

only met for $a_1 = a_2 = \ldots = a_n = 0$.

Let G be a finite group and V be a vector space over F. A representation of group G over V is a homomorphism $\gamma : G \to GL(V)$. In this paper, the vector spaces V was generalized to be modules M such that the representation of group G over M is a homomorphism $\gamma : G \to GL(M)$. Need to be remember that $GL(M)$ is an invertible automorphism of M to itself with $\gamma(g) \cdot m = g \cdot m$ for every $m \in M$ and $g \in G$ [3].

Definition 3: Equivalence Representation [3].

Let $\alpha_1 : G \to GL(M_1)$ and $\alpha_2 : G \to GL(M_2)$ be the representation group over modules. The two representation, α_1 and α_2 are equivalent if there is an isomorphism $f : M_1 \to M_2$ such that $f\alpha_1(g) = \alpha_2(g)f$ for every $g \in G$.

Definition 4: Invariant [4].

Let $\alpha : G \to GL(M)$ be a representation group over modules and N submodules of M. For every $g \in G$ and $n \in N$ such that $\alpha(g)n \in N$ then N is called invariant over representation α or $G-$invariant.
For every N submodules of M and $G-$invariant, a group representation $\beta : G \to GL(N)$ can be formed and it is called subrepresentation.

Let $\alpha : G \to GL(M)$ be a representation group over modules and N_1, N_2 are $G-$invariant, then the following condition are satisfy:

i. $N_1 + N_2$ is $G-$invariant.
ii. $N_1 \cap N_2$ is $G-$invariant.

But, the union of N_1, N_2 is not necessarily $G-$invariant.

Definition 5: Terminology Representation [3].

Let $\alpha : G \to GL(M)$ be a representation group over modules, then the following conditions are valid for representation α,

i. Representation α is called decomposable if the modules M can be expressed as the direct sum of $M = N_1 \oplus N_2$, for every N_1, N_2 are $G-$invariant submodules.
ii. Representation α is called irreducible if there is no nontrivial $G-$invariant submodules or the submodules only $\{0\}$ and M.
iii. Representation α is called completely reducible if for every $G-$invariant submodules can be expressed as the direct sum of irreducible $G-$invariant submodules of M.

3. GROUP REPRESENTATION OVER MODULES

In this section, the result of this study will be discussed. With regards to the equivalence properties of the representation, the following proposition has been obtained for equivalence of representation group over modules.

Proposition 1.

Let $\alpha_1 : G \to GL(M_1)$ and $\alpha_2 : G \to GL(M_2)$ are representation of group over modules. If M_1 and M_2 are free modules with the same dimension then representation α_1 and α_2 are equivalent.

Proof. Notice that M_1 and M_2 are free modules, so obviously having bases. Let $f : M_1 \to M_2$ as a modules homomorphism. Because the free modules have the same dimension then there exist isomorphism f such that $f(g \cdot m) = g \cdot f(m)$, then for every $g \in G$, $m \in M_1$, and $n \in M_2$ satisfy

$$(f\alpha_1(g))(m) = f(\alpha_1(g) \cdot m) = f(g \cdot m) = g \cdot f(m) = \alpha_2(g) f(m) = (\alpha_2(g) f)(m).$$

This mean that $f\alpha_1(g) = \alpha_2(g) f$. Based on the Definition 3, the representation α_1 and α_2 are equivalent.

Next, some facts about representation of group over modules will be given.

Proposition 2.

Let $\alpha : G \to GL(M)$ be a representation of group over modules and M is simple modules then α is irreducible representation also indecomposable representation.

Proof. It is clear because simple modules have no nonzero proper submodules and the only submodules are $\{0\}$ and M, therefore modules M cannot be expressed as direct sum of $G-$ invariant submodules.

Proposition 3.

Let $\alpha : G \to GL(M)$ be a representation of group over modules and M is free modules then α is decomposable representation.

Proof. Note that M is free modules, it mean M having a base. Let $J = \{e_1, e_2, \ldots, e_n\}$ be base of modules M, then there are $M_1, M_2, M_3, \ldots, M_n$ be subsets of M such that can be performed $M_1e_1, M_2e_2, M_3e_3, \ldots, M_ne_n$ submodules of M. Consequently, $M = M_1e_1 \oplus M_2e_2 \oplus M_3e_3 \oplus \ldots \oplus M_ne_n$ is obtained. Based on the Definition 5 (i), the representation of group over free modules is a decomposable representation.

Example 1. Let R be a ring and m, n be integers then $M_{m\times n}(R)$ be an $R-$ modules, then every representation of group over $M_{m\times n}(R)$ is decomposable representation.

- $M_{m\times n}(R)$ is free modules with $M_{m\times n}(E) = \{[E_{ij}] \mid i \in \{1,2,\ldots,n\}, j \in \{1,2,\ldots,m\}\}$ as the base [5]. Every $A \in M_{m\times n}(R)$ can be written as $A = RE_{11} + RE_{12} + RE_{13} \ldots + RE_{ij} + \ldots + RE_{mn}$. Because $M_{m\times n}(E)$ base of the $M_{m\times n}(R)$ then modules $M_{m\times n}(R)$ can be written as $M_{m\times n}(R) = RE_{11} \oplus RE_{12} \oplus RE_{13} \ldots \oplus RE_{ij} \oplus \ldots \oplus RE_{mn}$.

Let G be a finite group and M be Noetherian modules, then $\alpha : G \to GL(M)$ is a representation of group over Noetherian modules. The following representation of group condition fulfilled for Noetherian modules.

Lemma 1.

Let L be the maximal submodules of M and L is $G-$ invariant then every submodules of M also $G-$ invariant.

Proof. Let $L_1, L_2, L_3, L_4, L_5, \ldots, L_n$ be submodules of M. Notice that M is Noetherian modules, based on Definition 1, every submodules of M satisfy the ascending chain condition such that $L_1 \subseteq L_2 \subseteq L_3 \subseteq L_4 \subseteq L_5 \subseteq \ldots$ and it became stationary in maximal submodules L. Because submodules L is $G-$ invariant, so does the other submodules of M.

Corollary 1.

Let $\alpha: G \rightarrow GL(M)$ as a representation of group over Noetherian modules, then every subrepresentation formed from submodules of Noetherian modules M always contains $G-$invariant for every submodules.

Proof. Notice that submodules of submodules is also submodules and as the result of the definition of Noetherian modules, every submodules of Noetherian modules is also Noetherian. Based on Lemma 1, every submodules of Noetherian modules M are $G-$invariant. It is clear that every subrepresentation from representation α contains $G-$invariant.

4. CONCLUSION

Representation theory is parts of advanced topics in abstract algebra that deal with group. The representation of group G is a homomorphism $\gamma: G \rightarrow GL(V)$, where $GL(V)$ is an invertible automorphism from V to itself. The representation of group was generalized by exchanging the vector space V with module M. By changing the modules with free modules, it has been obtained that two representations of group over modules are equivalent if the dimension of the free modules are same. In addition, the representation of group over free modules is decomposable representation. Furthermore, it has been shown that for representation of group over Noetherian modules, the submodules are $G-$invariant if the maximal submodules is $G-$invariant.

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