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A Two-Warehouse Inventory Model for Maximum Lifetime Items Under Supplier's Trade Credit

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ABSTRACT

In this study, an inventory model is developed considering deteriorating items that have maximum lifetime in a two-warehouse environment. The dispatching policy adopted is First-in First-out (FIFO) against Last-in First-out (LIFO) due to the fact that freshness of items is considered more important than economic reasons. Trade credit was incorporated into the model to make it more practical. With the help of several realistic cases, cost functions were obtained. Gradient method was used to show the convexity of the cost functions and numerical example is given as an illustration of the model. From the sensitivity analysis, it was found that the bigger the lifetime of an item, the smaller the total cost incurred by the retailer.

Keywords: First-in-First-out, Last-in-First-out, Maximum lifetime items, Two-warehouse.

1 INTRODUCTION

In the past few decades, inventory problems for deteriorating items have been widely studied. Deterioration is the damage such that the item cannot be used for its original purpose. Most of the physical goods undergo decay or deterioration over time. Commodities such as fruits, vegetables, foodstuffs, drugs and so on, suffer from depletion by the direct spoilage while kept in store. Thus, decay or deterioration of physical goods in stock is a very realistic feature and inventory modelers felt the need to take this factor into consideration while developing inventory policies.

Many works have been carried out concerning the control of deteriorating inventory items. It is a common assumption in many inventory systems that products have indefinitely long lives. However, almost all items deteriorate over time but in some, the rate of deterioration is low and there is little need to consider the deterioration when determining economic lot size. The work in [1] presented the first economic order quantity (EOQ) model where the demand is constant, the warehouse is single, shortages are not allowed, trade credit not considered, and the item is non-deteriorating. The authors [2] were the first to extend [1] to the case of deteriorating items. Since then, a lot of models have been developed in the literature of EOQ models.

The concept of two-warehouse was adopted in order to make appropriate policy in a business situation when the retailer ordered consignment more than the stocking capacity of his/her own warehouse. In that situation, the retailer rented another warehouse with abundant space to keep the excess of the goods ordered. After stocking, to sell or dispatch, researchers have based their argument on which warehouse to sell the goods first.

In [3] goods are sold first from the rented warehouse before the own warehouse because of economic reasons which dispatching policy was assumed to be Last-in First-out (LIFO). For more on the works that considered LIFO, see [4] [5] [6] and [7].

All these models are developed considering the items to be continuously deteriorating. However, there are situations in which the items have maximum lifetime. In this regard, First-in First-out (FIFO) dispatching policy is mostly considered, against the LIFO, because of freshness of item. The work of [8] has shown that a FIFO issuing policy is optimal for perishable and deteriorating inventories in a single warehouse setting with unlimited capacity. The work in [9] in developed two-warehouse inventory model with FIFO dispatching policy in order to enhance the freshness of merchandise. In [10], developed an inventory model for deteriorating items under the FIFO dispatching policy the rented warehouse is assumed to charge a higher unit holding cost than the own warehouse. The rate of replenishment is finite, and shortages are allowed. In the model, minimum total relevant costs were determined as well as optimal reorder points and cycle times. In [11], the effect of future price increase for products with expiry dates and price sensitive demand under different payment policies was considered. The work in [12] developed an analytical solution of a modified single item continuous production model under constant deterioration, goal level and penalties.

2 MATHEMATICAL MODEL FORMULATION

2.1 Notations and Assumptions

The following are the notations and assumptions used in the model:
$I_o(t)$, $I_r(t)$ are the inventory levels of the own warehouse (OW) and rented warehouse (RW) respectively at time t.
Z is the stocking capacity of OW.
I_m is the total order quantity.
D is the constant demand rate.
t_1, T are the time at which inventory in OW and that in RW drop to zero respectively.
$\alpha(t)$ and $\beta(t)$ are the deterioration rates in OW and RW respectively with $\alpha(t) = \beta(t)=\frac{1}{1+m-t}$, where m is the maximum lifetime period (age) of an item before it expired completely.
h_o, h_r are the holding cost per unit item per unit time at OW and RW respectively.
A is ordering cost per order.
c, p are the purchasing cost per unit item and selling price of the item respectively.
TC is the total relevant costs per year.
M is the trade credit periods offered to the retailer by the supplier.
I_e, I_p are the interest earned and interest payable by the retailer respectively.

The following are assumed in the development of the model:

i. The model considers single items.
ii. The demand rate is assumed to be constant.
iii. The lead time is zero and shortages are not allowed.
iv. The OW has limited capacity and RW has unlimited capacity.
v. The dispatching policy is FIFO because of the freshness of items.

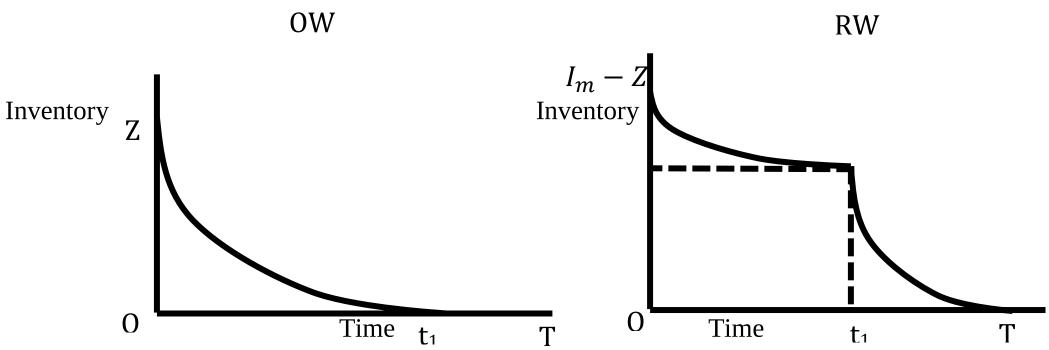

Figure 1: Pictorial presentation of the FIFO model.

2.2 Model Formulation

Goods ordered are stocked in OW first with the excess going to RW. Because the model follows FIFO, goods are retrieved from OW at first. Therefore, at $t = t_1$ the inventory at OW drops to zero due to demand and deterioration during the period $[0, t_1]$, while goods in RW during that period are depleted due to deterioration only. At $t = T$, both warehouses become empty due to depletion in RW by demands and deterioration during the period $[t_1, T]$.

These phenomena are represented by the following differential equations:

$$\frac{dI_o(t)}{dt} + \frac{1}{1+m-t} I_o(t) = -D \qquad 0 \leq t \leq t_1 \tag{1}$$

with boundary condition $I_o(t_1) = 0$

$$\frac{dI_r(t)}{dt} + \frac{1}{1+m-t} I_r(t) = 0 \qquad 0 \leq t \leq t_1 \tag{2}$$

with initial condition $I_r(0) = I_m - Z$

$$\frac{dI_r(t)}{dt} + \frac{1}{1+m-t} I_r(t) = -D \qquad t_1 < t \leq T \tag{3}$$

with boundary condition $I_r(T) = 0$

The solutions to equations (1), (2) and (3) are respectively given as

$$I_o(t) = (1 + m - t) D \ln\left(\frac{1+m-t}{1+m-t_1}\right), \qquad 0 \leq t \leq t_1 \tag{4}$$

$$I_r(t) = \frac{I_m - Z}{(1+m)}(1+m-t), \qquad 0 \le t \le t_1 \qquad (5)$$

$$I_r(t) = (1+m-t)D\ln\left(\frac{1+m-t}{1+m-T}\right) \qquad t_1 < t \le T \qquad (6)$$

In RW, continuity must be observed. Therefore, putting $t = t_1$ in Equations (5) and (6) and equating them, we find that

$$I_m = (1+m)D\ln\left(\frac{1+m-t_1}{1+m-T}\right) + Z \qquad (7)$$

which is the order quantity.

2.3 Annual Total Costs

The relevant costs in this model are

i. Ordering cost per order, OC;
ii. Stock holding cost per year, HC;
iii. Interest payable I_P and Interest earned I_E by the retailer.

Consequently, the total relevant costs per year is given by

$$TC(t_1, T) = OC + HC + I_P - I_E \qquad (8)$$

Evaluating the physical quantities we have

Ordering cost per order $= \frac{A}{T}$ (9)

Stock holding cost per year

The holding cost for the model

$$HC = \frac{Dh_0}{2T}\left[\frac{t_1^2}{2} - mt_1 - t_1 + (m+1)^2\ln\left(\frac{-m-1}{t_1-m-1}\right)\right] + \frac{Dh_r}{2T}\left[\frac{1}{2}(T^2 - t_1^2) - (T-t_1) - m(T-t_1) + (m+1)^2\ln\left(\frac{t_1-m-1}{T-m-1}\right)\right] \qquad (10)$$

Interest payable and Interest earned by the retailer

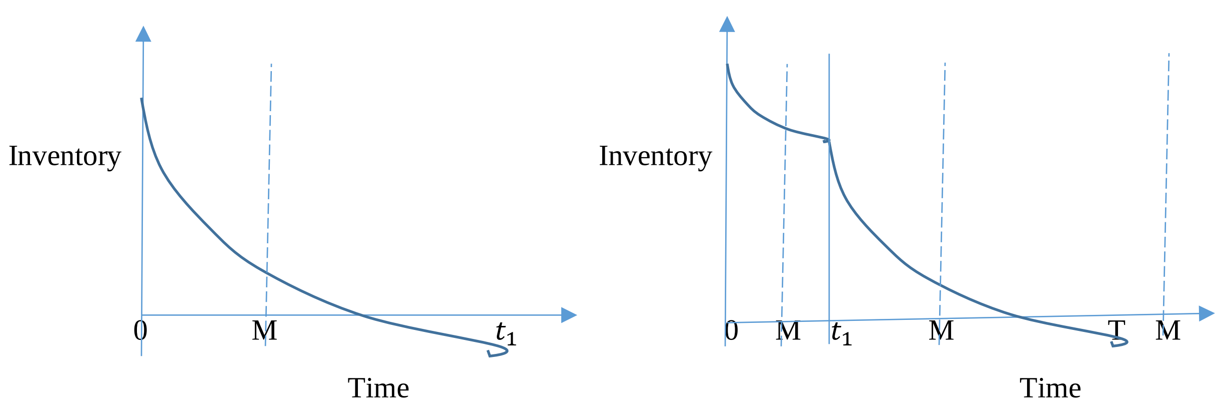

Figure 2: Graphical representation of the likely position of the trade credit period M.

To obtain the annual interest earned and annual interest payable by the retailer, three cases may arise; as shown in Fig. 2.

Case 1: $M \le t_1 < T$ (Trade credit period expired before the goods in OW finishes).

After the expiration of the trade credit period M, the retailer would pay interest on all unsold items in the warehouses which is given as:

$$IP_1 = \frac{cI_p}{T}\left[\int_M^{t_1} I_0(t)dt + \int_M^{t_1} I_r(t)dt + \int_{t_1}^{T} I_r(t)dt\right]$$

Using equation (4), (5) and (6), we see that

$$IP_1 = \frac{cI_pD}{T}\left(-\left(M + mM - \frac{M^2}{2}\right) ln\left(\frac{1+m-M}{1+m-t_1}\right) + \frac{(m+1)^2}{2} ln\left(\frac{M-m-1}{T-m-1}\right) + \frac{M}{2} + \frac{Mm}{2} - \frac{M^2}{4} - \frac{T}{2} - \frac{mT}{2} + \frac{T^2}{4}\right) \quad (11)$$

Likewise, the retailer has sold some items before the trade credit period expired, during the period $[0, M]$ and therefore earned interest on the sales revenue generated. Hence the interest earned by the retailer is given as:

$$IE_1 = \frac{pI_e}{T}\int_0^M Dtdt = \frac{pI_eDM^2}{2T} \quad (12)$$

Case 2: $t_1 < M \le T$ (Trade credit period is greater than the period when goods in OW finishes but before those in RW finishes)

During this period, the goods OW have finished, therefore, the retailer has to pay interest for the unsold items in RW after the time M, which is, using equation (6), given by:

$$IP_2 = \frac{cI_p}{T}\int_M^T I_r(t)dt = \frac{cI_pD}{T}\left(\frac{T^2}{4} - \frac{T}{2} - \frac{mT}{2} + \frac{M}{2} + \frac{Mm}{2} - \frac{M^2}{4} + \frac{(m+1)^2}{2} ln\left(\frac{M-m-1}{T-m-1}\right) - ln\left(\frac{1+m-M}{1+m-T}\right)\left(M + mM - \frac{M^2}{2}\right)\right) \quad (13)$$

In the same vein, the retailer will earn interest on the sales revenue generated during the period $(0, M)$ and is given by:

$$IE_2 = \frac{pI_e}{T}\int_0^M Dtdt = \frac{pI_eDM^2}{2T} \quad (14)$$

Case 3: $T < M$ (Trade credit period is greater than the replenishment cycle)

In this case, the items in both warehouses have finished before time M, therefore there is no interest payable and is given as,

$$IP_3 = 0 \tag{15}$$

However, the retailer has sold all items and generated revenue. So, the retailer would earn interest on the sales revenue generated during the period $(0, M)$ and is given by:

$$IE_3 = \frac{pI_e}{T}\left[\int_{t_1}^{T} Dtdt + DT(M-T)\right] = \frac{pI_eD}{2T}[TM - t_1^2] \tag{16}$$

Based on the cases, the TC is given as stated in equation (2.8) is

$$\text{TC}(t_1\ ,T)=\begin{cases} TC_1, & M < t_1 < T \\ TC_2, & t_1 < M < T \\ TC_3, & M > T \end{cases}$$

Using equations (9), (10), (11) and (12), we obtained,

$$TC_1 = \frac{1}{T}\left(A + \frac{Dh_0}{2}\left(\frac{t_1^2}{2} - mt_1 - t_1 + (m+1)^2 ln\left(\frac{-m-1}{t_1-m-1}\right)\right) + \frac{Dh_r}{2}\left(\frac{1}{2}(T^2 - t_1^2) - (T-t_1) - m(T - t_1) + (m+1)^2 ln\left(\frac{t_1-m-1}{T-m-1}\right)\right) + cI_pD\left(-\left(M + mM - \frac{M^2}{2}\right) ln\left(\frac{1+m-M}{1+m-t_1}\right) + \frac{(m+1)^2}{2} ln\left(\frac{M-m-1}{T-m-1}\right) + \frac{M}{2} + \frac{Mm}{2} - \frac{M^2}{4} - \frac{T}{2} - \frac{mT}{2} + \frac{T^2}{4}\right) - \frac{pI_eDM^2}{2}\right) \tag{17}$$

Again, using equations (9), (10), (13) and (14), we found that,

$$TC_2 = \frac{1}{T}\left(A + \frac{Dh_0}{2}\left(\frac{t_1^2}{2} - mt_1 - t_1 + (m+1)^2 \ln\left(\frac{-m-1}{t_1-m-1}\right)\right) + \frac{Dh_r}{2}\left(\frac{1}{2}(T^2 - t_1^2) - (T-t_1) - m(T - t_1) + (m+1)^2 ln\left(\frac{t_1-m-1}{T-m-1}\right)\right) + cI_pD\left(\frac{T^2}{4} - \frac{T}{2} - \frac{mT}{2} + \frac{M}{2} + \frac{Mm}{2} - \frac{M^2}{4} + \frac{(m+1)^2}{2} ln\left(\frac{M-m-1}{T-m-1}\right) - \left(M + mM - \frac{M^2}{2}\right) ln\left(\frac{1+m-M}{1+m-T}\right)\right) - \frac{pI_eDM^2}{2}\right) \tag{18}$$

Also, using equations, (9), (10), (15) and (16), we see that

$$TC_3 = \frac{1}{T}\left(A + \frac{Dh_0}{2}\left(\frac{t_1^2}{2} - mt_1 - t_1 + (m+1)^2 ln\left(\frac{-m-1}{t_1-m-1}\right)\right) + \frac{Dh_r}{2}\left(\frac{1}{2}(T^2 - t_1^2) - (T-t_1) - m(T - t_1) + (m+1)^2 ln\left(\frac{t_1-m-1}{T-m-1}\right)\right) - \frac{pI_eD}{2}[TM - t_1^2]\right) \tag{19}$$

3 ANALYSIS ON OPTIMIZATION

Since we are dealing with two decision variables t_1 and T, gradient method to establish the necessary and sufficient conditions for the existence and uniqueness of the optimal solutions.

The necessary conditions for TC_1 to be minimized are $\frac{\partial TC_1}{\partial t_1} = 0$ and $\frac{\partial TC_1}{\partial T} = 0$

Differentiating equation (17) with respect to t_1 and T simplifying and setting the result to zero, we respectively obtain

$$\frac{D}{2T(t_1-m-1)}\left\{t_1(t_1-2m-2)(h_o+h_r)+2cI_p\left(M+mM-\frac{M^2}{2}\right)\right\}=0 \tag{20}$$

and

$$\frac{1}{T}\left\{\frac{D}{2}\left[\frac{T(T-2m-2)}{(T-m-1)}\right]\left(h_r+cI_p\right)-TC_1\right\}=0 \tag{21}$$

The solution to the highly non-linear equations (20) and (21) give the values of t_1 and T. To confirm that the optimal solution (t_1^*, T^*), exist and unique, we show that

$$\left[\left(\frac{\partial^2 TC_1}{\partial T^2}\right)\left(\frac{\partial^2 TC_1}{\partial {t_1}^2}\right)-\left(\frac{\partial^2 TC_1}{\partial T\partial t_1}\right)\left(\frac{\partial^2 TC_1}{\partial t_1\,\partial T}\right)\right]\Bigg|_{(t_1^*,T^*)}>0$$

Differentiating (20) further with respect to t_1, we get

$$\frac{\partial^2 TC_1}{\partial t_1^2}=\frac{D(h_o+h_r)}{T}-\frac{1}{(t_1-m-1)}\left\{\frac{\partial TC_1}{\partial t_1}\right\}$$

Evaluating the immediate past equation at (t_1^*, T^*), we see that

$$\frac{\partial^2 TC_1}{\partial t_1^2}=\frac{D(h_o+h_r)}{T}-\frac{1}{(t_1-m-1)}\left\{\frac{\partial TC_1}{\partial t_1}\right\}>0 \tag{22}$$

since $\frac{\partial TC_1}{\partial T}\Big|_{(t_1^*,T^*)}=0$ from the necessary condition.

Differentiating (20) further with respect to T, and simplifying, we get

$$\frac{\partial^2 TC_1}{\partial T\partial t_1}=\frac{1}{T}\left\{-\frac{\partial TC_1}{\partial t_1}\right\}=0 \tag{23}$$

when evaluated at (t_1^*, T^*)

Differentiating (21) further with respect to T, we get

$$\frac{\partial^2 TC_1}{\partial T^2}=\frac{1}{T}\left\{\frac{D}{2}\left[\frac{(T-m-1)^2+(m+1)^2}{(T-m-1)^2}\right]\left(h_r+cI_p\right)-2\frac{\partial TC_1}{\partial T}\right\}$$

Evaluating the immediate past equation at (t_1^*, T^*), we see that

$$\frac{\partial^2 TC_1}{\partial T^2}=\frac{1}{T}\left\{\frac{D}{2}\left[\frac{(T-m-1)^2+(m+1)^2}{(T-m-1)^2}\right]\left(h_r+cI_p\right)-2\frac{\partial TC_1}{\partial T}\right\}\Bigg|_{(t_1^*,T^*)}$$

$$= \frac{1}{T}\left\{\frac{D}{2}\left[\frac{(T-m-1)^2+(m+1)^2}{(T-m-1)^2}\right]\left(h_r + cI_p\right) - 2\frac{\partial TC_1}{\partial T}\right\} > 0 \tag{24}$$

since $\left.\frac{\partial TC_1}{\partial T}\right|_{(t_1^*,T^*)} = 0$

Differentiating (21) further with respect to t_1, and simplifying, we get

$$\frac{\partial^2 TC_1}{\partial t_1 \partial T} = -\frac{1}{T}\left\{\frac{\partial TC_1}{\partial t_1}\right\} = 0 \tag{25}$$

when evaluated at (t_1^*, T^*).

The determinant of the two principal minors of the Hessian matrix, from equations (22) - (25) are positive, which implies the Hessian matrix is positive definite.

$$\left.\left[\left(\frac{\partial^2 TC_1}{\partial T^2}\right)\left(\frac{\partial^2 TC_1}{\partial {t_1}^2}\right) - \left(\frac{\partial^2 TC_1}{\partial T \partial t_1}\right)\left(\frac{\partial^2 TC_1}{\partial t_1 \partial T}\right)\right]\right|_{(t_1^*,T^*)} > 0$$

which shows that the cost function TC_1 is convex.

The necessary conditions for TC_2 to be minimized are $\frac{\partial TC_2}{\partial t_1} = 0$ and $\frac{\partial TC_2}{\partial T} = 0$

Using equation (18), differentiating with respect to t_1 and T, simplifying and setting the result equal to zero, we respectively see that,

$$\frac{D}{2T(t_1-m-1)}[t_1(t_1 - 2m - 2)](h_0 - h_r) = 0 \tag{26}$$

and

$$\frac{1}{T}\left\{\frac{D}{2(T-m-1)}\left([T(T-2m-2)]\left(h_r + cI_p\right) + cI_p(2M + 2mM - M^2)\right) - TC_2\right\} = 0 \tag{27}$$

The solution to the highly non-linear equations (26) and (27) gives the values of t_1 and T . To confirm that the optimal solution (t_1^*, T^*), exist and unique, we show that

$$\left.\left[\left(\frac{\partial^2 TC_2}{\partial T^2}\right)\left(\frac{\partial^2 TC_2}{\partial {t_1}^2}\right) - \left(\frac{\partial^2 TC_2}{\partial T \partial t_1}\right)\left(\frac{\partial^2 TC_2}{\partial t_1 \partial T}\right)\right]\right|_{(t_1^*,T^*)} > 0$$

Differentiating equation (26) further with respect to t_1 and simplifying, we get

$$\frac{\partial^2 TC_2}{\partial {t_1}^2} = \frac{D}{2T(t_1-m-1)^2}\left(\left(t_1 - (m+1)\right)^2 + (m+1)^2\right)(h_0 - h_r) > 0 \tag{28}$$

when evaluated at (t_1^*, T^*).

Again, differentiating equation (26) with respect to T, we obtain

$$\frac{\partial^2 TC_2}{\partial T \partial t_1} = -\frac{1}{T}\left\{\frac{\partial TC_2}{\partial t_1}\right\} \quad = 0 \tag{29}$$

when evaluated at (t_1^*, T^*).

Differentiating equation (27) further with respect to T and simplifying, we get

$$\frac{\partial^2 TC_2}{\partial T^2} = \frac{1}{T}\left\{\frac{D}{2(T-m-1)^2}\left([(T-m-1)^2 + (m+1)^2](h_r + cI_p) + cI_p(M^2 - 2M - 2mM)\right) - 2\frac{\partial TC_2}{\partial T}\right\}$$

Evaluating the immediate past equation at (t_1^*, T^*), we see that

$$\left.\frac{\partial^2 TC_2}{\partial T^2}\right|_{(t_1^*,T^*)} = \frac{1}{T}\left\{\frac{D}{2(T-m-1)^2}\left([(T-m-1)^2 + (m+1)^2](h_r + cI_p) + cI_p(M^2 - 2M - 2mM)\right) - 2\frac{\partial TC_2}{\partial T}\right\}\Bigg|_{(t_1^*,T^*)}$$

since $\left.\frac{\partial TC_2}{\partial T}\right|_{(t_1^*,T^*)} = 0$ from the necessary condition,

$$\frac{1}{T}\left\{\frac{D}{2(T-m-1)^2}\left([(T-m-1)^2 + (m+1)^2](h_r + cI_p) + cI_p(M^2 - 2M - 2mM)\right)\right\} > 0 \tag{30}$$

From simple algebra,

$$[(T-m-1)^2 + (m+1)^2](h_r + cI_p) > \ cI_p(M^2 - 2M - 2mM)$$

therefore,

$$\frac{1}{T}\left\{\frac{D}{2(T-m-1)^2}\left([(T-m-1)^2 + (m+1)^2](h_r + cI_p) + cI_p(M^2 - 2M - 2mM)\right)\right\} > 0$$

Now, differentiating equation (27) further with respect to t_1, and simplifying, we find that

$$\frac{\partial^2 TC_2}{\partial t_1\, \partial T} = \frac{1}{T}\left\{-\frac{\partial TC_2}{\partial t_1}\right\} = 0 \tag{31}$$

when evaluated at (t_1^*, T^*).

The determinant of the two principal minors of the Hessian matrix, from equations (28) - (31) are positive, which implies the Hessian matrix is positive definite.

$$\left[\left(\frac{\partial^2 TC_2}{\partial T^2}\right)\left(\frac{\partial^2 TC_2}{\partial {t_1}^2}\right) - \left(\frac{\partial^2 TC_2}{\partial T \partial t_1}\right)\left(\frac{\partial^2 TC_2}{\partial t_1\, \partial T}\right)\right]\Bigg|_{(t_1^*,T^*)} > 0$$

Which shows that the cost function TC_2 is a convex function.

The necessary conditions for TC_3 to be minimized are $\frac{\partial TC_3}{\partial t_1} = 0$ and $\frac{\partial TC_3}{\partial T} = 0$

Differentiating equation (19) with respect to t_1 and T, simplifying and setting the result to zero, we respectively obtain

$$\frac{D}{2T(t_1-m-1)}\{(h_o - h_r + 2pI_e){t_1}^2 - 2(m+1)(h_o - h_r + pI_e)t_1\} = 0 \tag{32}$$

and

$$\frac{1}{T}\left\{\frac{Dh_r}{2}\left[\frac{T(T-2m-2)}{(T-m-1)}\right] - \frac{pI_eDM}{2} - TC_3\right\} = 0 \tag{33}$$

The solution to the highly non-linear equations (32) and (33) gives the values of t_1 and T. To confirm that the optimal solution (t_1^*, T^*) exist and unique, we show that

$$\left[\left(\frac{\partial^2 TC_3}{\partial T^2}\right)\left(\frac{\partial^2 TC_3}{\partial {t_1}^2}\right) - \left(\frac{\partial^2 TC_3}{\partial T \partial t_1}\right)\left(\frac{\partial^2 TC_3}{\partial t_1\,\partial T}\right)\right]\Bigg|_{(t_1^*, T^*)} > 0$$

Differentiating equation (32) further with respect to t_1 and simplifying, we get

$$\frac{\partial^2 TC_3}{\partial {t_1}^2} = \frac{D}{T(t_1-m-1)}\left[\{(h_o - h_r + 2pI_e)t_1 - (m+1)(h_o - h_r + pI_e)\} - \frac{T}{D}\left\{\frac{\partial TC_3}{\partial t_1}\right\}\right]$$

$$\frac{\partial^2 TC_3}{\partial {t_1}^2} =$$

$$\frac{D}{T(t_1-m-1)}\left[\left\{(h_o - h_r + 2pI_e)\frac{2(m+1)(h_o-h_r+pI_e)}{(h_o-h_r+2pI_e)} - (m+1)(h_o - h_r + pI_e)\right\}\right] > 0 \tag{34}$$

when evaluated at (t_1^*, T^*) since $\frac{\partial TC_3}{\partial t_1}\Big|_{(t_1^*, T^*)} = 0$

Differentiating equation (32) further with respect to T and simplifying, we get

$$\frac{\partial^2 TC_3}{\partial T \partial t_1} = -\frac{1}{T}\left\{\frac{\partial TC_3}{\partial t_1}\right\} \qquad = 0 \tag{35}$$

when evaluated at (t_1^*, T^*).

Differentiating equation (33) further with respect to T and simplifying, we get

$$\frac{\partial^2 TC_3}{\partial T^2} = \frac{1}{T}\left\{\frac{Dh_r}{2}\left[\frac{\left(T-(m+1)\right)^2 + (m+1)^2}{(T-m-1)^2}\right]\right\} > 0 \tag{36}$$

since $\frac{\partial TC_3}{\partial T}\Big|_{(t_1^*, T^*)} = 0$ when evaluated (t_1^*, T^*).

Differentiating equation (33) further with respect to t_1 and simplifying, we get

$$\frac{\partial^2 TC_3}{\partial t_1\,\partial T} = \quad \frac{1}{T}\left\{-\frac{\partial TC_3}{\partial t_1}\right\} = 0 \tag{37}$$

when evaluated at (t_1^*, T^*).

The determinant of the two principal minors of the Hessian matrix, from equations (28) - (31) are positive, which implies the Hessian matrix is positive definite.

$$\left[\left(\frac{\partial^2 TC_3}{\partial T^2}\right)\left(\frac{\partial^2 TC_3}{\partial {t_1}^2}\right) - \left(\frac{\partial^2 TC_3}{\partial T \partial t_1}\right)\left(\frac{\partial^2 TC_3}{\partial t_1\,\partial T}\right)\right]\Bigg|_{(t_1^*, T^*)} > 0$$

Which shows that the cost function TC_3 is convex

4 NUMERICAL EXAMPLE

Given the parameters, $A = 1500,\ D = 2000,\ M = 0.5,\ m = 1, h_r = 3,\ h_o = 1,\ c = 10,\ p = 15,$ $I_e = 0.15,\ I_p = 0.12$ as obtained in [13].

Implementing the Gradient method on the MATLAB, we obtained $t_1^* = 1.2687$, $T^* = 1.5627$ and $TC^* = 1606.0150$ which present the least case among the three cases. This shows that it is better for the retailer to get a credit period that will be longer than the time when the goods stored in his/her own warehouse will finish.

Below are the results obtained for all the three cases in Table 1 below.

Table 1: Result of the Numerical Example for all the cases

Cases	t_1	T	TC
Case 1 : (Trade credit period expired before the goods in OW finishes)	0.1714	0.4505	4643.3326
Case 2 : (Trade credit period is greater than the period when goods in OW finish but before those in RW finishes)	1.2687	1.5627	1606.0150
Case 3 : (Trade credit period is greater than the replenishment cycle)	1.5440	2.6523	2111.3948

4.1 Sensitivity Analysis

Using the example above, we study the effect of the parameters changes on the optimal values of the t_1, T , and TCs. the results of the sensitivity analysis are shown and presented in Tables 2, 3 and 4.

Table 2 below presents the sensitivity analysis for case 1 when some important variables values are changed.

Table 2: Sensitivity analysis result of case 1

Parameters	%change in parameters	New t_1	% in t_1	New T	% in T	New TC_1	% in TC_1
	+10	0.1714	0	0.4771	5.9046	4966.7243	6.9646
	+5	0.1714	0	1.8723	315.6049	3189.7384	-31.3050
A	+0	0.1714	0	0.4505	0	4643.3326	0
	-5	0.1714	0	1.8717	315.4717	3157.8309	-31.9921
	-10	3.8286	2133.7229	0.6626	47.0810	5727.1950	23.3423
	+10	0.1714	0	1.8712	315.3607	3459.2429	-25.5009
	+5	0.1714	0	0.4370	-2.9967	4706.4734	1.3598

D	+0	0.1714	0	0.4505	0	4643.3326	0
	-5	0.1714	0	0.4648	3.1743	4575.0591	-1.4706
	-10	0.1714	0	1.8727	315.6937	2888.2961	-37.7969
	+10	0.1722	0.4667	1.9651	336.2042	3309.4939	-28.7259
	+5	0.1718	0.2334	1.9185	325.8602	3241.5033	-30.1902
m	+0	0.1714	0	0.4505	0	4643.3326	0
	-5	0.1710	-0.2334	1.8254	305.1942	3106.3742	-33.1003
	-10	3.6295	2017.5613	0.6200	37.6249	5344.8186	15.1074
	+10	0.1866	8.8681	1.8716	315.4495	3156.3166	-32.0249
	+5	3.8210	2129.2882	0.4201	-6.7481	5644.6463	21.5645
M	+0	0.1714	0	0.4505	0	4643.3326	0
	-5	0.1637	-4.4924	0.4575	166.9195	4727.8678	1.8206
	-10	0.1558	-9.1015	1.8723	315.6049	3189.7264	-31.3052

Table 3 below presents the sensitivity analysis for case 2 when some important variables values are changed.

Table 3: Sensitivity analysis result of case 2

Parameters	%change in parameters	New t_1	% in t_1	New T	% in T	New TC_2	% in TC_2
	+10	1.6544	30.4012	0.5647	0.3554	1647.7569	2.5991
	+5	5.6604	346.1575	0.5637	-55.5686	1626.8924	1.2999
A	+0	1.2687	0	0.5627	0	1606.0157	0
	-5	1.5002	18.2470	0.5617	-43.8300	1585.1248	-1.3007
	-10	4.9763	292.2361	0.5607	-0.3554	1564.2218	-2.6023
	+10	4.3789	245.1486	0.5609	-0.3199	1724.8348	7.3984
	+5	1.5041	18.5544	0.5617	-43.8300	1665.4304	3.6996
D	+0	1.22687	0	0.5627	0	1606.0150	0
	-5	3.6279	185.9541	0.5638	0.1955	1546.5867	-3.7004
	-10	1.8075	42.4687	0.5650	0.4087	1487.1433	-7.4017
	+10	1.1900	-6.2032	0.6407	13.8617	1735.9902	8.0930
	+5	2.2207	75.0374	0.6016	6.9131	1673.0535	4.1742
m	+0	1.2687	0	0.5627	0	1606.0150	0
	-5	2.6493	108.8201	0.6277	11.5514	5131.6024	219.5239
	-10	2.2478	77.1735	0.6547	16.3497	5169.0101	221.8532
	+10	1.2811	0.9774	0.6323	12.3689	5213.9530	224.6516

	+5	5.0808	300.4729	0.5572	-0.9774	1597.2619	-0.5450
M	+0	1.2687	0	0.5627	0	1006.0150	0
	-5	7.8216	516.505	0.5681	0.99597	1612.6785	0.4149
	-10	3.9455	210.9876	0.5733	1.8838	1617.3264	0.7043

Table 4 below presents the sensitivity analysis for case 3 when some important variables values are changed.

Table 4: Sensitivity analysis result of case 3

Parameters	%change in parameters	New t_1	% in t_1	New T	% in T	New TC_3	% in TC_3
	+10	1.5002	-2.8368	2.6494	-0.1093	2111.3988	0.00000189
	+5	1.5231	-1.3536	2.6508	-0.0566	2111.3969	0.0000995
A	+0	1.5440	0	2.6523	0	2111.3969	0
	-5	1.5623	1.1852	2.6538	0.0567	2111.3925	-0.0001
	-10	1.5775	2.1697	2.6554	0.1169	2111.3899	-0.0002
	+10	1.5750	2.0078	2.6551	0.1056	2322.5294	9.9997
	+5	1.5617	1.1334	2.6537	0.0528	2216.9623	4.9999
D	+0	1.54440	0	2.6523	0	2111.3948	0
	-5	1.5220	-1.4249	2.6507	-0.0603	2005.8272	-4.9998
	-10	1.4948	-3.1865	2.6490	-0.1244	1900.2592	-9.9998
	+10	2.6145	69.3329	3.1312	18.0560	2213.7692	4.8487
	+5	1.5293	-0.9521	2.6687	0.6183	2112.0236	0.0298
m	+0	1.5440	0	2.6523	0	2111.3969	0
	-5	4.1168	167.0649	2.6393	-0.4901	2110.6509	-0.0352
	-10	3.5982	35.663	2.9471	11.1149	2107.6788	-0.1760
	+10	1.5440	0	2.6523	0	2021.3948	-4.2626
	+5	1.5440	0	2.6523	0	2066.3948	-2.1313
M	+0	1.5440	0	2.6523	0	2111.3948	0
	-5	1.5440	0	2.6523	0	2156.3948	2.1313
	-10	1.5440	0	2.6523	0	2201.3948	4.2626

It can be seen from Tables 2 - 4 that:

i. The bigger the lifetime of an item, the smaller the total annual relevant cost incurred by the retailer.
ii. The total annual relevant cost is linearly decreasing with the increase in the lifespan of the deteriorating item. In real life as the lifespan of the item increases, the number of orders placed per year decreases, since the ordering cost per order is fixed, then the total annual ordering cost decreases linearly as the lifespan increases.
iii. The time at which goods ordered finishes at OW and in both warehouse changes with the increase in the lifetime of an item. This is expected in the real world because the longer an item is available, the more opportunities there are for it to be ordered and shipped from different warehouses.

iv. As demand D increases, t_1^* and T^* decreases while TC^* increases for all cases. This is expected in real life if the initial demand increases then, the retailer will order more goods, which results in more sales and an increase in total cost.

v. As the ordering cost A increases, t_1^*, T^* and TC^* also increases. In practical situations, when ordering cost is high, the retailer tends to order more goods, which leads to an increase in the replenishment cycle time and the total cost.

vi. As the credit period M increases, t_1^*, T^* and TC^*. In the real world this is expected because an increase in the credit period means an increase in the cycle time and an increase in the total cost of items.

5 CONCLUSIONS

In the study, two-warehouse inventory models for maximum lifetime items under supplier's trade credit were developed. FIFO dispatching policy was considered in the model in order to enhance freshness of the merchandise. This model gave us the most favorable replenishment policies for minimizing the total inventory cost. A numerical example is provided to evaluate the proposed model. Sensitivity analysis of the optimal solution with respect to key parameters is carried out. This model which is designed and analyzed can be extended in several ways such as holding cost to be inversely proportion, time varying deterioration rate, time dependent demand, increasing demand rate and so on.

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