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Two-warehouse Inventory Model for Deteriorating Items Under Two - Level Trade Credit Financing

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ABSTRACT

In this study, we consider an inventory system where both retailers and customers are presumed to be credit – worthy. Therefore, the retailer is given full trade credit of M days after taking the delivery of the consignment. Simultaneously, the retailer gives customers full trade credit of N days, where $M > N$. In most business transactions involving trade credit financing, bulk orders are involved which normally exceeds the stocking capacity of the own warehouse (OW) that necessitate the renting of another warehouse (RW) where excess of the goods ordered will be kept, and therefore, upstream and downstream trade credits are proposed to help in stimulating the demand of retailers and customers respectively. This is to decrease the on – hand inventory level which translates to reduction in holding cost. The relevant cost functions for the inventory system are determined and a numerical example is given. Sensitivity analysis is carried out to see the effect of changes in parameters on the optimal solution of the model.

Keywords: Credit – worthy, Holding cost, and Trade Credit.

1 INTRODUCTION

In classical inventory theory, instant payment is considered immediately after delivery of the consignment. This is not always the case in present day market. Some suppliers and retailers adopt some promotional tools to stimulate the demand of retailers and customers respectively. One of such tools considered in the literature is the trade credit. Trade credit is a kind of business transaction in which a certain fixed period of time is offered as a grace where payment of items delivered are not expected until after that fixed period of time. During the permissible delay in payment period, the retailer or the customer can deposit the sales revenue accrued to him/her in an interest bearing account for instance, so as to earn interest on the sales revenue generated. At the expiration of the period, the retailer is charged an interest on the unsold items at a rate higher than the rate of interest earned. Also, apart from stimulating the retailer's or customer's demand, trade credit serves as alternative to price or cash discount.

In the literature, the work of [1] was the first to consider permissible delay in payment between supplier and retailer. The supplier gives the retailer the benefit of the trade credit while the retailer does not offer same to the customers. In the model, Goyal considered selling and purchasing prices to be the same. In short reply to Goyal, [2] pointed it out as a mistake to assume the purchasing and selling prices to be the same. As an extension of Goyal's work, [3] developed an inventory system allowing delay in payment with appropriate purchasing and selling prices.

The work of [4] also extended [1] to consider deteriorating items. In all the models developed, shortages were not allowed. This has not taken care of some business situations, therefore, [5] extended the work of [1] by allowing shortages to occur.

The literature on permissible delay in payment considers the trade credit to be between the supplier and the retailer known as upstream trade credit. However, in some situations, despite the default risk factor associated with trade credit, the retailer also passes the trade credit to the customers as a promotional tool to enhance their demand. This is known as downstream trade credit. Thus, the adoption of both downstream and upstream trade credit simultaneously plays an important role in modern businesses and it is referred as two – level trade credit financing. In line with this, [6] developed an inventory model in which the supplier offers the retailer trade credit and in turn the retailer passes the grace to the customers. The retailer gives $(0, N)$ period to the customers to settle their account. On the other hand, [7] looked at the other perspective of [6] by giving the customers' $N + t$ trade credit period.

As a result of permissible delay in payment, the retailer usually orders goods in large quantity so as to earn much interest over the sales accrued during the allowed period. This translates to higher profit. Also, if the retailer fears scarcity of the item in the near future, bulk orders may be considered. In this situation, the retailer might have excess of goods ordered after exceeding the maximum stocking capacity of the own warehouse (OW) which will necessitate renting another warehouse (RW) of unlimited capacity. This is referred to in the literature, as two – warehouse inventory system. For the authors who worked on two – warehouse with condition of permissible delay in payment; see [8] [9] [10] [11] [12] [13] and so on. All these authors considers upstream trade credit only which suggested the need to incorporate downstream trade credit. This has lots of benefits to the retailer such as decrease in on – hand stock level which translates to reduction in the inventory holding cost. It also results in decrease in the number of deteriorated items especially if the deterioration rate is considered to be quantity dependent.

Due to the large quantity of orders that is involved in two – warehouse inventory system, it is pertinent to check credit riskiness of the parties in the supply chain management. In this regard, [14] investigated the credit riskiness of a customer by proposing partial downstream to curtail the menace. For papers that work on credit riskiness, see [15] [16].

In this study, we are extending the two – warehouse inventory system for deteriorating items considered by [11] to consider two – levels trade credit. That is, the supplier offers the retailer the benefit of permissible delay in payment and the retailer, simultaneously, passes it to the customers.

The structure of the work is, in section 2 Notations and Assumptions are given whereas in section 3, we present the model formulation. In section 4, optimization and analysis is given and in Section 5, we present Numerical example and sensitivity analysis. Conclusion and recommendations are given in the last section.

1.1 Notations and Assumptions

The following are the notations used in the model:

$I_r(t)$, $I_o(t)$ are the inventory levels of the RW and OW respectively at time t.

D is the constant demand rate of the item in each of the warehouses.

W is the maximum quantity that can be stored in OW.

t_w and T are the times at which inventories in RW and OW drop to zero respectively.

α, β are the deterioration rates in OW and RW respectively, with $\alpha > \beta$.

h_r , h_o are the holding costs per unit per unit time of RW and OW respectively.

A is the ordering cost per order.

I_P , I_e are the interest payable and interest earned return rates respectively.

c, p are the purchasing cost and selling price of the item respectively.

M and N are the trade credit periods offered to the retailer by the supplier and to the customer by the retailer respectively, where $M>N$.

TC is the total annual relevant costs of the model to be minimized.

All other notations not defined here will be defined in due course.

The following are the assumptions made in building the model:

a) Demand and deterioration rates are assumed to be constant in both warehouses.

b) Deterioration rate in RW is less than that in OW, i.e. $\alpha > \beta$ due to higher preserving facilities in RW and charges higher holding cost in RW than in OW, i.e. $h_r > h_o$. This give rise to the assumption, $h_r - h_o > c(\alpha - \beta)$.

c) The demand in a warehouse is greater than the deterioration rate in the warehouse, i.e. $D > \alpha W$ and $D > \beta(Q - W)$ for OW and RW respectively.

d) Due to the high holding cost of the RW, i.e. $h_r > h_o$, the goods in OW are dispatched only after the inventory in RW drops to zero.

e) Shortage is not allowed and lead time is assumed to be zero.

f) Replenishment cycle is finite.

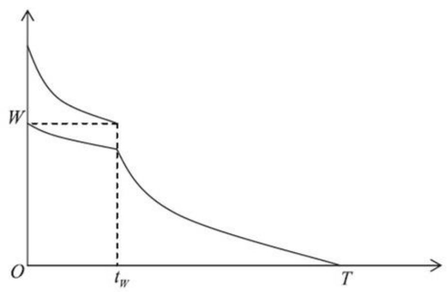

Fig 1: pictorial presentation of the model.

2 MODEL FORMULATION

We first sell goods from RW since the model is a Last-In-First-Out, LIFO. At $t = t_w$ the inventory at RW drops to zero due to demand and deterioration during the period $[0, t_w]$, while goods in OW at the same period are depleted due to deterioration only. At $t = T$, both warehouses become empty due to depletion in OW by demand and deterioration during the period $[t_w, T]$. These phenomena are represented by the following differential equations:

$$\frac{dI_r(t)}{dt} + \beta I_r(t) = -D \qquad 0 \le t \le t_w \tag{1}$$

$$\frac{dI_o(t)}{dt} + \alpha I_o(t) = 0 \qquad 0 \le t \le t_w \tag{2}$$

$$\frac{dI_o(t)}{dt} + \alpha I_o(t) = -D \quad , \qquad t_w \le t \le T \tag{3}$$

with the boundary condition $I_r(t_w) = 0$ for RW in (1), initial condition $I_o(0) = W$ for OW in (2) and boundary condition $I_o(T) = 0$ for OW in (3).

The solutions to equations (1) - (3) using the initial and boundary conditions are given as

$$I_r(t) = \frac{D}{\beta}\left(e^{\beta(t_w - t)} - 1\right), \qquad 0 \le t \le t_w \tag{4}$$

$$I_o(t) = We^{-\alpha t} \qquad 0 \le t \le t_w \tag{5}$$

$$I_o(t) = \frac{D}{\alpha}\left(e^{\alpha(T-t)} - 1\right), \qquad t_w \le t \le T \tag{6}$$

For the model to stand, there must be continuity in OW at $t = t_w$. Therefore, substituting $t = t_w$ into equations (5) and (6) and equating them, we get the following after simplification:

$$T = t_w + \frac{1}{\alpha}\ln\left(\frac{\alpha}{D}We^{-\alpha t_w} + 1\right) \tag{7}$$

For us to get the Annual Total Relevant Costs (TC), we calculate the following costs and add up.

a) Annual Ordering Cost which is $\frac{A}{T}$ (8)

b) Annual Holding Cost

The total annual holding cost for both RW and OW is given by

$$\frac{Dh_r}{\beta^2 T}\left(e^{\beta t_w} - \beta t_w - 1\right) + \frac{h_o}{T}\left(\frac{W}{\alpha}(1 - e^{-\alpha t_w}) + \frac{D}{\alpha^2}(e^{\alpha(T-t_w)} - \alpha(T - t_w) - 1)\right) \tag{9}$$

c) Annual Deterioration Cost.

The annual total deterioration cost in both warehouse is

$$\frac{c}{T}\left(\beta\int_0^{t_w} I_r(t)dt + \alpha\left[\int_0^{t_w} I_o(t)dt + \int_{t_w}^{T} I_o(t)dt\right]\right) = \frac{DC}{\beta T}\left(e^{\beta t_w} - \beta t_w - 1\right) + \frac{CW}{T}(1 - e^{-\alpha t_w}) + \frac{DC}{\alpha T}\left(e^{\alpha(T-t_w)} - \alpha(T - t_w) - 1\right) \tag{10}$$

d) Annual Interest Payable I_P and Interest Earned I_E

Based on the values of N, M, t_w and T and restricting $N < t_w$, four cases can occur as follows;

(1) $N < M \leq t_w < T$ (2) $N \leq t_w \leq M < T$ (3) $T \leq M < T + N$ (4) $T + N \leq M$.

Case 1: $N < M \leq t_w < T$

Before the time M, the retailer will make sales and generate revenue. After the time M, the retailer will pay interest on all unsold items in stock. Therefore, the annual interest payable by the retailer using equations (4), (5) and (6) is

$$I_{P1} = \frac{cI_p}{T}\left(\frac{D}{\beta^2}\left(e^{\beta(t_w-M)} - \beta(t_w + N - M) - e^{-\beta N}\right) + \frac{W}{\alpha}\left(e^{-\alpha M} - e^{-\alpha t_w}\right) + \frac{D}{\alpha^2}\left(e^{\alpha(T-t_w)} - \alpha(T + N - t_w) - e^{-\alpha N}\right)\right) \tag{11}$$

The retailer will earn interest on the sales revenue recovered from the customers during the period $[N, M]$ and the interest earned is given by

$$I_{E1} = \frac{pI_e}{T}\left(\int_N^M D(t - N)dt\right) = \frac{pDI_e}{2T}(M - N)^2 \tag{12}$$

Therefore, the total annual relevant costs for this case is the sum of the ordering cost OC, the holding cost HC, the deterioration cost, DC and interest charged I_C minus the interest earned I_E and so it is given as follows:

$$TC_1 = OC + HC + DC + I_{P1} - I_{E1}$$

Using equations (8), (9), (10), (11) and (12) we obtain

$$TC_1 = \frac{1}{T}\Big(A + \frac{D}{\beta^2}\Big((h_r + c\beta)\left(e^{\beta t_w} - \beta t_w - 1\right) + cI_p\left(e^{\beta(t_w-M)} - \beta(t_w + N - M) - e^{-\beta N}\right)\Big) + \frac{W}{\alpha}\Big((h_o + c\alpha)(1 - e^{-\alpha t_w}) + cI_p(e^{-\alpha M} - e^{-\alpha t_w})\Big) + \frac{D}{\alpha^2}\Big((h_o + c\alpha)\left(e^{\alpha(T-t_w)} - \alpha(T - t_w) - 1\right) + cI_p\left(e^{\alpha(T-t_w)} - \alpha(T + N - t_w) - e^{-\alpha N}\right)\Big) - \frac{1}{2}pDI_e(M - N)^2\Big) \tag{13}$$

Case 2: $N \leq t_w \leq M < T$

In this case, goods in RW have finished before the date of permissible delay. If $M \leq t_w + N$, there is outstanding payment from the last customers that bought goods from RW on credit basis. Therefore, the retailer will pay interest for the period $(M, t_w + N)$ for the goods bought from RW and $[M, T + N]$ for the goods bought from OW. Therefore, the annual interest payable by the retailer using equations (4) and (6) is

$$I_{P2} = \frac{cI_p}{T}\left(\frac{D}{\beta^2}\left(e^{\beta(t_w-M)} - \beta(t_w + N - M) - e^{-\beta N}\right) + \frac{D}{\alpha^2}\left(e^{\alpha(T-M)} - \alpha(T + N - M) - e^{-\alpha N}\right)\right) \tag{14}$$

Likewise, the retailer will earn interest for the period (N, M) on the revenue generated. Therefore, the interest earned is given as

$$I_{E2} = \frac{pI_e}{T}\int_N^M D(t - N)dt = \frac{pDI_e}{2T}(M - N)^2 \tag{15}$$

The total annual relevant costs of the system in this case is given by

$$TC_2 = OC + HC + DC + I_{P2} - I_{E2}$$

Using the equations (8), (9), (10), (14) and (15), we get

$$TC_2 = \frac{1}{T}\Big(A + \frac{D}{\beta^2}\Big((h_r + c\beta)\big(e^{\beta t_w} - \beta t_w - 1\big) + cI_p\big(e^{\beta(t_w - M)} - \beta(t_w + N - M) - e^{-\beta N}\big)\Big) + \frac{W}{\alpha}(h_o + c\alpha)(1 - e^{-\alpha t_w}) + \frac{D}{\alpha^2}\Big((h_o + c\alpha)\big(e^{\alpha(T - t_w)} - \alpha(T - t_w) - 1\big) + cI_p\big(e^{\alpha(T-M)} - \alpha(T + N - M) - e^{-\alpha N}\big)\Big) - \frac{1}{2}pDI_e(M - N)^2\Big) \quad (16)$$

Case 3: $T \leq M < T + N$

In this case, both warehouses are empty before the time M. Therefore, the retailer will only pay interest on the outstanding payments from the last customer. Therefore, the annual interest payable by the retailer using equation (6) is

$$I_{P3} = \frac{DcI_p}{\alpha^2 T}\left(e^{\alpha(T-M)} - \alpha(T + N - M) - e^{-\alpha N}\right) \quad (17)$$

The retailer will earn interest on the sales revenue recovered from the customers during $[N, M]$. Therefore, the annual interest is given by

$$I_{E3} = \frac{pI_e}{T}\left(\int_N^T D(t - N)dt + DT(M - T)\right) = \frac{pDI_e}{2T}\left((T - N)^2 + 2T(M - T)\right) \quad (18)$$

The total annual relevant costs for the model in this case is given by

$$TC_3 = OC + HC + DC + I_{P3} - I_{E3}$$

Using the equations (8), (9), (10), (17) and (18), we see that

$$TC_3 = \frac{1}{T}\Bigg(A + \frac{D}{\beta^2}(h_r + c\beta)\big(e^{\beta t_w} - \beta t_w - 1\big) + \frac{W}{\alpha}(h_o + c\alpha)(1 - e^{-\alpha t_w}) + \frac{D}{\alpha^2}\Big((h_o + c\alpha)\big(e^{\alpha(T - t_w)} - \alpha(T - t_w) - 1\big) + cI_p\big(e^{\alpha(T-M)} - \alpha(T + N - M) - e^{-\alpha N}\big)\Big) - \frac{1}{2}pDI_e\big((T - N)^2 + 2T(M - T)\big)\Bigg) \quad (19)$$

Case 4: $T + N \leq M$

In this case, the goods have finished in both warehouses before the time M and the retailer has no outstanding payment from the customers. Therefore, the annual interest payable by the retailer is given by

$$I_{P4} = 0 \quad (20)$$

The retailer had already sold all the items and had also received all payments from the customers, therefore, the annual interest earned by the retailer is given by

$$I_{E4} = \frac{pI_e}{T}\left(\int_N^T D(t - N)dt + DT(T + N - T) + D(T + N)(M - T - N)\right) = \frac{pDI_e}{2T}\left((T - N)^2 + 2TN + 2(T + N)(M - T - N)\right) \quad (21)$$

Therefore, the annual relevant costs for the model in this case is given by

$TC_4 = OC + HC + DC + I_{P4} - I_{E4}$

Using the equations (8), (9), (10), (20) and (21) we get

$$TC_4 = \frac{1}{T}\Bigg(A + \frac{D}{\beta^2}(h_r + c\beta)\left(e^{\beta t_w} - \beta t_w - 1\right) + \frac{W}{\alpha}(h_o + c\alpha)(1 - e^{-\alpha t_w}) + \frac{D}{\alpha^2}(h_o + c\alpha)\left(e^{\alpha(T-t_w)} - \alpha(T - t_w) - 1\right) - \frac{1}{2}pDI_e\left((T-N)^2 + 2TN + 2(T+N)(M-T-N)\right)\Bigg) \quad (22)$$

3 OPTIMIZATION AND ANALYSIS

The necessary conditions for TC_1 to have a minimum are $\frac{\partial TC_1}{\partial t_w} = 0$ and $\frac{\partial TC_1}{\partial T} = 0$.

Using equation (13), differentiating and setting the derivative to zero, we get

$$\frac{\partial TC_1}{\partial t_w} = \frac{1}{T}\Bigg(\frac{D}{\beta}\left((h_r + c\beta)\left(e^{\beta t_w} - 1\right) + cI_p\left(e^{\beta(t_w - M)} - 1\right)\right) + W\left(h_o + c\alpha + cI_p\right)e^{-\alpha t_w} + \frac{D}{\alpha}\left(h_o + c\alpha + cI_p\right)\left(1 - e^{\alpha(T-t_w)}\right)\Bigg) = 0 \quad (23)$$

Using (13) again, we obtain

$$\frac{\partial TC_1}{\partial T} = \frac{1}{T}\left(\frac{D}{\alpha}\left(\left(h_o + c\alpha + cI_p\right)\left(e^{\alpha(T-t_w)} - 1\right)\right) - TC_1\right) = 0 \quad (24)$$

Equations (23) and (24) are highly non – linear and their solutions give the required values of t_w and T. To confirm that the solutions exist and are unique, we let (t_w^{1*}, T_1^*) be the optimal solutions to (23) and (24). If we confirm that the determinant of the Hessian matrix $\begin{bmatrix} \frac{\partial^2 TC_1}{\partial t_w^2} & \frac{\partial^2 TC_1}{\partial T \partial t_w} \\ \frac{\partial^2 TC_1}{\partial t_w \partial T} & \frac{\partial^2 TC_1}{\partial T^2} \end{bmatrix} = \left\{\frac{\partial^2 TC_1}{\partial t_w^2}\frac{\partial^2 TC_1}{\partial T^2} - \frac{\partial^2 TC_1}{\partial t_w \partial T}\frac{\partial^2 TC_1}{\partial T \partial t_w}\right\}\Big|_{(t_w^{1*}, T_1^*)} > 0$, then the optimal solutions exist and are unique. Thus,

Using left hand side of equation (23), we see that

$$\frac{\partial^2 TC_1}{\partial t_w^2} = \frac{1}{T}\left(D\left((h_r + c\beta)e^{\beta t_w} + cI_p e^{\beta(t_w - M)}\right) - \alpha W\left(h_o + c\alpha + cI_p\right)e^{-\alpha t_w}\right) + D\left(h_o + c\alpha + cI_p\right)e^{\alpha(T-t_w)} \quad (25)$$

Evaluating equation (25) at the point (t_w^{1*}, T_1^*) we get

$\frac{\partial^2 TC_1}{\partial t_w^2}\Big|_{(t_w^{1*}, T_1^*)} = \frac{1}{T}\left(D\left((h_r + c\beta)e^{\beta t_w} + cI_p e^{\beta(t_w - M)}\right) - \alpha W\left(h_o + c\alpha + cI_p\right)e^{-\alpha t_w}\right) + D\left(h_o + c\alpha + cI_p\right)e^{\alpha(T-t_w)}\Big|_{(t_w^{1*}, T_1^*)} > \frac{1}{T}\left(\left(De^{\alpha T} - \alpha W\right)\left(h_o + c\alpha + cI_p\right)e^{-\alpha t_w}\right)\Big|_{(t_w^{1*}, T_1^*)} > 0$ since $D - \alpha W > 0$ from assumption (c) and $e^{\alpha T} \geq 1$ for any value of T, meaning that $(De^{\alpha T} - \alpha W) > 0$.

Using left hand side of (24), we see that

$$\frac{\partial^2 TC_1}{\partial T^2} = \frac{1}{T}\left(D\left(\left(h_o + c\alpha + cI_p\right)e^{\alpha(T-t_w)}\right) - 2\frac{\partial TC_1}{\partial T}\right) \quad (26)$$

Evaluating equation (26) at the point (t_w^{1*}, T_1^*) we see that

$$\left.\frac{\partial^2 TC_1}{\partial T^2}\right|_{(t_w^{1*},T_1^*)} = \frac{1}{T}\left(D\left(\left(h_o + c\alpha + cI_p\right)e^{\alpha(T-t_w)}\right)\right)\Big|_{(t_w^{1*},T_1^*)}$$

since $\left.\frac{\partial TC_1}{\partial T}\right|_{(t_w^{1*},T_1^*)} = 0$. Thus, value of $\left.\frac{\partial^2 TC_1}{\partial T^2}\right|_{(t_w^{1*},T_1^*)} > 0$.

Using the left hand side of equation (23), we obtain

$$\frac{\partial^2 TC_1}{\partial T \partial t_w} = -\frac{1}{T}\left(D\left(h_o + c\alpha + cI_p\right)e^{\alpha(T-t_w)} + \frac{\partial TC_1}{\partial t_w}\right) \quad (27)$$

Also, using left hand side of equation (24), we get

$$\frac{\partial^2 TC_1}{\partial t_w \partial T} = -\frac{1}{T}\left(D\left(h_o + c\alpha + cI_p\right)e^{\alpha(T-t_w)} + \frac{\partial TC_1}{\partial t_w}\right) \quad (28)$$

Evaluating (27) and (28) at the point (t_w^{1*}, T_1^*) and noting that $\left.\frac{\partial TC_1}{\partial t_w}\right|_{(t_w^{1*},T_1^*)} = 0$, we find that

$$\left.\frac{\partial^2 TC_1}{\partial T \partial t_w}\right|_{(t_w^{1*},T_1^*)} = -\frac{1}{T}\left(D\left(h_o + c\alpha + cI_p\right)e^{\alpha(T-t_w)}\right)\Big|_{(t_w^{1*},T_1^*)} = \left.\frac{\partial^2 TC_1}{\partial t_w \partial T}\right|_{(t_w^{1*},T_1^*)}$$

Theorem 1: The cost function in equation (13) is a convex function.

Proof: Equations (25) – (28) confirm that the required Hessian Matrix is positive definite, hence TC_1 is convex.

The necessary conditions for TC_2 to have a minimum are $\frac{\partial TC_2}{\partial t_w} = 0$ and $\frac{\partial TC_2}{\partial T} = 0$

Using (16), differentiating and setting the derivative to zero, we get

$$\frac{\partial TC_2}{\partial t_w} = \frac{1}{T}\Bigg(\frac{D}{\beta}\left(\left(h_r + c\beta\right)\left(e^{\beta t_w} - 1\right) + cI_p\left(e^{\beta(t_w-M)} - 1\right)\right) + W\left(h_o + c\alpha\right)e^{-\alpha t_w} + \frac{D}{\alpha}\left(h_o + c\alpha\right)\left(1 - e^{\alpha(T-t_w)}\right)\Bigg) = 0 \quad (29)$$

Also using (16),

$$\frac{\partial TC_2}{\partial T} = \frac{1}{T}\left(\frac{D}{\alpha}\left(\left(h_o + c\alpha\right)\left(e^{\alpha(T-t_w)} - 1\right) + cI_p\left(e^{\alpha(T-M)} - 1\right)\right) - TC_2\right) = 0 \quad (30)$$

The solutions of (29) and (30) say (t_w^{2*}, T_2^*) give the values of t_w and T. To show that the solutions exist and are unique, we show that the determinant of the corresponding Hessian matrix of the cost function is positive definite, that is, $\left.\left\{\frac{\partial^2 TC_2}{\partial t_w^2}\frac{\partial^2 TC_2}{\partial T^2} - \frac{\partial^2 TC_2}{\partial t_w \partial T}\frac{\partial^2 TC_2}{\partial T \partial t_w}\right\}\right|_{(t_w^{2*},T_2^*)} > 0$.

Using the left hand side of equation (29), we see that

$$\frac{\partial^2 TC_2}{\partial t_w^2} = \frac{1}{T}\left(D\left((h_r + c\beta)e^{\beta t_w} + cI_p e^{\beta(t_w - M)}\right) - \alpha W(h_o + c\alpha)e^{-\alpha t_w} + D(h_o + c\alpha)e^{\alpha(T-t_w)}\right) \quad (31)$$

$\left.\frac{\partial^2 TC_2}{\partial t_w^2}\right|_{(t_w^{2*}, T_2^*)} = \frac{1}{T}\left(D\left((h_r + c\beta)e^{\beta t_w} + cI_p e^{\beta(t_w - M)}\right) - \alpha W(h_o + c\alpha)e^{-\alpha t_w} + D(h_o + c\alpha)e^{\alpha(T-t_w)}\right)\Big|_{(t_w^{2*}, T_2^*)} > \frac{1}{T}\left((De^{\alpha T} - \alpha W)(h_o + c\alpha)e^{-\alpha t_w}\right)\Big|_{(t_w^{2*}, T_2^*)} > 0$ since $D - \alpha W > 0$ and $e^{\alpha T} \geq 1$ for any value of T.

Using the left hand side of (30), we see that

$$\frac{\partial^2 TC_2}{\partial T^2} = \frac{1}{T}\left(D\left((h_o + c\alpha)e^{\alpha(T-t_w)} + cI_p e^{\alpha(T-M)}\right) - 2\frac{\partial TC_2}{\partial T}\right) \quad (32)$$

$\left.\frac{\partial^2 TC_2}{\partial T^2}\right|_{(t_w^{2*}, T_2^*)} = \frac{1}{T}\left(D\left((h_o + c\alpha)e^{\alpha(T-t_w)} + cI_p e^{\alpha(T-M)}\right)\right)\Big|_{(t_w^{2*}, T_2^*)}$ since $\left.\frac{\partial TC_2}{\partial T}\right|_{(t_w^{2*}, T_2^*)} = 0.$

Thus, $\left.\frac{\partial^2 TC_2}{\partial T^2}\right|_{(t_w^{2*}, T_2^*)} > 0.$

Using the left hand side of equation (29), we obtain

$$\frac{\partial^2 TC_2}{\partial T \partial t_w} = -\frac{1}{T}\left(D(h_o + c\alpha)e^{\alpha(T-t_w)} + \frac{\partial TC_2}{\partial t_w}\right) \quad (33)$$

Also, using the left hand side of (30), we see that

$$\frac{\partial^2 TC_2}{\partial t_w \partial T} = -\frac{1}{T}\left(D(h_o + c\alpha)e^{\alpha(T-t_w)} + \frac{\partial TC_2}{\partial t_w}\right) \quad (34)$$

Evaluating (33) and (34) at the point (t_w^{2*}, T_2^*) and noting that $\left.\frac{\partial TC_2}{\partial t_w}\right|_{(t_w^{2*}, T_2^*)} = 0$ we see that

$$\left.\frac{\partial^2 TC_2}{\partial T \partial t_w}\right|_{(t_w^{2*}, T_2^*)} = -\frac{1}{T}\left(D(h_o + c\alpha)e^{\alpha(T-t_w)}\right)\Big|_{(t_w^{2*}, T_2^*)} = \left.\frac{\partial^2 TC_2}{\partial t_w \partial T}\right|_{(t_w^{2*}, T_2^*)}$$

Theorem 2: The cost function in equation (16) is a convex function.

Proof: The proof is clear using equations (31) – (34).

The necessary conditions for TC_3 to have a minimum are $\frac{\partial TC_3}{\partial t_w} = 0$ and $\frac{\partial TC_3}{\partial T} = 0$

Using (19), differentiating and setting the derivative to zero, we get

$$\frac{\partial TC_3}{\partial t_w} =$$

$$\frac{1}{T}\left(\frac{D}{\beta}(h_r + c\beta)\left(e^{\beta t_w} - 1\right) + W(h_o + c\alpha)e^{-\alpha t_w} + \frac{D}{\alpha}(h_o + c\alpha)\left(1 - e^{\alpha(T-t_w)}\right)\right) = 0 \quad (35)$$

Also using (19),

$\frac{\partial TC_3}{\partial T} =$

$$\frac{1}{T}\left(\frac{D}{\alpha}\Big((h_o + c\alpha)\big(e^{\alpha(T-t_w)} - 1\big) + cI_p\big(e^{\alpha(T-M)} - 1\big)\Big) - pDI_e(M - N - T) - TC_3\right) = 0 \quad (36)$$

The solutions to equations (35) and (36) give the required values of t_w and T. To show that the solutions to the equations exist and are unique, we show that $\left\{\frac{\partial^2 TC_3}{\partial t_w^2}\frac{\partial^2 TC_3}{\partial T^2} - \frac{\partial^2 TC_3}{\partial t_w \partial T}\frac{\partial^2 TC_3}{\partial T \partial t_w}\right\}\Big|_{(t_w^{3*},T_3^*)} >$ 0.

Let (t_w^{3*},T_3^*) be the optimal solutions of (35) and (36), then using the left hand side of equation (35) we obtain

$$\frac{\partial^2 TC_3}{\partial t_w^2} = \frac{1}{T}\left(D(h_r + c\beta)e^{\beta t_w} - \alpha W(h_o + c\alpha)e^{-\alpha t_w} + D(h_o + c\alpha)e^{\alpha(T-t_w)}\right) \quad (37)$$

Evaluating equation (37) at the point (t_w^{3*},T_3^*), we see that

$$\frac{\partial^2 TC_3}{\partial t_w^2}\Big|_{(t_w^{3*},T_3^*)} = \frac{1}{T}\left(D(h_r + c\beta)e^{\beta t_w} - \alpha W(h_o + c\alpha)e^{-\alpha t_w} + D(h_o + c\alpha)e^{\alpha(T-t_w)}\right)\Big|_{(t_w^{3*},T_3^*)} > \frac{1}{T}\left((De^{\alpha T} - \alpha W)(h_o + c\alpha)e^{-\alpha t_w}\right)\Big|_{(t_w^{3*},T_3^*)} > 0$$

Using the left hand side of equation (36), we see that

$$\frac{\partial^2 TC_3}{\partial T^2} = \frac{1}{T}\left(D\left((h_o + c\alpha)e^{\alpha(T-t_w)} + cI_p e^{\alpha(T-M)}\right) + DpI_e - 2\frac{\partial TC_3}{\partial T}\right) \quad (38)$$

and $\frac{\partial^2 TC_3}{\partial T^2}\Big|_{(t_w^{3*},T_3^*)} = \frac{1}{T}\left(D\left((h_o + c\alpha)e^{\alpha(T-t_w)} + cI_p e^{\alpha(T-M)}\right) + DpI_e\right)\Big|_{(t_w^{3*},T_3^*)} > 0$

since $\frac{\partial TC_3}{\partial T}\Big|_{(t_w^{3*},T_3^*)} = 0$

Using the left hand side of equation (35), we see that

$$\frac{\partial^2 TC_3}{\partial T \partial t_w} = -\frac{1}{T}\left(D(h_o + c\alpha)e^{\alpha(T-t_w)} + \frac{\partial TC_3}{\partial t_w}\right) \quad (39)$$

Using the left hand side of equation (36), we find that

$$\frac{\partial^2 TC_3}{\partial t_w \partial T} = -\frac{1}{T}\left(D(h_o + c\alpha)e^{\alpha(T-t_w)} + \frac{\partial TC_3}{\partial t_w}\right) \quad (40)$$

Evaluating equations (39) and (40) at the point (t_w^{3*},T_3^*), and noting that $\frac{\partial TC_3}{\partial t_w}\Big|_{(t_w^{3*},T_3^*)} = 0$ we get

$$\frac{\partial^2 TC_3}{\partial T \partial t_w}\Big|_{(t_w^{3*},T_3^*)} = -\frac{1}{T}\left(D(h_o + c\alpha)e^{\alpha(T-t_w)}\right)\Big|_{(t_w^{3*},T_3^*)} = \frac{\partial^2 TC_3}{\partial t_w \partial T}\Big|_{(t_w^{3*},T_3^*)}$$

Theorem 3: The cost function in equation (19) is a convex function.

Proof: the proof is clear using equations (37) – (40)

The necessary conditions for TC_4 to have a minimum are $\frac{\partial TC_4}{\partial t_w} = 0$ and $\frac{\partial TC_4}{\partial T} = 0$

Using (22), differentiating and setting the result to zero, we get

$$\frac{\partial TC_4}{\partial t_w} = \frac{1}{T}\left(\frac{D}{\beta}(h_r + c\beta)\left(e^{\beta t_w} - 1\right) + W(h_o + c\alpha)e^{-\alpha t_w} + \frac{D}{\alpha}(h_o + c\alpha)\left(1 - e^{\alpha(T - t_w)}\right)\right) = 0 \quad (41)$$

Also using (22), we get

$$\frac{\partial TC_4}{\partial T} = \frac{1}{T}\left(\frac{D}{\alpha}(h_o + c\alpha)\left(e^{\alpha(T - t_w)} - 1\right) - pDI_e(M - 2N - T) - TC_4\right) = 0 \quad (42)$$

The solutions to equations (41) and (42) give the required values of t_w and T. To show that the solutions exist and are unique, we show that the determinant of the corresponding Hessian matrix evaluated at the optimal point (t_w^{4*},T_4^*) is positive definite.

Using the left hand side of (41), we get

$$\frac{\partial^2 TC_4}{\partial t_w^2} = \frac{1}{T}\left(D(h_r + c\beta)e^{\beta t_w} - \alpha W(h_o + c\alpha)e^{-\alpha t_w} + D(h_o + c\alpha)e^{\alpha(T - t_w)}\right) \quad (43)$$

Evaluating equation (43) at the point (t_w^{4*},T_4^*) we have

$$\left.\frac{\partial^2 TC_4}{\partial t_w^2}\right|_{(t_w^{4*},T_4^*)} = \frac{1}{T}\left(D(h_r + c\beta)e^{\beta t_w} - \alpha W(h_o + c\alpha)e^{-\alpha t_w} + D(h_o + c\alpha)e^{\alpha(T - t_w)}\right)\Big|_{(t_w^{4*},T_4^*)} >$$
$$\frac{1}{T}\left((De^{\alpha T} - \alpha W)(h_o + c\alpha)e^{-\alpha t_w}\right)\Big|_{(t_w^{4*},T_4^*)} > 0$$

Using the left hand side of equation (42), we obtain

$$\frac{\partial^2 TC_4}{\partial T^2} = \frac{1}{T}\left(D(h_o + c\alpha)e^{\alpha(T - t_w)} + pDI_e - 2\frac{\partial TC_4}{\partial T}\right) \quad (44)$$

Evaluating equation (44) at the point (t_w^{4*},T_4^*) we get

$$\left.\frac{\partial^2 TC_4}{\partial T^2}\right|_{(t_w^{4*},T_4^*)} = \frac{1}{T}\left(D(h_o + c\alpha)e^{\alpha(T - t_w)} + pDI_e\right)\Big|_{(t_w^{4*},T_4^*)} > 0 \text{ since } \left.\frac{\partial TC_4}{\partial T}\right|_{(t_w^{4*},T_4^*)} = 0$$

Using the left hand side of equation (41), we find that

$$\frac{\partial^2 TC_4}{\partial T \partial t_w} = -\frac{1}{T}\left(D(h_o + c\alpha)e^{\alpha(T - t_w)} + \frac{\partial TC_4}{\partial t_w}\right) \quad (45)$$

Using the left hand side of equation (42), we find that

$$\frac{\partial^2 TC_4}{\partial t_w \partial T} = -\frac{1}{T}\left(D(h_o + c\alpha)e^{\alpha(T - t_w)} + \frac{\partial TC_4}{\partial t_w}\right) \quad (46)$$

Evaluating the equations (45) and (46) at the point (t_w^{4*},T_4^*), and noting that $\left.\frac{\partial TC_4}{\partial t_w}\right|_{(t_w^{4*},T_4^*)} = 0$ we have

$$\left.\frac{\partial^2 TC_4}{\partial T \partial t_w}\right|_{(t_w^{4*},T_4^*)} = -\frac{1}{T}\left(D(h_o + c\alpha)e^{\alpha(T - t_w)}\right)\Big|_{(t_w^{4*},T_4^*)} = \left.\frac{\partial^2 TC_4}{\partial t_w \partial T}\right|_{(t_w^{4*},T_4^*)}$$

Theorem 4: The cost function in equation (22) is a convex function.

Proof: The proof of the theorem is clear using equations (43) – (46).

4 EMPIRICAL REALIZATON OF THE MODEL

Suppose an inventory system has the following parameters; $A = 1500$, $D = 2000$, $W = 100$, $c = 10$, $p = 15$, $h_r = 3$, $h_o = 1$, $\beta = 0.06$, $\alpha = 0.1$, $M = 0.5$, $N = 0.25$, $I_e = 0.12$, $I_p = 0.15$.

We find out that the optimal time period (given in days) in which goods finish at RW is $t_w^* = 184\ (0.5041)$ days and both warehouses become empty at $T^* = 201(0.5515\ )$ days and the associated optimal cost is $TC^* = 4047.25$.

Table 1: Optimal solutions of the model in respect of different cost functions.

Cases	t_w	T	TC
Case 1	0.5041	0.5515	4047.25
Case 2	0.5041	0.5515	4047.29
Case 3	0.3890	0.4370	4190.76
Case 4	0.3890	0.4370	5289.92

Based on the results obtained from this example, it is better to consider market situation in which the retailer is given grace period smaller than the time at which goods in the rented warehouse finish. Therefore, it is advisable based on the result obtained, for the retailer to accept a shorter period of time as a grace possibly due to high interest charge if given longer period of time.

4.1 Sensitivity Analysis on the Example

Using the example above, we study the effect of parameter changes (sensitivity analysis) of the parameters *W, A,* and *D* on the optimal values of the example. We determine optimal values for the new parameters $W \in (100,\ 250,\ 400)$, $A \in (1500,\ 2000,\ 2500)$ and $D \in (2000,\ 2500,\ 3000)$, as follows:

Table 2: Sensitivity analysis for the parameters (*W, A, and D*)

W	A	D	t_w^*	T^*	TC^*
	1500	2000	0.5041	0.5515	4047.245
		2500	0.4575	0.4957	4378.649
		3000	0.4247	0.4566	4653.238
		2000	0.5836	0.6306	4892.253

100	2000	2500	0.5315	0.5694	5317.470
		3000	0.4904	0.5221	5675.639
	2500	2000	0.6548	0.7015	5641.444
		2500	0.5845	0.5321	6150.570
		3000	0.5479	0.5795	6583.738
250	1500	2000	0.4356	0.5546	3852.312
		2500	0.4027	0.4983	4179.288
		3000	0.3781	0.4580	4450.593
	2000	2000	0.5151	0.6330	4694.242
		2500	0.4740	0.5689	5115.393
		3000	0.4438	0.5232	5470.516
	2500	2000	0.5863	0.7035	5441.701
		2500	0.5397	0.6340	5946.965
		3000	0.5014	0.5803	6377.202

Table 2 cont'd

W	A	D	t_w^*	T^*	TC^*
400	1500	2000	0.3699	0.5608	3685.694
		2500	0.3479	0.5013	4005.472
		3000	0.3315	0.4597	4271.355
	2000	2000	0.4493	0.6387	4520.438
		2500	0.4219	0.5741	4935.193
		3000	0.3973	0.5246	5285.489
		2000	0.5205	0.7086	5263.287

	2500	2500	0.4849	0.6362	5762.647
		3000	0.4548	0.5814	6188.465

Table 3: Sensitivity Analysis on the trade credit periods ($M>N$)

M	N	t_w^*	T^*	TC^*
0.75	0.5	0.4877	0.5352	3140.583
	0.25	0.5260	0.5734	3530.410
	0.125	0.5069	0.5543	3329.010
0.50	0.25	0.5041	0.5515	4047.245
	0.125	0.5041	0.5515	4045.273
0.25	0.125	0.4932	0.5406	4694.080

From Tables 2 and 3 above, we can deduce the following:

i) The retailer incurs the highest total relevant cost when the capacity of own warehouse is the smallest, that is, W=100. This is expected because the larger part of the stock ordered by the retailer has been kept in the rented warehouse and the holding cost is bigger in the rented warehouse based on the assumptions of the model.

ii) The retailer incurs the smallest total relevant cost when the stocking capacity of own warehouse is the highest, i.e. W=400. This is also expected as it is the reverse case of (i) above.

iii) For each value of W, as the ordering cost A increases, TC also increases which is also expected since the larger the set up cost, the larger will be the TC. On the other hand, for each value of A, as the demand increases, TC also increases which is not expected because larger demand is supposed to reduce the stock which should translate to reduction in holding cost as well as the TC. The reason for increase in TC is probably due to the interest incurred by the retailer.

iv) As for the cycle length T, as W increases, the T value increases which is due to the fact that bigger size in W result to larger cycle T. However, as D increases across different values of A, T decreases. This is expected since larger demand is supposed to shorten the replenishment cycle.

v) Looking at the values of t_w, one notes that as W increases, the t_w values decrease and this is because once the size of W is increased, the amount of items stocked in RW decreases and therefore, this results in a decrease in t_w.

vi) Note that, increase/decrease in ordering cost doesn't have effect on the length of the replenishment cycle.

vii) The retailer incurs more cost if he/she gives to the customers a trade credit period 50% less than what was offered to him/her by the supplier. The less the period of the downstream the less the total cost which is expected as instant payment to the retailer will helps in increasing the interest earned on the sales revenue.

5 CONCLUSION AND RECOMMENDATIONS

In the study, an EOQ model for capacity constrained warehouse has been developed and discussed having looked at situations when both the retailer and retailer's customers are assumed credit – worthy. In the work, full trade credit has been incorporated at both upstream and downstream level to reflect the happenings of some market situations. The numerical example shows that, it is better for the retailer to accept shorter periods of permissible delay in payment possibly due to high interest charged if given longer period of time. From the sensitivity analysis carried out, it is worthy to note that any adjustment on the crucial parameters, *W, A,* and *D* from table 2 and as well the less the downstream period the less the total cost from table 3, and hence results in changes on the optimal solution as discussed earlier.

It is recommended that, this work is extended to consider situations when the demand is no longer deterministic but stochastic or price/stock dependent. It can be extended to incorporate shortages. The holding cost and deterioration cost can also be taken as time – varying. Risk of default in payment can also be considered.

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